Frontiers in Combinatorics and Number Theory
Vol. 3, 2026, pp. 68-87 (July 27, 2026)
doi:10.3934/fcnt.2026022

UNIVERSAL SUMS VIA PRODUCTS OF RAMANUJAN'S THETA FUNCTIONS

Nasser Abdo Saeed Bulkhali[*1] and Zhi-Wei Sun[2]

[1]Department of Mathematics, Faculty of Education-Zabid,
Hodeidah University, Hodeidah, Yemen

[2]School of Mathematics, Nanjing University, Nanjing 210093, China

(Communicated by Jiang Zeng)

Abstract. An integer-valued polynomial $P(x,y,z)$ is said to be universal (over $\mathbb{Z}$) if each nonnegative integer can be written as $P(x,y,z)$ with $x,y,z\in\mathbb{Z}$. In this paper, we mainly introduce a new technique to determine the universality of some sums in the form $x(a_1x+a_2)/2+y(b_1y+b_2)/2+z(c_1z+c_2)/2$ (with $a_1-a_2, b_1-b_2, c_1-c_2$ all even) conjectured by Sun, using various identities of Ramanujan's theta functions. For example, we prove that $x(3x+1)+y(3y+2)+2z(3z+2)$ and $x(4x+r)+y(3y+1)/2+z(7z+1)/2$ $(r=1,3)$ are universal.

1. **Introduction.** Let $\mathbb{N}=\{0,1,2,\ldots\}$ and $\mathbb{Z}^+=\{1,2,3,\ldots\}$. For an integer-valued polynomial $P(x_1,\ldots,x_n)$, we call it *universal* (over $\mathbb{Z}$) if

$$\{P(x_1,\ldots,x_n):\ x_1,\ldots,x_n\in\mathbb{Z}\}=\mathbb{N},$$

and call $P(x_1,\ldots,x_n)$ *universal over* $\mathbb{N}$ if

$$\{P(x_1,\ldots,x_n):\ x_1,\ldots,x_n\in\mathbb{N}\}=\mathbb{N}.$$

When

$$\mathbb{N}\setminus\{P(x_1,\ldots,x_n):\ x_1,\ldots,x_n\in\mathbb{Z}\}$$

is finite, we say that $P(x_1,\ldots,x_n)$ is *almost universal* (over $\mathbb{Z}$).

For each integer $m\geqslant 3$, the numbers

$$p_m(n)=(m-2)\binom{n}{2}+n=\frac{(m-2)n^2-(m-4)n}{2}\ (n=0,1,2,\ldots)$$

are called *m-gonal numbers*, and the numbers $p_m(x)$ with $x\in\mathbb{Z}$ are usually called *generalized m-gonal numbers*. Note that

$$p_m(-x)=\frac{x((m-2)x+m-4)}{2}. \tag{1}$$

2020 *Mathematics Subject Classification.* Primary: 11D72, 11E20, 11E25; Secondary: 05A30, 11F27.

Key words and phrases. Ramanujan's theta functions, sums of squares, triangular numbers, ternary quadratic forms.

The initial version of this paper was posted to arXiv in 2024 as a preprint with the ID `arXiv:2410.14605`.

The second author is supported by the Natural Science Foundation of China (grant no. 12371004).

*Corresponding author: Nasser Abdo Saeed Bulkhali.

In particular,

$p_3(x) = x(x+1)/2$	$p_3(-x) = x(x-1)/2$
$p_4(x) = x^2$	$p_4(-x) = x^2$
$p_5(x) = x(3x-1)/2$	$p_5(-x) = x(3x+1)/2$
$p_7(x) = x(5x-3)/2$	$p_7(-x) = x(5x+3)/2$
$p_8(x) = x(3x-2)$	$p_8(-x) = x(3x+2)$

Obviously,

$$\{p_6(x):\ x \in \mathbb{Z}\} = \{x(2x+1):\ x \in \mathbb{Z}\} = \{p_3(x):\ x \in \mathbb{Z}\} = \{p_3(n):\ n \in \mathbb{N}\}. \quad (2)$$

Fermat claimed that $p_m(x_1) + \cdots + p_m(x_m)$ is universal over $\mathbb{N}$. This was confirmed by Lagrange, Gauss, and Cauchy for the cases $m = 4$, $m = 3$, and $m \geqslant 5$, respectively. In 1862, Liouville (cf. [8, p. 23]) determined all universal sums $ap_3(x)+bp_3(y)+cp_3(z)$ with $a, b, c \in \mathbb{Z}^+$, namely, those triples (a, b, c) with $a \leqslant b \leqslant c$ such that $ap_3(x) + bp_3(y) + cp_3(z)$ is universal are

$$(1, 1, 1),\ (1, 1, 2),\ (1, 1, 4),\ (1, 1, 5),\ (1, 2, 2),\ (1, 2, 3),\ (1, 2, 4).$$

In 2007, Z.-W. Sun [19] initiated the study of universal sums of the types $ax^2 + by^2 + cp_3(z)$ and $ax^2 + bp_3(y) + cp_3(z)$ with $a, b, c \in \mathbb{Z}^+$, and the investigations were finished via the three papers [19, 11, 18].

In 2010, B. Kane and Sun [16] used modular forms and the theory of ternary quadratic forms to characterize those tuples $(a, b, c) \in (\mathbb{Z}^+)^3$ such that $ax^2 + by^2 + cp_3(z)$ is almost universal; they also investigated those tuples $(a, b, c) \in (\mathbb{Z}^+)^3$ such that $ax^2+bp_3(y)+cp_3(z)$ is almost universal, as well as those tuples $(a, b, c) \in (\mathbb{Z}^+)^3$ such that $ap_3(x) + bp_3(y) + cp_3(z)$ is almost universal. Via solving a conjecture of Kane and Sun [16], W. K. Chan and B.-K. Oh [5] gave a complete characterization of those triples $(a, b, c) \in (\mathbb{Z}^+)^3$ with $ax^2+bp_3(y)+cp_3(z)$ (or $ap_3(x)+bp_3(y)+cp_3(z)$) almost universal.

In 2015, Sun [20] investigated universal sums of the type $ap_i(x) + bp_j(y) + cp_k(z)$ with $a, b, c \in \mathbb{Z}^+$ and $i, j, k \in \{3, 4, 5, \dots\}$. In particular, for $k \in \{4, 5, 7, 8, 9, \dots\}$ and $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$, Sun [20] showed that if $ap_k(x) + bp_k(y) + cp_k(z)$ is universal, then $k = 5$, $a = 1$, and (b, c) is among the following 20 ordered pairs:

$$(1, c)\,(c \in \{1, \dots, 10\} \setminus \{7\}),\ (2, c)\,(c \in \{2, 3, 4, 6, 8\}), (3, c)\,(c = 3, 4, 6, 7, 8, 9).$$

For these ordered pairs (b, c), it is now known (cf. [12, 20, 17]) that $p_5(x)+bp_5(y)+cp_5(z)$ is indeed universal.

In 2017, Sun [22] proved that, for integers $0 \leqslant b \leqslant c \leqslant d \leqslant a$ with $a > 2$, the sum $x(ax + b) + y(ay + c) + z(az + d)$ is universal if and only if the quadruple (a, b, c, d) is among

$$(3, 0, 1, 2),\ (3, 1, 1, 2),\ (3, 1, 2, 2),\ (3, 1, 2, 3),\ (4, 1, 2, 3).$$

Sun [22] also proved that $x(ax + 1) + y(by + 1) + z(cz + 1)$ is universal if (a, b, c) is among the following seven triples:

$$(1, 2, 3),\ (1, 2, 4),\ (1, 2, 5),\ (2, 2, 4),\ (2, 2, 5),\ (2, 3, 3),\ (2, 3, 4).$$

J. Ju and Oh [14] showed that $x(ax + 1) + y(by + 1) + z(cz + 1)$ is also universal for $(a, b, c) = (2, 2, 6),\ (2, 3, 5),\ (2, 3, 7)$ as conjectured by Sun [22].

