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A CONGRUENCE FOR PRIMES

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Abstract. With the help of the Pell sequence we obtain the following new congruence for odd primes:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod p.$$

1. Introduction

Congruences for primes are of great interest. Some examples are

$$a^p \equiv a \pmod p \quad \text{(Fermat's little theorem)},$$
$$(p-1)! \equiv -1 \pmod p \quad \text{(Wilson's theorem)},$$
$$\binom{2p-1}{p} \equiv 1 \pmod{p^3} \text{ for primes } p > 3 \quad \text{(Wolstenholme's theorem)}.$$

In this paper we shall establish the following congruence for odd primes:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod p$$

where $[\cdot]$ is the greatest integer function.

2. Some preparations

Let

$$\begin{bmatrix} n \\ r \end{bmatrix} = \sum_{\substack{k=0 \\ k \equiv r \pmod 8}}^{n} \binom{n}{k}.$$

Using

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{and} \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

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one can easily prove that

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}, \qquad \begin{bmatrix} n+1 \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} + \begin{bmatrix} n \\ r-1 \end{bmatrix}.$$

So we have

$$\begin{aligned}\begin{bmatrix} n+2 \\ 2r \end{bmatrix} &= \begin{bmatrix} n+1 \\ 2r \end{bmatrix} + \begin{bmatrix} n+1 \\ 2r-1 \end{bmatrix} = \begin{bmatrix} n \\ 2r \end{bmatrix} + \begin{bmatrix} n \\ 2r-1 \end{bmatrix} + \begin{bmatrix} n \\ 2r-1 \end{bmatrix} + \begin{bmatrix} n \\ 2r-2 \end{bmatrix} \\ &= \begin{bmatrix} n \\ 2r \end{bmatrix} + 2\begin{bmatrix} n \\ 2((n+1)/2-r) \end{bmatrix} + \begin{bmatrix} n \\ 2(r-1) \end{bmatrix}.\end{aligned}$$

By induction on n, one can prove the following result (due to Zhi-Hong Sun [1]) involving the Pell sequence $\{P_n\}$ ($P_0 = 0$, $P_1 = 1$, $P_{k+1} = 2P_k + P_{k-1}$) and its companion $\{Q_n\}$ ($Q_0 = 2$, $Q_1 = 2$, $Q_{k+1} = 2Q_k + Q_{k-1}$).

Lemma 1. *Let $n > 0$ be odd. We have*

(i) *if $n \equiv 1 \pmod 8$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} + (-1)^r 2^{(n-5)/2} + (-1)^{[r/2]+(n-1)/8} 2^{(n-5)/4} P_{(n+(-1)^r)/2};$$

(ii) *if $n \equiv 3 \pmod 8$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} - (-1)^r 2^{(n-5)/2} + (-1)^{[r/2]+(n-3)/8} 2^{(n-11)/4} Q_{(n-(-1)^r)/2};$$

(iii) *if $n \equiv 5 \pmod 8$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} - (-1)^r 2^{(n-5)/2} + (-1)^{[(r+1)/2]+(n+3)/8} 2^{(n-5)/4} P_{(n-(-1)^r)/2};$$

(iv) *if $n \equiv 7 \pmod 8$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} + (-1)^r 2^{(n-5)/2} + (-1)^{[(r+1)/2]+(n+1)/8} 2^{(n-11)/4} Q_{(n+(-1)^r)/2}.$$

