

## SUPERCONGRUENCES FOR CENTRAL TRINOMIAL COEFFICIENTS

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ABSTRACT. For each  $n = 0, 1, 2, \dots$  the central trinomial coefficient  $T_n$  is the coefficient of  $x^n$  in the expansion of  $(x^2 + x + 1)^n$ . In 2016 the second author conjectured that for any prime  $p > 3$  and positive integer  $n$  the quotient  $(T_{pn} - T_n)/(pn)^2$  is a  $p$ -adic integer. In this paper we confirm this conjecture and prove further that

$$\frac{T_{pn} - T_n}{(pn)^2} \equiv \frac{T_{n-1}}{6} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p},$$

where  $\left(\frac{p}{3}\right)$  is the Legendre symbol and  $B_{p-2}(x)$  is the Bernoulli polynomial of degree  $p - 2$ .

### 1. INTRODUCTION

Let  $b, c \in \mathbb{Z}$  and  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ . As in [13, 14], we denote by  $T_n(b, c)$  the coefficient of  $x^n$  in the expansion of  $(x^2 + bx + c)^n$ . It is easy to see that

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k. \tag{1.1}$$

When  $b^2 - 4c \neq 0$ , it is known that

$$T_n(b, c) = (\sqrt{b^2 - 4c})^n P_n \left( \frac{b}{\sqrt{b^2 - 4c}} \right), \tag{1.2}$$

where

$$P_n(x) := \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k$$

is the Legendre polynomial of degree  $n$ .

Those  $T_n := T_n(1, 1)$  ( $n \in \mathbb{N}$ ) are called *central trinomial coefficients*. Such numbers play important roles in enumerative combinatorics; for example,  $T_n$  is the number of lattice paths from the point  $(0, 0)$  to  $(n, 0)$  with only allowed steps  $(1, 1)$ ,  $(1, -1)$  and  $(1, 0)$  (cf. [12, A002426]). For some known congruences involving central trinomial coefficients or their generalizations, the reader may consult [1, 3, 8, 9, 13, 14, 18].

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In 2016 the second author conjectured (cf. [15, A277640]) that for any prime  $p > 3$  and  $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$  we have

$$\frac{T_{pn} - T_n}{(pn)^2} \in \mathbb{Z}_p, \quad (1.3)$$

where  $\mathbb{Z}_p$  denotes the ring of  $p$ -adic integers. In this paper we confirm this conjecture and establish the following further result.

**Theorem 1.1.** *Let  $p > 3$  be a prime and let  $n$  be a positive integer. Then*

$$\frac{T_{pn} - T_n}{(pn)^2} \equiv \frac{T_{n-1}}{6} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}, \quad (1.4)$$

where  $\left(\frac{p}{3}\right)$  is the Legendre symbol and  $B_{p-2}(x)$  is the Bernoulli polynomial of degree  $p-2$ .

This theorem has the following consequence.

**Corollary 1.1.** *For any prime  $p > 3$  and  $a \in \mathbb{Z}^+$ , we have*

$$\frac{T_{p^a} - T_{p^{a-1}}}{p^{2a}} \equiv \frac{1}{6} \left(\frac{p}{3}\right)^a B_{p-2} \left(\frac{1}{3}\right) \pmod{p}. \quad (1.5)$$

*Remark 1.1.* By Corollary 1.1, for any prime  $p > 3$  and  $a \in \mathbb{Z}^+$  we have

$$T_{p^a} \equiv T_{p^{a-1}} \equiv \dots \equiv T_{p^0} = 1 \pmod{p^2}.$$

In 2016, the second author conjectured (cf. Comments in [12, A002426]) that for any integer  $n > 3$ , the congruence  $T_n \equiv 1 \pmod{n^2}$  holds if and only if  $n$  is prime. This conjecture, if true, provides an interesting characterization of primes via central trinomial coefficients.