For convenience, for any $a_1, b_1, c_1 \in \mathbb{Z}^+$ and $a_2, b_2, c_2 \in \mathbb{Z}$ with $a_1 - a_2, b_1 - b_2, c_1 - c_2$ all even, whenever

$$\frac{x(a_1x + a_2)}{2} + \frac{y(b_1y + b_2)}{2} + \frac{z(c_1z + c_2)}{2} \quad (3)$$

is universal (resp., almost universal), we call the tuple $(a_1, a_2, b_1, b_2, c_1, c_2)$ universal (resp., almost universal). For any $a \in \mathbb{Z}^+$, by (2) we have

$$\{ap_3(x): \ x \in \mathbb{Z}\} = \left\{\frac{x(ax+a)}{2}: \ x \in \mathbb{Z}\right\} = \left\{\frac{x(4ax+2a)}{2}: \ x \in \mathbb{Z}\right\}.$$

Sun [23, 24] investigated those *standard* universal tuples $(a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{N}^6$ with $a_1 - a_2, b_1 - b_2, c_1 - c_2 \in 2\mathbb{Z}^+$, $a_1 \geqslant b_1 \geqslant c_1 \geqslant 2$, and $a_2 \geqslant b_2$ if $a_1 = b_1$, and $b_2 \geqslant c_2$ if $b_1 = c_1$; he showed that such tuples $(a_1, a_2, b_1, b_2, c_1, c_2)$ are among the 12082 tuples listed in [23, Appendix]. Those tuples in the list marked with $*$ have been proved to be universal, and Sun [23, 24] conjectured that all the remaining tuples in the list are indeed universal. In this direction, H.-L. Wu and Sun [25] made certain progress. As Sun [22] proved that $x(2x + 1) + y(3y + 1) + z(4z + 1)$ is universal, the tuple $(8, 2, 6, 2, 4, 2)$ not marked with $*$ in the list is indeed universal. Note that all previous works in this direction depend heavily on the theory of ternary quadratic forms.

In Section 2, we will confirm that some open tuples listed in [23, Appendix] are indeed universal by using the theory of ternary quadratic forms.

Ramanujan's general theta function is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \quad \text{with } |ab| < 1. \tag{4}$$

In particular, Ramanujan [1, Entry 22] introduced $\varphi(q)$ and $\psi(q)$ for $|q| < 1$:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{5}$$

$$\psi(q) := f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{n(2n+1)} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{6}$$

For various formulas of Ramanujan's theta functions, one may consult the excellent introductory book of B.C. Berndt [3, Chapter 1].

In this paper, we find that many universal sums can be studied via identities of Ramanujan's theta functions. We establish the following theorem connecting universal sums with Ramanujan's theta functions.

Theorem 1.1. *Let $i, j, s, t, u, v \in \mathbb{Z}$ with $i + j, s + t, u + v$ positive. Suppose that*

$$f(q^i, q^j) f(q^s, q^t) f(q^u, q^v) = \sum_{r=1}^{k} m_r q^{r-1} f(q^{ka_{1r}}, q^{ka_{2r}}) f(q^{kb_{1r}}, q^{kb_{2r}}) f(q^{kc_{1r}}, q^{kc_{2r}}), \tag{7}$$

where $a_{1r}, a_{2r}, b_{1r}, b_{2r}, c_{1r}, c_{2r} \in \mathbb{Z}$ and $k, m_r, a_{1r} + a_{2r}, b_{1r} + b_{2r}, c_{1r} + c_{2r} \in \mathbb{Z}^+$.

(i) *The tuple $(i + j, i - j, s + t, s - t, u + v, u - v)$ is universal if and only if*

$$(a_{1r} + a_{2r}, a_{1r} - a_{2r}, b_{1r} + b_{2r}, b_{1r} - b_{2r}, c_{1r} + c_{2r}, c_{1r} - c_{2r})$$

is universal for all $r = 1, \ldots, k$.

(ii) *The tuple $(i + j, i - j, s + t, s - t, u + v, u - v)$ is almost universal if and only if the tuple*

$$(a_{1r} + a_{2r}, a_{1r} - a_{2r}, b_{1r} + b_{2r}, b_{1r} - b_{2r}, c_{1r} + c_{2r}, c_{1r} - c_{2r})$$

is almost universal for all $r = 1, \ldots, k$.

Utilizing the above theorem and some known identities on the theta functions, we establish the following theorem whose results were conjectured by Sun [23].

Theorem 1.2. *All the tuples*

$$(8,2,5,1,3,1),\ (8,2,5,3,4,2),\ (8,2,6,2,3,1),\ (8,2,6,4,4,2),\ (8,2,6,4,6,2),$$
$$(8,2,7,1,3,1), (8,4,4,0,3,1),\ (8,4,6,4,4,0),\ (8,6,5,1,3,1),\ (8,6,6,4,4,2),$$
$$(8,6,6,4,6,2),\ (8,6,7,1,3,1),\ (9,5,4,2,2,0),\ (9,5,6,2,3,1),\ (9,5,8,4,3,1),$$
$$(9,7,6,4,3,1), (9,7,8,4,4,2),\ (9,c,3,1,2,0)\,(c=1,5,7),$$
$$(9,c,5,d,3,1)\,(c\in\{1,5,7\},\ d\in\{1,3\}),\ (12,6,6,4,2,0),\ (12,6,6,4,4,2),$$
$$(12,6,8,4,6,4),\ (12,6,9,c,3,1)\,(c=1,5,7),\ (12,8,4,2,3,1),\ (12,8,6,4,6,2),$$
$$(12,8,9,3,3,1),\ (12,8,12,6,3,1),\ (16,8,6,2,2,0),\ (16,8,8,4,6,4)$$

are universal. In other words, the sums

$$x(4x+1)+\frac{y(3y+1)}{2}+\frac{z(5z+1)}{2},\ x(4x+1)+y(2y+1)+\frac{z(5z+3)}{2},$$
$$x(4x+1)+y(3y+1)+\frac{z(3z+1)}{2},\ x(2x+1)+y(3y+2)+z(4z+1),$$
$$x(4x+1)+y(3y+1)+z(3z+2),\ x(4x+1)+\frac{y(3y+1)}{2}+\frac{z(7z+1)}{2},$$
$$2x^2+y(y+1)+\frac{z(3z+1)}{2},\ 2x^2+y(y+1)+z(3z+2),$$