From Lemma 1 it follows that for odd $n > 0$

$$\begin{aligned}
n\sum_{k=1}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1} =&4\sum_{\substack{k=1\\4|k}}^{n}\frac{n}{k}\binom{n-1}{k-1}\\
=&4\sum_{\substack{k=0\\4|k}}^{n}\binom{n}{k}-4=4\left(\begin{bmatrix}n\\0\end{bmatrix}+\begin{bmatrix}n\\4\end{bmatrix}\right)-4\\
=&4\times(2^{n-2}+(-1)^{(n^2-1)/8}2^{(n-3)/2})-4\\
=&2(2^{n-1}-1)+2(-1)^{(n^2-1)/8}\\
&\times(2^{(n-1)/2}-(-1)^{(n^2-1)/8}),\\
n\sum_{k=1}^{[(n+1)/4]}\frac{1}{2k-1}\binom{n-1}{4k-3} =&2\sum_{\substack{k=0\\4|k+2}}^{n}\frac{n}{k}\binom{n-1}{k-1}=2\left(\begin{bmatrix}n\\2\end{bmatrix}+\begin{bmatrix}n\\6\end{bmatrix}\right)\\
=&2(2^{n-2}-(-1)^{(n^2-1)/8}2^{(n-3)/2})\\
=&2^{n-1}-1-(-1)^{(n^2-1)/8}(2^{(n-1)/2}-(-1)^{(n^2-1)/8}).
\end{aligned}$$

So we have

Lemma 2. *Let $n > 0$ be odd. Then*

$$F_n+(-1)^{(n^2-1)/8}E_n=\frac{1}{2}\sum_{k=1}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1},$$

$$F_n-(-1)^{(n^2-1)/8}E_n=\sum_{k=1}^{[(n+1)/4]}\frac{1}{2k-1}\binom{n-1}{4k-3}$$

where

$$F_n=\frac{2^{n-1}-1}{n}\quad\text{and}\quad E_n=\frac{2^{(n-1)/2}-(-1)^{(n^2-1)/8}}{n}.$$

For the sequences $\{P_n\}$ and $\{Q_n\}$, it is well known that

$$P_n=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n-(1-\sqrt{2})^n],\qquad Q_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n.$$

¿From this one can easily derive that

$$P_{n+1}^2+P_n^2=P_{2n+1}=\tfrac{1}{8}(Q_{n+1}^2+Q_n^2).$$

This property will be used below.

Lemma 3. *Suppose that $n>0$ is odd. Then*

$$\begin{aligned}2^{(n-1)/2}P_n=&1+\frac{n}{2}\sum_{k=1}^{[(n-1)/4]}\frac{(-1)^k}{k}\binom{n-1}{4k-1}\\&+\frac{n^2}{4}\Bigg[\Bigg(\sum_{k=1}^{[(n-1)/4]}\frac{(-1)^k}{2k}\binom{n-1}{4k-1}\Bigg)^2\\&\qquad+\Bigg(\sum_{k=1}^{[(n+1)/4]}\frac{(-1)^{k-1}}{2k-1}\binom{n-1}{4k-3}\Bigg)^2\Bigg].\end{aligned}$$

Proof. By Lemma 1 we have

$$\begin{aligned}&\left(\begin{bmatrix}n\\2r\end{bmatrix}-2^{n-3}-(-1)^{r+(n^2-1)/8}2^{(n-5)/2}\right)^2\\&\quad+\left(\begin{bmatrix}n\\2(r+1)\end{bmatrix}-2^{n-3}-(-1)^{r+1+(n^2-1)/8}2^{(n-5)/2}\right)^2\\&=\begin{cases}2^{(n-5)/2}(P_{(n+1)/2}^2+P_{(n-1)/2}^2)=2^{(n-5)/2}P_n & \text{if } n\equiv 1 \pmod 4,\\ 2^{(n-11)/2}(Q_{(n+1)/2}^2+Q_{(n-1)/2}^2)=2^{(n-5)/2}P_n & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Putting $r=0$ and noticing that

$$\begin{bmatrix}n\\0\end{bmatrix}=1+\sum_{\substack{k=1\\8\mid k}}^{n-1}\binom{n}{k}=1+nC_n,\qquad \begin{bmatrix}n\\2\end{bmatrix}=\sum_{\substack{k=1\\k\equiv 2\ (\mathrm{mod}\ 8)}}^{n-1}\binom{n}{k}=nD_n$$