For  $n = 0, 1, 2, \dots$ , we define

$$a_n := \sum_{k=0}^n \binom{n}{k} \binom{n-k}{k} \binom{n+k}{k}. \quad (1.6)$$

Note that

$$a_n = \sum_{k=0}^n \binom{n+k}{2k} \binom{n-k}{k} \binom{2k}{k} = \sum_{k=0}^n \binom{n+k}{3k} \binom{3k}{k} \binom{2k}{k}. \quad (1.7)$$

For  $|x| < 1$ , we have

$$\begin{aligned} & \sum_{k=0}^{\infty} \binom{2k}{k} \binom{3k}{k} \frac{x^{2k}}{(1-x)^{3k+1}} \\ &= \sum_{k=0}^{\infty} \binom{2k}{k} \binom{3k}{k} x^{2k} \sum_{l=0}^{\infty} \binom{3k+l}{3k} x^l \\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{3k}{k} \binom{n+k}{3k} = \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

So the sequence  $(a_n)_{n \geq 0}$  coincides with [5, A208425]. Our second theorem on this sequence confirms a conjecture of the second author (cf. Comments in [5, A208425]) formulated in 2016.

**Theorem 1.2.** *Let  $p > 3$  be a prime and let  $n$  be a positive integer. Then*

$$\frac{a_{pn} - a_n}{(pn)^3} \in \mathbb{Z}_p. \quad (1.8)$$

We are going to prove Theorem 1.1 and Corollary 1.1 in the next section. Theorem 1.2 will be proved in Section 3. Our proofs involve the cubic root

$$\omega := \frac{-1 + \sqrt{-3}}{2}$$

of unity.

The reader may consult [2, 10, 17] for some other similar known supercongruences, and [16, Conjecture 82] for some conjectural supercongruences.

## 2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.1

**Lemma 2.1.** *For any  $n \in \mathbb{N}$  we have*

$$(-1)^n T_n = \sum_{k=0}^n \binom{n}{k}^2 \omega^{2k-n} = \sum_{k=0}^n \binom{n}{k}^2 \bar{\omega}^{2k-n}. \quad (2.1)$$

*Proof.* For  $n \in \mathbb{N}$  it is known (cf. [4, (3.134)]) that

$$P_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{x+1}{2}\right)^{n-k} \left(\frac{x-1}{2}\right)^k.$$

Combining this with (1.2) we see that

$$\begin{aligned} T_n &= \sqrt{-3}^n P_n\left(\frac{1}{\sqrt{-3}}\right) = \sqrt{-3}^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{1/\sqrt{-3}+1}{2}\right)^{n-k} \left(\frac{1/\sqrt{-3}-1}{2}\right)^k \\ &= \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{1+\sqrt{-3}}{2}\right)^{n-k} \left(\frac{1-\sqrt{-3}}{2}\right)^k. \end{aligned}$$

Clearly,

$$\frac{1 + \sqrt{-3}}{2} = -\bar{\omega} = -\omega^2.$$

So, from the above we obtain

$$T_n = (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \omega^{2n-k}.$$

Taking conjugates of both sides, we get

$$T_n = (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \bar{\omega}^{2n-k}.$$

As  $2n - k \equiv 2k - n \pmod{3}$ , we see that (2.1) holds.  $\square$

**Lemma 2.2.** *Let  $p > 3$  be a prime. Then*

$$\sum_{\substack{0 < k < p \\ 3|k+p}} \frac{1}{k^2} \equiv -\frac{1}{9} \binom{p}{3} B_{p-2} \left( \frac{1}{3} \right) \pmod{p}. \quad (2.2)$$

*Proof.* Note that

$$\sum_{\substack{0 < k < p \\ 3|k+p}} \frac{1}{k^2} + \sum_{\substack{0 < k < p \\ 3|k-p}} \frac{1}{k^2} + \sum_{\substack{0 < k < p \\ 3|k}} \frac{1}{k^2} = \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}$$

by Wolstenholme's congruence [19]. As

$$\sum_{\substack{0 < k < p \\ 3|k-p}} \frac{1}{k^2} = \sum_{\substack{0 < k < p \\ 3|p-k}} \frac{1}{(p-k)^2} = \sum_{\substack{0 < j < p \\ 3|j}} \frac{1}{j^2},$$

by the above we obtain

$$\sum_{\substack{0 < k < p \\ 3|k+p}} \frac{1}{k^2} \equiv -2 \sum_{\substack{0 < k < p \\ 3|k}} \frac{1}{k^2} = -\frac{2}{9} \sum_{j=1}^{\lfloor (p-1)/3 \rfloor} \frac{1}{j^2} \pmod{p}. \quad (2.3)$$