$$x(4x+3)+\frac{y(3y+1)}{2}+\frac{z(5z+1)}{2},\ x(2x+1)+y(3y+2)+z(4z+3),$$
$$x(3x+1)+y(3y+2)+z(4z+3),\ x(4x+3)+\frac{y(3y+1)}{2}+\frac{z(7z+1)}{2},$$
$$x^2+\frac{y(y+1)}{2}+\frac{z(9z+5)}{2},\ x(3x+1)+\frac{y(3y+1)}{2}+\frac{z(9z+5)}{2},$$
$$x(x+1)+\frac{y(3y+1)}{2}+\frac{z(9z+5)}{2},\ x(3x+2)+\frac{y(3y+1)}{2}+\frac{z(9z+7)}{2},$$
$$x(x+1)+\frac{y(y+1)}{2}+\frac{z(9z+7)}{2},\ x^2+\frac{y(3y+1)}{2}+\frac{z(9z+c)}{2}\ (c=1,5,7),$$
$$\frac{x(3x+1)}{2}+\frac{y(5y+d)}{2}+\frac{z(9z+c)}{2}\ (c\in\{1,5,7\},\ d\in\{1,3\}),$$
$$x^2+3y(2y+1)+z(3z+2),\ x(ax+1)+3y(2y+1)+z(3z+2)\,(a=1,2),$$
$$3\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+\frac{z(9z+c)}{2}\,(c=1,5,7),$$
$$\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+2z(3z+2),\ x(3x+1)+y(3y+2)+2z(3z+2),$$
$$2x(3x+2)+\frac{y(3y+1)}{2}+3\frac{z(3z+1)}{2},\ 3\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+2z(3z+2),$$
$$x^2+2y(y+1)+z(3z+1),\ x(x+1)+2y(y+1)+z(3z+2)$$

are all universal.

Remark 1.1. It seems that it is almost hopeless to prove Theorem 1.2 completely via the theory of ternary quadratic forms. For example, the tuple $(12,8,12,6,3,1)$ is related to the ternary quadratic form $x^2+9y^2+16z^2$. There are four classes in the genus of $x^2+9y^2+16z^2$, and the three classes not containing $x^2+9y^2+16z^2$ have representatives

$$x^2+y^2+144z^2,\ 2x^2+2y^2+37z^2-2yz+2xz,\ 4x^2+5y^2+10z^2+2yz+4xz+4xy$$

respectively. Also, the tuples $(8,2,7,1,3,1)$ and $(8,6,7,1,3,1)$ are related to the ternary quadratic form $6x^2+14y^2+21z^2$. There are three classes in the genus of $6x^2+14y^2+21z^2$, and the two classes not containing $6x^2+14y^2+21z^2$ have representatives

$$3x^2+14y^2+42z^2 \text{ and } 5x^2+17y^2+21z^2-16yz+4xz+2xy$$

respectively.

In view of the corollary in [9, p. 56], for a particular polynomial $P(x,y,z)=x(a_1x+a_2)+y(b_1y+b_2)+z(c_1z+c_2)\in\mathbb{Z}[x,y,z]$, it is easy to check whether $P(x,y,z)$ is almost universal by the theory of ternary quadratic forms.

Using the second part of Theorem 1.1, we deduce the following new result on almost universal sums.

Theorem 1.3. *All the sums*

$$\begin{gathered}3x^2+3\frac{y(y+1)}{2}+\frac{z(3z+1)}{2},\ 3\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+z(3z+2),\\ 3x^2+2y(y+1)+\frac{z(3z+1)}{2},\ 2x(x+1)+\frac{y(3y+1)}{2}+z(3z+2),\\ 2x^2+3y^2+\frac{z(3z+1)}{2},\ 2x^2+\frac{y(3y+1)}{2}+z(3z+2),\\ x(x+1)+y(3y+1)+\frac{z(3z+1)}{2},\ x(x+1)+3y(y+1)+\frac{z(3z+1)}{2},\\ 2x^2+y(3y+1)+\frac{z(3z+1)}{2},\ 2x^2+3y(y+1)+\frac{z(3z+1)}{2}\end{gathered}$$

are almost universal.

We will prove Theorems 1.1-1.3 in Sections 3-5 respectively.

2. **Universal sums via the theory of quadratic forms.** For two integer-valued polynomials $P(x_1,\ldots,x_n)$ and $Q(x_1,\ldots,x_n)$, if

$$\{P(x_1,\ldots,x_n):\ x_1,\ldots,x_n\in\mathbb{Z}\}=\{Q(x_1,\ldots,x_n):\ x_1,\ldots,x_n\in\mathbb{Z}\}$$

then we say that P and Q are equivalent and denote this by $P(x_1,\ldots,x_n)\sim Q(x_1,\ldots,x_n)$. For $a_1,b_1,c_1,a_1',b_1',c_1'\in\mathbb{Z}^+$ and $a_2,b_2,c_2,a_2',b_2',c_2'\in\mathbb{Z}$ with $a_1-a_2,b_1-b_2,c_1-c_2,a_1'-a_2',b_1'-b_2',c_1'-c_2'$ all even, if

$$\frac{x(a_1x+a_2)}{2}+\frac{y(b_1y+b_2)}{2}+\frac{z(c_1z+c_2)}{2}\sim\frac{x(a_1'x+a_2')}{2}+\frac{y(b_1'y+b_2')}{2}+\frac{z(c_1'z+c_2')}{2},$$

then we say that the two tuples $(a_1,a_2,b_1,b_2,c_1,c_2)$ and $(a_1',a_2',b_1',b_2',c_1',c_2')$ are equivalent, and denote this by

$$(a_1,a_2,b_1,b_2,c_1,c_2)\sim(a_1',a_2',b_1',b_2',c_1',c_2').$$

As Euler observed,

$$p_3(x)+p_3(y)\sim x^2+2p_3(y) \tag{8}$$

(cf. [8, p. 11]). H.-L. Wu and Z.-W. Sun [25, (1)] proved that

$$p_3(x)+p_5(y)\sim p_5(x)+3p_5(y). \tag{9}$$

Lemma 2.1. (Sun [24, Theorem 1.9]) (i) *For any $a \in \mathbb{Z}^+$ and $b \in \mathbb{N}$ with $b \leqslant a/2$, we have*

$$x(ax+b)+y(ay+a-b) \sim ap_3(x)+\frac{y(ay+a-2b)}{2}. \tag{10}$$

In particular,

$$2p_5(x)+p_8(y) \sim 3p_3(x)+p_5(y). \tag{11}$$

(ii) *We have*

$$x^2+p_3(y) \sim p_5(x)+2p_5(y), \tag{12}$$

$$p_3(x)+2p_3(y) \sim p_5(x)+p_8(y), \tag{13}$$

$$x^2+4p_3(y) \sim 4p_5(x)+p_8(y), \tag{14}$$

$$p_3(x)+p_3(y) \sim \frac{x(5x+1)}{2}+\frac{y(5y+3)}{2}. \tag{15}$$

Remark 2.1. We can also deduce Lemma 2.1(i) by using the identity

$$f(a,ab^2)f(b,a^2b) = f(a,b)\psi(ab) \tag{16}$$

in [1, Entry 30 (i)].

For convenience, for $a,b,c \in \mathbb{Z}^+$ we define

$$E(a,b,c)=\mathbb{N}\setminus\{ax^2+by^2+cz^2:\ x,y,z\in\mathbb{Z}\}.$$

Dickson [7, pp. 112-113] listed 102 regular ternary quadratic forms $ax^2+by^2+cz^2$ for which the structure of $E(a,b,c)$ is known explicitly.

Theorem 2.2. *All the tuples*

$(8,2,3,1,2,0)$, $(8,2,3,1,3,1)$, $(8,2,4,2,2,0)$, $(8,2,4,2,4,0)$, $(8,2,5,1,4,2)$, $(8,2,5,3,5,1)$, $(8,4,6,4,6,2)$, $(8,4,8,2,3,1)$, $(8,4,8,4,6,4)$, $(8,6,3,1,2,0)$, $(8,6,4,2,4,0)$, $(8,6,4,2,4,2)$, $(8,6,5,1,4,2)$, $(8,6,8,4,3,1)$

are universal.