where

$$C_n=\sum_{k=1}^{[(n-1)/8]}\frac{1}{8k}\binom{n-1}{8k-1}\quad\text{and}\quad D_n=\sum_{k=1}^{[(n+5)/8]}\frac{1}{8k-6}\binom{n-1}{8k-7},$$

we then get

$$\begin{aligned}2^{(n+3)/2}P_n =&\left(4\begin{bmatrix}n\\0\end{bmatrix}-2^{n-1}-(-1)^{(n^2-1)/8}2^{(n-1)/2}\right)^2\\&+\left(4\begin{bmatrix}n\\2\end{bmatrix}-2^{n-1}+(-1)^{(n^2-1)/8}2^{(n-1)/2}\right)^2\\=&[4+4nC_n-(1+nF_n)-(-1)^{(n^2-1)/8}((-1)^{(n^2-1)/8}+nE_n)]^2\\&+[4nD_n-(1+nF_n)+(-1)^{(n^2-1)/8}((-1)^{(n^2-1)/8}+nE_n)]^2\\=&[2+n(4C_n-F_n-(-1)^{(n^2-1)/8}E_n)]^2\\&+n^2[4D_n-F_n+(-1)^{(n^2-1)/8}E_n]^2\\=&4+4n(4C_n-F_n-(-1)^{(n^2-1)/8}E_n)\\&+n^2[(4C_n-F_n-(-1)^{(n^2-1)/8}E_n)^2\\&+(4D_n-F_n+(-1)^{(n^2-1)/8}E_n)^2].\end{aligned}$$

Using Lemma 2 we find that

$$\begin{aligned}4C_n-F_n-(-1)^{(n^2-1)/8}E_n =&\sum_{k=1}^{[(n-1)/8]}\frac{1}{2k}\binom{n-1}{8k-1}-\frac{1}{2}\sum_{k=1}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1}\\=&\sum_{\substack{k=1\\2\mid k}}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1}-\frac{1}{2}\left[\sum_{\substack{k=1\\2\mid k}}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1}+\sum_{\substack{k=1\\2\nmid k}}^{[(n-1)/4]}\frac{1}{k}\binom{n-1}{4k-1}\right]\\=&\frac{1}{2}\sum_{k=1}^{[(n-1)/4]}\frac{(-1)^k}{k}\binom{n-1}{4k-1},\end{aligned}$$

$$\begin{aligned}&4D_n-F_n+(-1)^{(n^2-1)/8}E_n\\&=4\sum_{k=1}^{[(n+5)/8]}\frac{1}{8k-6}\binom{n-1}{8k-7}-\sum_{k=1}^{[(n+1)/4]}\frac{1}{2k-1}\binom{n-1}{4k-3}\\&=2\sum_{\substack{k=1\\2\mid k+1}}^{[(n+1)/4]}\frac{1}{2k-1}\binom{n-1}{4k-3}-\sum_{k=1}^{[(n+1)/4]}\frac{1}{2k-1}\binom{n-1}{4k-3}\\&=\sum_{k=1}^{[(n+1)/4]}\frac{(-1)^{k-1}}{2k-1}\binom{n-1}{4k-3}.\end{aligned}$$

This concludes the proof. □

Lemma 4. *Let $n>0$ be odd. Then*

$$P_n = 2^{(n-1)/2} + 2^{(n-3)/2} n \sum_{k=1}^{(n-1)/2} \frac{1}{k\cdot 2^k}\binom{n-1}{2k-1}.$$

Proof.