By [7, (9)],

$$\sum_{j=1}^{\lfloor (p-1)/3 \rfloor} \frac{1}{j^2} \equiv \sum_{j=1}^{\lfloor (p-1)/3 \rfloor} j^{p-3} \equiv -\frac{1}{p-2} B_{p-2} \left( \left\{ \frac{p}{3} \right\} \right) \pmod{p},$$

where  $\{p/3\}$  is the fractional part of  $p/3$ . Note that

$$B_{p-2} \left( \left\{ \frac{p}{3} \right\} \right) = \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right)$$

since

$$B_{p-2} \left( \frac{2}{3} \right) = (-1)^{p-2} B_{p-2} \left( \frac{1}{3} \right)$$

(cf. [6, p. 248]). Therefore

$$\sum_{j=1}^{\lfloor (p-1)/3 \rfloor} \frac{1}{j^2} \equiv \frac{1}{2} \binom{p}{3} B_{p-2} \left( \frac{1}{3} \right) \pmod{p}$$

and hence by (2.3) we have

$$\sum_{\substack{0 < k < p \\ 3|k+p}} \frac{1}{k^2} \equiv -\frac{2}{9} \sum_{j=1}^{\lfloor (p-1)/3 \rfloor} \frac{1}{j^2} \equiv -\frac{1}{9} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}.$$

This concludes the proof.  $\square$

*Proof of Theorem 1.1.* Let  $k \in \{0, \dots, pn\}$  with  $p \nmid k$ . Then

$$\begin{aligned} \binom{pn}{k} &= \frac{pn}{k} \prod_{0 < j < k} \frac{pn-j}{j} \\ &= \frac{pn}{k} \prod_{0 < i \leq \lfloor (k-1)/p \rfloor} \frac{pn-pi}{pi} \times \prod_{\substack{0 < j < k \\ p \nmid j}} \left(\frac{pn}{j} - 1\right) \\ &\equiv \frac{pn}{k} \binom{n-1}{\lfloor (k-1)/p \rfloor} (-1)^{\delta_k} \pmod{p^{2+2\nu_p(n)}} \end{aligned}$$

where  $\delta_k := |\{0 < j < k : p \nmid j\}|$  and  $\nu_p(n) := \max\{a \in \mathbb{N} : p^a \mid n\}$ . It follows that

$$\sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \binom{pn}{k}^2 \omega^{2k-pn} \equiv \sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \frac{p^2 n^2}{k^2} \binom{n-1}{\lfloor (k-1)/p \rfloor}^2 \omega^{2k-pn} \pmod{p^{3+2\nu_p(n)}}. \quad (2.4)$$

By Lemma 2.1,

$$(-1)^{n-1} T_{n-1} = \sum_{k=0}^{n-1} \omega^{2k-n+1} \binom{n-1}{k}^2 = \sum_{k=0}^{n-1} \bar{\omega}^{2k-n+1} \binom{n-1}{k}^2.$$

As  $p \equiv \pm 1 \pmod{3}$ , we get

$$\sum_{k=0}^{n-1} \omega^{p(2k-n+1)} \binom{n-1}{k}^2 = (-1)^{n-1} T_{n-1}.$$

Therefore

$$\begin{aligned} &\sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \frac{\omega^{2k-pn}}{k^2} \binom{n-1}{\lfloor (k-1)/p \rfloor}^2 \\ &= \omega^{-pn} \sum_{k=0}^{n-1} \sum_{j=1}^{p-1} \frac{\omega^{2(pk+j)}}{(pk+j)^2} \binom{n-1}{k}^2 \equiv \omega^{-pn} \sum_{k=0}^{n-1} \omega^{2pk} \binom{n-1}{k}^2 \sum_{j=1}^{p-1} \frac{\omega^{2j}}{j^2} \\ &\equiv \omega^{-p} \sum_{k=0}^{n-1} \omega^{p(2k-n+1)} \binom{n-1}{k}^2 \sum_{j=1}^{p-1} \frac{\omega^{2j}}{j^2} = \omega^{-p} (-1)^{n-1} T_{n-1} \sum_{j=1}^{p-1} \frac{\omega^{2j}}{j^2} \pmod{p}. \end{aligned}$$