Proof. (a) To prove the universality of $(8,6,4,2,4,2)$ (or the sum $x(4x+3)+y(y+1)/2+z(z+1)/2$), we observe that

$$\begin{aligned} &n = x(4x+3)+\frac{y(y+1)}{2}+\frac{z(z+1)}{2} \\ \iff\ &16n+13=(8x+3)^2+2(2y+1)^2+2(2z+1)^2. \end{aligned}$$

As $E(1,2,2)=\{4^s(8t+7):\ s,t\in\mathbb{N}\}$ by Dickson [7, pp. 112-113], for each $n\in\mathbb{N}$ we can write $16n+13=w^2+2u^2+2v^2$ with $w,u,v\in\mathbb{Z}$. As $2\nmid w$, we have $2(u^2+v^2)\equiv 13-w^2\equiv 13-1\equiv 4 \pmod 8$, and hence $2\nmid uv$. Thus, $u=2y+1$ and $v=2z+1$ for some $y,z\in\mathbb{Z}$. Since

$$w^2\equiv 13-2(u^2+v^2)\equiv 13-2(1+1)=3^2 \pmod{16},$$

we have $w\not\equiv\pm1 \pmod 8$, and hence we can write w or $-w$ as $8x+3$ with $x\in\mathbb{Z}$. Thus, $16n+13=(8x+3)^2+2(2y+1)^2+2(2z+1)^2$ as desired.

Similarly, $(8,2,4,2,4,2)$ (or $x(2x+1)+y(2y+1)+z(4z+1)$) is universal, as pointed out in [22]. Since $(8,2,5,3,5,1)\sim(8,2,4,2,4,2)$ by (15), the tuple $(8,2,5,3,5,1)$ is also universal.

(b) To prove the universality of $(8,2,3,1,2,0)$ (or the sum $x^2+y(4y+1)+z(3z+1)/2$), we note that

$$\begin{aligned} &n = x^2 + y(4y+1) + \frac{z(3z+1)}{2} \\ \iff\ &48n+5 = 48x^2 + 3(8y+1)^2 + 2(6z+1)^2. \end{aligned}$$

As

$$E(2,3,48) = \bigcup_{k\in\mathbb{N}} \{8k+1, 8k+7, 16k+6, 16k+10, 64k+24\} \cup \{9^s(3t+1):\ s,t\in\mathbb{N}\}$$

by Dickson [7, pp. 112-113], for each $n\in\mathbb{N}$ we can write $48n+5 = 48x^2+3u^2+2v^2$ with $x,u,v\in\mathbb{Z}$. Clearly, $2\nmid u$ and $2v^2 \equiv 5-3u^2 \equiv 2 \pmod 8$, and thus $2\nmid uv$. As $3u^2 \equiv 5-2v^2 \equiv 5-2 = 3 \pmod{16}$, we can write u or $-u$ as $8y+1$ with $y\in\mathbb{Z}$. Since $\gcd(v,6)=1$, we can write v or $-v$ as $6z+1$ with $z\in\mathbb{Z}$. Hence, $48n+5 = 48x^2+3(8y+1)^2+2(6z+1)^2$ as desired.

Similarly, by using $E(2,3,48)$, we can prove the universality of $(8,6,3,1,2,0)$.

(c) By Dickson [7, pp. 112-113], we have

$$\begin{aligned} E(2,2,3) &= \{8k+1:\ k\in\mathbb{N}\} \cup \{9^s(9t+6):\ s,t\in\mathbb{N}\}, \\ E(1,2,16) &= \bigcup_{k\in\mathbb{N}} \{8k+5, 8k+7, 16k+10\} \cup \{4^s(16t+14):\ s,t\in\mathbb{N}\}, \\ E(1,2,32) &= \bigcup_{k\in\mathbb{N}} \{16k+14, 8k+5, 2(8k+5), 4(8k+5)\} \cup \{4^s(8t+7):\ s,t\in\mathbb{N}\}, \\ E(2,3,12) &= \{16k+6:\ k\in\mathbb{N}\} \cup \{9^s(3t+1):\ s,t\in\mathbb{N}\}, \\ E(2,3,8) &= \bigcup_{k\in\mathbb{N}} \{8k+1, 8k+7, 32k+4\} \cup \{9^s(9t+6):\ s,t\in\mathbb{N}\}, \\ E(2,5,10) &= \{8k+3:\ k\in\mathbb{N}\} \cup \bigcup_{s,t\in\mathbb{N}} \{25^s(5t+1), 25^s(5t+4)\}, \\ E(3,3,4) &= \{4k+1:\ k\in\mathbb{N}\} \cup \{9^s(3t+2):\ s,t\in\mathbb{N}\}. \end{aligned}$$

In view of this, we can easily prove that all the tuples

$$\begin{gathered} (8,2,3,1,3,1), (8,6,3,1,3,1), (8,2,4,2,2,0), (8,2,4,2,4,0), (8,6,4,2,4,0), \\ (8,2,5,1,4,2), (8,6,5,1,4,2), (8,4,6,4,6,2), (8,4,8,2,3,1), (8,4,8,4,6,4), \\ (8,6,8,4,3,1) \end{gathered}$$

are universal. This concludes the proof. □

Lemma 2.3. (Sun [20, Lemma 3.2]) *Let $w = x^2+3y^2 \equiv 4 \pmod{8}$ with $x,y\in\mathbb{Z}$. Then, there are odd integers u and v such that $w = u^2+3v^2$.*

Lemma 2.4. (F. Ge and Sun [10, Lemma 2.2]) *Let $w = x^2+3y^2$ with x,y odd and $3\nmid x$. Then, there are integers u and v relatively prime to 6 such that $w = u^2+3v^2$.*

Theorem 2.5. *All the tuples*

$$\begin{gathered} (9,1,4,2,3,1),\ (9,3,9,1,3,1),\ (9,5,4,2,4,2),\ (9,5,5,3,5,1), \\ (9,5,8,4,2,0),\ (9,5,9,3,3,1),\ (9,7,4,2,4,2),\ (9,7,5,3,5,1), \\ (9,7,8,4,2,0),\ (9,c,3,1,3,1)\ (c=1,5,7),\ (10,6,10,2,6,4) \end{gathered}$$

are universal.

Proof. In view of (15) and the universality of $(8,4,8,4,6,4)$ mentioned in Theorem 2.2, we have the universality of $(10,6,10,2,6,4)$.

To prove the universality of $(9,1,3,1,3,1)$ (or $x(3x+1)/2+y(3y+1)/2+z(9z+1)/2$), we note that

$$n=\frac{x(3x+1)}{2}+\frac{y(3y+1)}{2}+\frac{z(9z+1)}{2}$$
$$\iff 72n+7=3(6x+1)^2+3(6y+1)^2+(18z+1)^2.$$

As $E(1,3,3)=\{9^s(3t+2):\ s,t\in\mathbb{N}\}$ by [7, pp. 112-113], for each $n\in\mathbb{N}$ we can write $72n+7=w^2+3u^2+3v^2$ with $w,u,v\in\mathbb{Z}$. As $w^2\not\equiv 7 \pmod 4$, u or v is odd. Without loss of generality, we assume that v is odd. Since $w^2+3u^2\equiv 7-3v^2\equiv 4 \pmod 8$, by Lemma 2.3 we can write w^2+3u^2 as $w_0^2+3u_0^2$ with w_0 and u_0 odd integers. Note that $72n+7=w_0^2+3u_0^2+3v^2$. As $3\nmid w_0$, by Lemma 2.4 we can write $w_0^2+3u_0^2$ as $w_1^2+3u_1^2$ with $w_1,u_1\in\mathbb{Z}$ and $\gcd(w_1u_1,6)=1$. As $2\nmid w_1v$ and $3\nmid w_1$, by Lemma 2.4 we can write $w_1^2+3v^2$ as $w_2^2+3v_1^2$ with $w_2,v_1\in\mathbb{Z}$ and $\gcd(w_2v_1,6)=1$. Write u_1 or $-u_1$ as $6x+1$ with $x\in\mathbb{Z}$, and write v_1 or $-v_1$ as $6y+1$ with $y\in\mathbb{Z}$. Then, $72n+7=w_2^2+3(6x+1)^2+3(6y+1)^2$. As $w_2^2\equiv 7-3-3=1 \pmod{36}$, we can write w_2 or $-w_2$ as $18z+1$. Thus, $72n+1=3(6x+1)^2+3(6y+1)^2+(18z+1)^2$ as desired.