$$\begin{aligned}
P_n - 2^{(n-1)/2} =& \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n] - 2^{(n-1)/2} \\
=& \frac{1}{2\sqrt{2}} \times 2 \sum_{\substack{k=0\\ 2\nmid k}}^{n} \binom{n}{n-k}(\sqrt{2})^k - (\sqrt{2})^{n-1} \\
=& \sum_{i=1}^{(n-1)/2} \binom{n}{n+1-2i} 2^{i-1} \\
=& \frac{1}{2}\sum_{i=1}^{(n-1)/2} \frac{n}{(n+1)/2-i}\binom{n-1}{2((n+1)/2-i)-1} 2^{(n-1)/2-((n+1)/2-i)} \\
=& \frac{n}{2}\sum_{k=1}^{(n-1)/2} \frac{1}{k}\binom{n-1}{2k-1} 2^{(n-1)/2-k} \\
=& 2^{(n-3)/2} n \sum_{k=1}^{(n-1)/2} \frac{1}{k\cdot 2^k}\binom{n-1}{2k-1}. \quad \square
\end{aligned}$$

3. The main result

Theorem. *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k\cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

Proof. From Lemmas 3 and 4 it follows that

$$\begin{aligned}2^{p-2}\sum_{k=1}^{(p-1)/2}\frac{1}{k\cdot 2^k}\binom{p-1}{2k-1}=&\frac{2^{(p-1)/2}P_p-2^{p-1}}{p}\\ =&-\frac{2^{p-1}-1}{p}+\frac{1}{2}\sum_{k=1}^{[(p-1)/4]}\frac{(-1)^k}{k}\binom{p-1}{4k-1}\\ &+\frac{p}{4}\Bigg[\Bigg(\sum_{k=1}^{[(p-1)/4]}\frac{(-1)^k}{2k}\binom{p-1}{4k-1}\Bigg)^2\\ &\quad+\Bigg(\sum_{k=1}^{[(p+1)/4]}\frac{(-1)^{k-1}}{2k-1}\binom{p-1}{4k-3}\Bigg)^2\Bigg].\end{aligned}$$

Since p is an odd prime, we have

$$2^{p-1}\equiv 1 \pmod p,\quad \binom{p-1}{n}=\frac{(p-1)(p-2)\cdots(p-n)}{1\times 2\times\cdots\times n}\equiv(-1)^n \pmod p,$$

and hence

$$\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k\cdot 2^k}(-1)^{2k-1}\equiv-\frac{1}{2}\times\frac{2^p-2}{p}+\frac{1}{2}\sum_{k=1}^{[(p-1)/4]}\frac{(-1)^k}{k}(-1)^{4k-1} \pmod p.$$

Thus

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{1}{k\cdot 2^k}\equiv&\frac{\sum_{k=1}^{p-1}\binom{p}{k}}{p}+\sum_{k=1}^{[(p-1)/4]}\frac{(-1)^k}{k}\equiv\sum_{k=1}^{p-1}\frac{1}{k}\binom{p-1}{k-1}\\ &+\sum_{k=1}^{[(p-1)/4]}\frac{(-1)^{p-k}}{p-k} \pmod p\\ \equiv&\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}+\sum_{k=p-[(p-1)/4]}^{p-1}\frac{(-1)^k}{k}\\ \equiv&\sum_{k=1}^{p-1-[(p-1)/4]}\frac{(-1)^{k-1}}{k}\equiv\sum_{k=1}^{[3p/4]}\frac{(-1)^{k-1}}{k} \pmod p.\end{aligned}$$

This completes the proof. □

Remark. The congruence we proved was first conjectured by the author's brother Zhi-Hong Sun in the summer of 1988. Although he failed to prove the conjecture, he was able to show the congruence

$$P_p - 2^{(p-1)/2} \equiv -2^{(p-3)/2} p \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \pmod{p^2}$$

and confirmed the conjecture in the case $p \equiv 1 \pmod 8$.

To end this paper we mention another beautiful congruence (for odd primes)

$$\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv \sum_{k=1}^{[(p-1)/6]} \frac{(-1)^k}{k} \pmod p.$$

It will be published in [2]. For related work the reader can consult [3] and [4].

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