In light of this and the equality  $\omega + \bar{\omega} = -1$ , we get

$$\begin{aligned}
& \sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \frac{\omega^{2k-pn} + \bar{\omega}^{2k-pn}}{k^2} \binom{n-1}{\lfloor (k-1)/p \rfloor}^2 \\
& \equiv (-1)^{n-1} T_{n-1} \sum_{j=1}^{p-1} \frac{\omega^{2j-p} + \bar{\omega}^{2j-p}}{j^2} \equiv (-1)^{n-1} T_{n-1} \left( \sum_{\substack{0 < j < p \\ p \mid 2j-p}} \frac{2}{j^2} - \sum_{\substack{0 < j < p \\ 3 \nmid 2j-p}} \frac{1}{j^2} \right) \\
& \equiv (-1)^{n-1} T_{n-1} \left( \sum_{\substack{0 < j < p \\ 3 \mid 2j-p}} \frac{3}{j^2} - \sum_{j=1}^{p-1} \frac{1}{j^2} \right) \equiv 3(-1)^{n-1} T_{n-1} \sum_{\substack{0 < j < p \\ 3 \mid j+p}} \frac{1}{j^2} \pmod{p}.
\end{aligned}$$

Combining this with (2.4) and Lemma 2.2, we obtain

$$\sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \binom{pn}{k}^2 \frac{\omega^{2k-pn} + \bar{\omega}^{2k-pn}}{k^2} \equiv (-1)^n \frac{T_{n-1}}{6} \left(\frac{p}{3}\right) (pn)^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^{3+2\nu_p(n)}}. \quad (2.5)$$

In view of Lemma 2.1 and (2.5), we have

$$\begin{aligned}
(-1)^{pn} T_{pn} - (-1)^n T_n &= \sum_{k=0}^{pn} \binom{pn}{k}^2 \frac{\omega^{2k-pn} + \bar{\omega}^{2k-pn}}{2} - \sum_{k=0}^n \binom{n}{k}^2 \frac{\omega^{2k-n} + \bar{\omega}^{2k-n}}{2} \\
&= \sum_{k=0}^n \binom{pn}{pk}^2 \frac{\omega^{p(2k-n)} + \bar{\omega}^{p(2k-n)}}{2} - \sum_{k=0}^n \binom{n}{k}^2 \frac{\omega^{2k-n} + \bar{\omega}^{2k-n}}{2} \\
&\quad + \sum_{\substack{0 \leq k \leq pn \\ p \nmid k}} \binom{pn}{k}^2 \frac{\omega^{2k-pn} + \bar{\omega}^{2k-pn}}{k^2} \\
&\equiv \sum_{k=0}^n \left( \binom{pn}{pk}^2 - \binom{n}{k}^2 \right) \frac{\omega^{2k-n} + \bar{\omega}^{2k-n}}{2} \\
&\quad + (-1)^n \frac{T_{n-1}}{6} \left(\frac{p}{3}\right) (pn)^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^{3+2\nu_p(n)}}
\end{aligned}$$

since  $p \equiv \pm 1 \pmod{3}$ . For each  $k \in \{0, \dots, n\}$ , it is known (see, e.g., [11]) that

$$\binom{pn}{pk} = \binom{n}{k} (1 + p^3 nk(n-k)q_k) \quad (2.6)$$

for some  $q_k \in \mathbb{Z}_p$ , hence

$$\binom{pn}{pk}^2 - \binom{n}{k}^2 = 2p^3 \binom{n}{k}^2 nk(n-k)q_k + p^6 \left( \binom{n}{k} nk(n-k)q_k \right)^2 \equiv 0 \pmod{p^{3+3\nu_p(n)}}$$

since  $\binom{n}{k}k = n\binom{n-1}{n-k}$  and  $\binom{n}{k}(n-k) = n\binom{n-1}{k}$ . So we finally get

$$T_{pn} - T_n \equiv \frac{T_{n-1}}{6} \left(\frac{p}{3}\right) (pn)^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^{3+2\nu_p(n)}}$$

which is equivalent to (1.4).  $\square$

*Proof of Corollary 1.1.* Applying Theorem 1.1 with  $n = p^{a-1}$ , we get

$$\frac{T_{p^a} - T_{p^{a-1}}}{p^{2a}} \equiv \frac{T_{p^{a-1}-1}}{6} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}.$$