By similar arguments, we can show that both $(9,5,3,1,3,1)$ and $(9,7,3,1,3,1)$ are also universal.

To prove the universality of $(9,1,4,2,3,1)$ (or $x(x+1)/2+y(3y+1)/2+z(9z+1)/2$), we note that

$$n=\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+\frac{z(9z+1)}{2}$$
$$\iff 72n+13=9(2x+1)^2+3(6y+1)^2+(18z+1)^2.$$

As $E(1,1,3)=\{9^s(9t+6):\ s,t\in\mathbb{N}\}$ by [7, pp. 112-113], for each $n\in\mathbb{N}$ we can write $72n+13=u^2+v^2+3w^2$ with $u,v,w\in\mathbb{Z}$. As $3w^2\not\equiv 13 \pmod 4$, u or v is odd. Without loss of generality, we assume that $2\nmid u$. As $v^2+3w^2\equiv 13-u^2\equiv 4 \pmod 8$, by Lemma 2.3 we may write $v^2+3w^2=v_1^2+3w_1^2$ with v_1 and w_1 odd integers. Thus, $72n+13=u^2+v_1^2+3w_1^2$ and $2\nmid uv_1w_1$. As $u^2+v_1^2\equiv 13\equiv 1 \pmod 3$, u or v_1 is divisible by 3. Without loss of generality, we assume that $u=3(2x+1)$ with $x\in\mathbb{Z}$. Since $2\nmid v_1w_1$ and $3\nmid v_1$, by Lemma 2.4 we can write $v_1^2+3w_1^2$ as $v_2^2+3w_2^2$ with $v_2,w_2\in\mathbb{Z}$ and $\gcd(v_2w_2,6)=1$. We may write w_2 or $-w_2$ as $6y+1$ with $y\in\mathbb{Z}$. Thus, $72n+13=9(2x+1)^2+3(6y+1)^2+v_2^2$. As $v_2^2\equiv 13-9-3=1 \pmod{36}$, we may write v_2 or $-v_2$ as $18z+1$ with $z\in\mathbb{Z}$. So, $72n+13=9(2x+1)^2+3(6y+1)^2+(18z+1)^2$ as desired.

In view of (9), from the universality of $(9,1,4,2,3,1)$, we deduce that the tuple $(9,3,9,1,3,1)$ is universal. The tuple $(9,5,4,2,3,1)$ is known to be universal (cf. [23, Theorem 1.2(ii)]), so $(9,5,9,3,3,1)$ is also universal with the aid of (9).

To prove the universality of $(9,5,4,2,4,2)$ (or $x(x+1)/2+y(y+1)/2+z(9z+5)/2$), we note that

$$n=\frac{x(x+1)}{2}+\frac{y(y+1)}{2}+\frac{z(9z+5)}{2}$$
$$\iff 72n+43=9(2x+1)^2+9(2y+1)^2+(18z+5)^2.$$

As $E(1,1,1)=\{4^s(8t+7):\ s,t\in\mathbb{N}\}$ (cf. [7, pp. 112-113], for each $n\in\mathbb{N}$ we can write $72n+43=u^2+v^2+w^2$ with $u,v,w\in\mathbb{Z}$. As $u^2+v^2+w^2\equiv 43\equiv 3 \pmod 8$, we have $2\nmid uvw$. Since $u^2+v^2+w^2\equiv 1 \pmod 3$, without loss of generality we may assume that both u and v are divisible by 3. Thus, $u = 3(2x+1)$ and $v = 3(2y+1)$ for some $x, y \in \mathbb{Z}$. As $w^2 \equiv 43 - 9 - 9 = 5^2 \pmod{36}$, we may write w or $-w$ as $18z + 5$ with $z \in \mathbb{Z}$. So, $72n + 43 = 9(2x+1)^2 + 9(2y+1)^2 + (18z+5)^2$ as desired.

Similarly, we can prove that $(9, 7, 4, 2, 4, 2)$ is universal.

For $c = 5, 7$, we have

$$(9, c, 5, 3, 5, 1) \sim (9, c, 4, 2, 4, 2) \sim (9, c, 8, 4, 2, 0)$$

by (15) and (8). Since $(9, 5, 4, 2, 4, 2)$ and $(9, 7, 4, 2, 4, 2)$ are universal, the tuples

$$(9, 5, 5, 3, 5, 1),\ (9, 5, 8, 4, 2, 0),\ (9, 7, 5, 3, 5, 1),\ (9, 7, 8, 4, 2, 0)$$

are also universal.

In view of the above, we have completed the proof of Theorem 2.5. □

The following known lemma can be found in [20, Lemma 2.1].

Lemma 2.6. *Let $w = x^2 + my^2$ be a positive integer with $m \in \{2, 5, 8\}$ and $x, y \in \mathbb{Z}$. Then, we can write w in the form $u^2 + mv^2$ with $u, v \in \mathbb{Z}$ such that u or v is not divisible by 3.*

The following lemma follows from [21, Lemma 2.2(i)] (or the comments around [20, (1.4)]).

Lemma 2.7. *If $x, y, z \in \mathbb{Z}$ are not all divisible by 3, then $9(x^2 + y^2 + z^2) = u^2 + v^2 + w^2$ for some $u, v, w \in \mathbb{Z}$ not all divisible by 3.*

Theorem 2.8. *All the tuples*

$$(12, 4, 3, 1, 2, 0),\ (12, 4, 6, 4, 6, 2),\ (12, 8, 3, 1, 3, 1),$$
$$(12, 6, 8, 2, 3, 1),\ (12, 6, 8, 6, 3, 1),\ (12, 8, 3, 1, 2, 0),\ (12, 8, 5, 3, 5, 1),$$
$$(12, 8, 4, 2, 4, 2),\ (12, 8, 8, 4, 2, 0),\ (12, 8, 8, 4, 3, 1),\ (12, 8, 12, 4, 2, 0)$$

are universal.

Proof. To prove the universality of $(12, 4, 3, 1, 2, 0)$ (or $x^2 + 2y(3y+1) + z(3z+1)/2$), we observe that

$$\begin{aligned} n &= x^2 + 2y(3y+1) + \frac{z(3z+1)}{2} \\ \iff 24n + 5 &= 24x^2 + 4(6y+1)^2 + (6z+1)^2. \end{aligned}$$

By Dickson [7, pp. 112-113],

$$E(1, 4, 24) = \bigcup_{k \in \mathbb{N}} \{4k+2, 4k+3\} \cup \{9^s(9t+3) :\ s, t \in \mathbb{N}\}.$$

Thus, for each $n \in \mathbb{N}$ we can write $24n + 5 = 24x^2 + 4u^2 + v^2$ with $x, u, v \in \mathbb{Z}$. Clearly, $2 \nmid v$. As $4u^2 \equiv 5 - v^2 \equiv 4 \pmod 8$, u is also odd. Since $u^2 + v^2 \equiv 4u^2 + v^2 \equiv 5 \equiv 2 \pmod 3$, we have $3 \nmid uv$. As uv is relatively prime to 6, we can write u or $-u$ as $6y+1$, and write v or $-v$ as $6z+1$, where y and z are integers. Now, $n = 24x^2 + 4(6y+1)^2 + (6z+1)^2$ as desired.

To prove the universality of $(12, 6, 8, 2, 3, 1)$ (or $3x(x+1)/2 + y(3y+1)/2 + z(4z+1)$), we note that

$$\begin{aligned} n &= 3\frac{x(x+1)}{2} + \frac{y(3y+1)}{2} + z(4z+1) \\ \iff 48n + 23 &= 18(2x+1)^2 + 2(6y+1)^2 + 3(8z+1)^2. \end{aligned}$$

By Dickson [7, pp. 112-113],

$$E(2,3,18)=\bigcup_{k\in\mathbb{N}}\{3k+1,8k+1\}\cup\{9^s(9t+3):\ s,t\in\mathbb{N}\}.$$

Thus, for each $n\in\mathbb{N}$, we can write $48n+23=18u^2+2v^2+3w^2$ with $u,v,w\in\mathbb{Z}$. Clearly, $2\nmid w$. Since $2(u^2+v^2)\equiv 23-3w^2\equiv 4 \pmod 8$, we have $2\nmid uv$. Note also that $3\nmid v$. So, we may write $u=2x+1$ with $x\in\mathbb{Z}$, and v or $-v$ as $6y+1$ with $y\in\mathbb{Z}$. As $3w^2\equiv 23-2u^2-2v^2\equiv 23-2-2\equiv 3 \pmod{16}$, we may write w or $-w$ as $8z+1$ with $z\in\mathbb{Z}$. Thus, $48n+23=18(2x+1)^2+2(6y+1)^2+3(8z+1)^2$ as desired. Similarly, $(12,6,8,6,3,1)$ is universal.

By similar arguments, we can prove that $(12,8,3,1,2,0)$ is universal by using

$$E(1,16,24)=\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3,8k+5,64k+8\}\cup\{9^s(9t+3):\ s,t\in\mathbb{N}\}$$

(cf. [7, pp. 112-113]). We can also prove that $(12,8,8,4,3,1)$ is universal by using

$$E(1,6,16)=\bigcup_{k\in\mathbb{N}}\{8k+3,8k+5,16k+2,16k+14,64k+8\}\cup\{9^s(9t+3):\ s,t\in\mathbb{N}\}$$

(cf. [7, pp. 112-113]), and that the three tuples

$$(12,8,5,3,5,1),\ (12,8,4,2,4,2) \text{ and } (12,8,8,4,2,0)$$

(which are equivalent in view of (8) and (15)) are universal by using

$$E(3,8,12)=\bigcup_{k\in\mathbb{N}}\{4k+1,4k+2\}\cup\{9^s(3t+1):\ s,t\in\mathbb{Z}\}$$

(cf. [7, pp. 112-113]). Also, we can prove that $(12,8,12,4,2,0)$ is universal by using

$$E(1,4,6)=\{16k+2:\ k\in\mathbb{N}\}\cup\{9^s(9t+3):\ s,t\in\mathbb{N}\}$$

(cf. [7, pp. 112-113]).

To prove the universality of $(12,4,6,4,6,2)$ (or $2x(3x+1)+y(3y+1)+z(3z+2)$), we note that

$$n=2x(3x+1)+y(3y+1)+z(3z+2)$$
$$\iff 12n+7=2(6x+1)^2+(6y+1)^2+4(3z+1)^2.$$

As $E(1,2,4)=\{4^s(16t+14):\ s,t\in\mathbb{N}\}$ (cf. [7, pp. 112-113]), for each $n\in\mathbb{N}$ we can write $12n+7$ as $u^2+2v^2+4w^2=u^2+2(v^2+2w^2)$ with $u,v,w\in\mathbb{Z}$. By Lemma 2.6, without loss of generality, we may assume that u or v is not divisible by 3, and also v or w is not divisible by 3. If $3\mid v$, then $3\nmid uw$, and also

$$2\equiv u^2+w^2\equiv u^2+4w^2\equiv 7\equiv 1 \pmod 3.$$

Therefore, we must have $3\nmid v$. As $u^2+w^2\equiv u^2+4w^2\equiv 7-2v^2\equiv 2 \pmod 3$, and we also have $3\nmid uw$. As u is relatively prime to 6, we may write u or $-u$ as $6y+1$ with $y\in\mathbb{Z}$. Note that $2v^2\equiv 7-u^2\equiv 2 \pmod 4$. So, we may write v or $-v$ as $6x+1$ with $x\in\mathbb{Z}$. Clearly, we may write w or $-w$ as $3z+1$ with $z\in\mathbb{Z}$. Therefore, $12n+7=2(6x+1)^2+(6y+1)^2+4(3z+1)^2$ as desired.

To prove the universality of $(12,8,3,1,3,1)$ (or $2x(3x+2)+y(3y+1)/2+z(3z+1)/2$), we note that

$$n=2x(3x+2)+\frac{y(3y+1)}{2}+\frac{z(3z+1)}{2}$$
$$\iff 24n+18=16(3x+1)^2+(6y+1)^2+(6z+1)^2.$$

As $E(1,1,1) = \{4^s(8t+7):\ s,t \in \mathbb{N}\}$ (cf. [7, pp. 112-113]), for each $n \in \mathbb{N}$ we can write $24n+18$ as a sum of three squares. In view of Lemma 2.7, we may write $24n+18 = u^2+v^2+w^2$ with $u,v,w \in \mathbb{Z}$ not all divisible by 3. As $u^2+v^2+w^2 \equiv 0 \pmod 3$, we must have $3 \nmid uvw$. Without loss of generality, we may assume that w is even. As $u^2+v^2 \equiv 18 \equiv 2 \pmod 4$, we have $2 \nmid uv$. Note that $w^2 \equiv 18-u^2-v^2 \equiv 0 \pmod 8$ and hence $w \equiv 0 \pmod 4$. We may write w or $-w$ as $4(3x+1)$ with $x \in \mathbb{Z}$. Also, we may write u or $-u$ as $6y+1$ with $y \in \mathbb{Z}$, and write v or $-v$ as $6z+1$ with $z \in \mathbb{Z}$. Thus, $24n+18 = 16(3x+1)^2+(6y+1)^2+(6z+1)^2$ as desired. This ends the proof. $\square$

Theorem 2.9. *All the tuples*

$$(16,8,6,4,4,0),\ (16,8,6,4,4,2),\ (18,6,6,4,6,2),$$
$$(20,10,5,1,4,2),\ (20,10,5,3,4,2),\ (24,12,6,2,2,0)$$

are universal.

Proof. In view of (8), we have $(16,8,6,4,4,0) \sim (8,4,8,4,6,4)$. As $(8,4,8,4,6,4)$ is universal by Theorem 2.2, the tuple $(16,8,6,4,4,0)$ is also universal.