So it suffices to prove

$$T_{p^{a-1}-1} \equiv \left(\frac{p}{3}\right)^{a-1} \pmod{p}. \quad (2.7)$$

Clearly (2.7) holds for  $a = 1$ . Below we assume  $a > 1$ . For each  $k = 1, \dots, (p^{a-1} - 1)/2$ , we have

$$\binom{p^{a-1} - 1}{2k} = \prod_{j=1}^{2k} \left(\frac{p^{a-1}}{j} - 1\right) \equiv (-1)^{2k} = 1 \pmod{p}$$

and

$$\frac{\binom{(p^{a-1}-1)/2}{k}}{\binom{-1/2}{k}} = \prod_{0 \leq j < k} \left(1 - \frac{p^{a-1}}{2j+1}\right) \equiv 1 \pmod{p}.$$

Thus

$$\begin{aligned} T_{p^{a-1}-1} &= \sum_{k=0}^{(p^{a-1}-1)/2} \binom{p^{a-1}-1}{2k} \binom{2k}{k} = \sum_{k=0}^{(p^{a-1}-1)/2} \binom{p^{a-1}-1}{2k} \binom{-1/2}{k} (-4)^k \\ &\equiv \sum_{k=0}^{(p^{a-1}-1)/2} \binom{(p^{a-1}-1)/2}{k} (-4)^k = (1-4)^{(p^{a-1}-1)/2} = (-3)^{\frac{p-1}{2} \sum_{0 \leq r < a-1} p^r} \\ &\equiv \left(\frac{-3}{p}\right)^{a-1} = \left(\frac{p}{3}\right)^{a-1} \pmod{p} \end{aligned}$$

with the aid of the theory of quadratic residues. This proves the desired (2.7).

In view of the above, we have completed the proof of Corollary 1.1.  $\square$

### 3. PROOF OF THEOREM 1.2

**Lemma 3.1.** *For any  $n \in \mathbb{N}$  we have*

$$a_n = \sum_{k=0}^n \binom{n}{k}^3 (-\omega)^{2k-n} = \sum_{k=0}^n \binom{n}{k}^3 (-\bar{\omega})^{2k-n}. \quad (3.1)$$

*Proof.* Recall MacMahon's identity [4, (6.7)]

$$\sum_{k=0}^n \binom{n}{k}^3 x^k = (1+x)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{3k} \binom{2k}{k} \binom{3k}{k} \left( \frac{x}{(1+x)^2} \right)^k.$$

Putting  $x = \omega^2$  and noting  $(x+1)^2 = x$ , we get

$$\sum_{k=0}^n \binom{n}{k}^3 \omega^{2k} = (-\omega)^n \sum_{k=0}^n \binom{n+k}{3k} \binom{2k}{k} \binom{3k}{k}$$

and hence

$$\sum_{k=0}^n \binom{n}{k}^3 (-\omega)^{2k-n} = a_n$$

by (1.7). Taking conjugates of both sides of the last equality, we obtain

$$\sum_{k=0}^n \binom{n}{k}^3 (-\bar{\omega})^{2k-n} = a_n.$$

So (3.1) is valid. □

*Proof of Theorem 1.2.* For  $k \in \{0, \dots, n\}$ , we have (2.6) for some  $q_k \in \mathbb{Z}_p$ , hence

$$\binom{pn}{pk}^3 = \binom{n}{k}^3 (1 + p^3 nk(n-k)q_k)^3 \equiv \binom{n}{k}^3 \pmod{p^{3+3\nu_p(n)}}$$

since  $k \binom{n}{k} = n \binom{n-1}{n-k}$  and  $(n-k) \binom{n}{k} = n \binom{n-1}{k}$ . Thus

$$\begin{aligned} a_{pn} &= \sum_{k=0}^n \binom{pn}{pk}^3 (-\omega)^{2pk-pn} + \sum_{\substack{0 < k < pn \\ p \nmid k}} \frac{(pn)^3}{k^3} \binom{pn-1}{k-1}^3 (-\omega)^{2k-pn} \\ &\equiv \sum_{k=0}^n \binom{n}{k}^3 (-\omega)^{\binom{p}{3}(2k-n)} (-\omega)^{(p-\binom{p}{3})(2k-n)} \\ &= \sum_{k=0}^n \binom{n}{k}^3 (-\omega^{\binom{p}{3}})^{2k-n} = a_n \pmod{p^{3+3\nu_p(n)}} \end{aligned}$$

and hence (1.8) follows. □

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