To prove the universality of $(16,8,6,4,4,2)$ (or $x(x+1)/2+2y(y+1)+z(3z+2)$), we note that

$$\begin{aligned} & n = \frac{x(x+1)}{2} + 2y(y+1) + z(3z+2) \\ \iff & 24n+23 = 3(2x+1)^2+12(2y+1)^2+8(3z+1)^2. \end{aligned}$$

As

$$E(3,8,12) = \bigcup_{k\in\mathbb{N}} \{4k+1, 4k+2\} \cup \{9^s(3t+1):\ s,t \in \mathbb{N}\}$$

by [7, pp. 112-113], for each $n \in \mathbb{N}$ we can write $24n+23$ as $3u^2+12v^2+8w^2$ with $u,v,w \in \mathbb{Z}$. Clearly u is odd. As $12v^2 \equiv 23-3u^2 \equiv 20 \not\equiv 0 \pmod 8$, v is odd. So, $u = 2x+1$ and $v = 2y+1$ for some $x,y \in \mathbb{Z}$. Since $8w^2 \equiv 23 \pmod 3$, we can write w or $-w$ as $3z+1$ with $z \in \mathbb{Z}$. Therefore, $24n+23 = 3(2x+1)^2+12(2y+1)^2+8(3z+1)^2$ as desired.

By (9), we have $(18,6,6,4,6,2) \sim (8,4,6,4,6,2)$. Since $(8,4,6,4,6,2)$ is universal by Theorem 2.2, the tuple $(18,6,6,4,6,2)$ is also universal.

To prove the universality of $(20,10,5,1,4,2)$ (or the sum $x(x+1)/2+5y(y+1)/2+z(5z+1)/2$), we note that

$$\begin{aligned} & n = \frac{x(x+1)}{2} + 5\frac{y(y+1)}{2} + \frac{z(5z+1)}{2} \\ \iff & 40n+31 = 5(2x+1)^2+25(2y+1)^2+(10z+1)^2. \end{aligned}$$

By Dickson [7, pp. 112-113],

$$E(1,5,25) = \bigcup_{k\in\mathbb{N}} \{5k+2, 5k+3, 25k+10, 25k+15\} \cup \{4^s(8t+3):\ s,t \in \mathbb{N}\}.$$

Thus, for any $n \in \mathbb{N}$ we can write $40n+31$ as $w^2+5u^2+25v^2$ with $w,u,v \in \mathbb{Z}$. If w is even, then

$$u^2+v^2 \equiv 5u^2+25v^2 \equiv 31-w^2 \equiv 3 \pmod 4$$

which is impossible. Note also that $w^2 \equiv 31 \equiv 1 \pmod 5$ and hence $w \equiv \pm 1 \pmod 5$. Thus, w or $-w$ can be written as $10z+1$ with $z \in \mathbb{Z}$. As $u^2+v^2 \equiv 31-w^2 \equiv$

2 (mod 4), we have $2 \nmid uv$. Thus, $u = 2x + 1$ and $v = 2y + 1$ for some $x, y \in \mathbb{Z}$. Therefore,

$$40n + 31 = w^2 + 5u^2 + 25v^2 = 5(2x + 1)^2 + 25(2y + 1)^2 + (10z + 1)^2$$

as desired.

Similarly, we can prove that the tuple $(20, 10, 5, 3, 4, 2)$ is universal.

In view of (11), we have $(24, 12, 6, 2, 2, 0) \sim (12, 8, 12, 4, 2, 0)$. As $(12, 8, 12, 4, 2, 0)$ is universal by Theorem 2.8, the tuple $(24, 12, 6, 2, 2, 0)$ is also universal.

In view of the above, we have completed the proof of Theorem 2.9. □

3. **An approach via Ramanujan's theta functions.**

Now we turn to a new approach to universal sums via Ramanujan's theta functions.

Proof of Theorem 1.1. Let $|q| < 1$. For $a, b \in \mathbb{Z}$ with $a + b > 0$, clearly $|q^a q^b| < 1$ and

$$f(q^a, q^b) = \sum_{x=-\infty}^{\infty} q^{ax(x+1)/2 + bx(x-1)/2} = \sum_{x=-\infty}^{\infty} q^{x((a+b)x+a-b)/2}.$$

For any $n \in \mathbb{N}$, let $R(n)$ denote the number of triples $(x, y, z) \in \mathbb{Z}^3$ satisfying the equation

$$\frac{x((i + j)x + i - j)}{2} + \frac{y((s + t)y + s - t)}{2} + \frac{z((u + v)z + u - v)}{2} = n,$$

and let $R_r(n)$ (with $1 \leqslant r \leqslant k$) denote the number of ways to write n as

$$\frac{x((a_{1r} + a_{2r})x + a_{1r} - a_{2r})}{2} + \frac{y((b_{1r} + b_{2r})y + b_{1r} - b_{2r})}{2} + \frac{z((c_{1r} + c_{2r})z + c_{1r} - c_{2r})}{2}$$

with $(x, y, z) \in \mathbb{Z}^3$.

For any fixed $n \in \mathbb{N}$ and $r \in \{1, \ldots, k\}$, by comparing coefficients of q^{kn+r-1} on both sides of (7), we obtain

$$R(kn + r - 1) = m_r R_r(n). \tag{17}$$

Thus, for any $n_0 \in \mathbb{N}$, we have

$$\begin{aligned} &R_r(n) > 0 \text{ for all } r = 1, \ldots, k \text{ and } n = n_0, n_0 + 1, \ldots \\ &\iff R(n) > 0 \text{ for every integer } n \geqslant kn_0. \end{aligned} \tag{18}$$

When $n_0 = 0$, this yields part (i) of Theorem 1.1. As (18) holds for any $n_0 \in \mathbb{N}$, we also have part (ii) of Theorem 1.1. This concludes the proof. □

Remark 3.1. In view of (17), $|\{n \in \mathbb{N} : R(n) = 0\}| = 1$, if and only if for certain $1 \leqslant r_0 \leqslant k$ we have $|\{n \in \mathbb{N} : R_{r_0}(n) = 0\}| = 1$ and $R_r(n) > 0$ for all $r \in \{1, \ldots, k\} \setminus \{r_0\}$ and $n \in \mathbb{N}$.

In 2023, Ju [13] classified all almost universal sums of triangular numbers with one exception. Furthermore, he provided an effective criterion for almost universality with one exception of an arbitrary sum of triangular numbers, which might be considered as a natural generalization of the 15-theorem of Conway, Miller, and Schneeberger (cf. [6]).

4. **Proof of Theorem 1.2.** To apply Theorem 1.1, we need to make certain preparations.

Let a and b be complex numbers with $|ab| < 1$. The Ramanujan theta function $f(a,b)$ satisfies the following basic properties [1]:

$$f(a,b) = f(b,a) \tag{19}$$

and

$$f(1,a) = 2f(a,a^3). \tag{20}$$

The functions $\varphi(q)$ given by (5), $\psi(q)$ given by (6),

$$X(q) := f(q,q^2) = \sum_{x=-\infty}^{\infty} q^{x(3x+1)/2} \tag{21}$$

and

$$Y(q) := f(q,q^5) = \sum_{x=-\infty}^{\infty} q^{x(3x+2)} \tag{22}$$

are related to the squares, the triangular numbers, the generalized pentagonal numbers, and the generalized octagonal numbers, respectively. In general, the theta function $f(q,q^{m-3})$ with $m \geqslant 3$ is related to the generalized m-gonal numbers $p_m(x)$ $(x \in \mathbb{Z})$.

Let $|ab| < 1$. The following general formula is due to Ramanujan (cf. [1, Entry 31]:

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right), \tag{23}$$

where

$$U_k = a^{k(k+1)/2} b^{k(k-1)/2} \quad \text{and} \quad V_k = a^{k(k-1)/2} b^{k(k+1)/2}.$$

This result implies the following lemma.

Lemma 4.1. *For $|q| < 1$, we have*

$$\varphi(q) = \varphi\left(q^4\right) + 2q\psi\left(q^8\right), \tag{24}$$

$$\psi(q) = f\left(q^6,q^{10}\right) + qf\left(q^2,q^{14}\right), \tag{25}$$

$$Y(q) = X\left(q^8\right) + qY\left(q^4\right), \tag{26}$$

$$X(q) = f\left(q^{12},q^{15}\right) + qf\left(q^6,q^{21}\right) + q^2 f\left(q^3,q^{24}\right). \tag{27}$$

Let $|q| < 1$. In [2, (50)], setting $k = 3$, $l = 2$, and $\varepsilon_1 = \varepsilon_2 = 1$, we obtain

$$X(q)X(q^2) = \varphi(q^9)X(q^3) + qX(q^3)Y(q^3) + 2q^2\psi(q^9)Y(q^3). \tag{28}$$

In [2, (15)], setting $k = 3$, $l = 2$, $g = 5$, $h = 1$, $u = 2$, $v = 1$, and $\varepsilon_1 = \varepsilon_2 = 1$, we obtain

$$X(q)Y(q) = X(q^3)X(q^6) + 2q\psi(q^9)X(q^6) + 2q^2\psi(q^{18})X(q^3). \tag{29}$$

Lemma 4.1 and the identities (28) and (29) will be used later.

Proof of Theorem 1.2. Let $|q| < 1$. Multiplying both sides of the identity (25) by $f(q^6,q^4)f(q^4,q^2)$, we get

$$f(q^6,q^4)f(q^4,q^2)f(q^3,q) = f(q^{10},q^6)f(q^6,q^4)f(q^4,q^2) + qf(q^{14},q^2)f(q^6,q^4)f(q^4,q^2). \tag{30}$$

Thus, (7) holds with

$$i = 6,\ j = 4,\ s = 4,\ t = 2,\ u = 3,\ v = 1,\ k = 2,$$

$$a_{11}=5,\ a_{21}=3,\ b_{11}=3,\ b_{21}=2,\ c_{11}=2,\ c_{21}=1,$$
$$a_{12}=7,\ a_{22}=1,\ b_{12}=3,\ b_{22}=2,\ c_{12}=2,\ c_{22}=1.$$

Since the tuple

$$(i+j,i-j,s+t,s-t,u+v,u-v)=(10,2,6,2,4,2)$$

(or the sum $x(2x+1)+y(3y+1)+z(5z+1)$) is universal as was conjectured by Sun [22] and confirmed by Ju and Oh [14], by applying Theorem 1.1(i), we get that the tuples

$$(a_{11}+a_{21},a_{11}-a_{21},b_{11}+b_{21},b_{11}-b_{21},c_{11}+c_{21},c_{11}-c_{21})=(8,2,5,1,3,1),$$

and

$$(a_{12}+a_{22},a_{12}-a_{22},b_{12}+b_{22},b_{12}-b_{22},c_{12}+c_{22},c_{12}-c_{22})=(8,6,5,1,3,1),$$

are universal.

Using (25), we find that

$$\psi(q)\psi(q^2)f(q^2,q^8)=f\left(q^6,q^{10}\right)\psi(q^2)f(q^2,q^8)+qf\left(q^2,q^{14}\right)\psi(q^2)f(q^2,q^8),$$

which can be written in the form

$$\begin{aligned}f(q^8,q^2)f(q^6,q^2)f(q^3,q)=&\ f\left(q^{10},q^6\right)f(q^8,q^2)f(q^6,q^2)\\&+qf\left(q^{14},q^2\right)f(q^8,q^2)f(q^6,q^2).\end{aligned}\tag{31}$$

Thus, (7) holds with

$$i=8,\ j=2,\ s=6,\ t=2,\ u=3,\ v=1,\ k=2,$$
$$a_{11}=5,\ a_{21}=3,\ b_{11}=4,\ b_{21}=1,\ c_{11}=3,\ c_{21}=1.$$

Sun [20] determined the universality of the tuple $(i+j,i-j,s+t,s-t,u+v,u-v)=(10,6,8,4,4,2)$ (or the sum $p_3+2p_3+2p_7$). Thus, using the first term of the identity (31) with the help of Theorem 1.1(i), we see that the tuple

$$(a_{11}+a_{21},a_{11}-a_{21},b_{11}+b_{21},b_{11}-b_{21},c_{11}+c_{21},c_{11}-c_{21})=(8,2,5,3,4,2)$$

is universal.

From (25), we obtain

$$\psi(q)X\left(q^2\right)f\left(q^6,q^8\right)=X\left(q^2\right)f\left(q^6,q^{10}\right)f\left(q^6,q^8\right)+qX\left(q^2\right)f\left(q^2,q^{14}\right)f\left(q^6,q^8\right).\tag{32}$$

Ju and Oh [14] proved that the sum $x(2x+1)+y(3y+1)+z(7z+1)$ (or the tuple $(14,2,6,2,4,2)$) is universal as conjectured by Sun [22]. Thus, the identity (32) with the help of Theorem 1.1(i) yields the universality of the tuples $(8,2,7,1,3,1)$ and $(8,6,7,1,3,1)$.

By (25), we have

$$\psi(q)X(q^2)X(q^4)=X(q^2)X(q^4)f\left(q^6,q^{10}\right)+qX(q^2)X(q^4)f\left(q^2,q^{14}\right).\tag{33}$$

Sun [19] proved that the tuple $(8,4,4,2,4,0)$ is universal. As $(8,4,4,2,4,0)\sim(12,4,6,2,4,2)$ by (12), the tuple $(12,4,6,2,4,2)$ is universal. Combining this with the first term of (33) and Theorem 1.1(i), we obtain the universality of the tuple $(8,2,6,2,3,1)$.

Using (25), we find that

$$\psi(q)\psi\left(q^2\right)Y\left(q^2\right)=\psi\left(q^2\right)Y\left(q^2\right)f\left(q^6,q^{10}\right)+q\psi\left(q^2\right)Y\left(q^2\right)f\left(q^2,q^{14}\right).\tag{34}$$

As conjectured by Sun [20, Conjecture 1.13] and confirmed by Ju, Oh, and Seo [15], the tuple $(12,8,8,4,4,2)$ is universal. Combining this with the identity (34) and Theorem 1.1(i), we get the universality of the tuples $(8,2,6,4,4,2)$ and $(8,6,6,4,4,2)$.

In view of (25),

$$\psi(q)X\left(q^4\right)Y\left(q^2\right)=X\left(q^4\right)Y\left(q^2\right)f\left(q^6,q^{10}\right)+qX\left(q^4\right)Y\left(q^2\right)f\left(q^2,q^{14}\right). \quad (35)$$

It is known (cf. [23]) that the tuple $(24,12,6,2,4,2)$ is universal. By (11),

$$(24,12,6,2,4,2)\sim(12,8,12,4,4,2).$$

Thus, the tuple $(12,8,12,4,4,2)$ is also universal. Combining this with (35) and Theorem 1.1(i), we obtain the universality of $(8,2,6,4,6,2)$ and $(8,6,6,4,6,2)$.

Using (24), we find that

$$\begin{aligned}\varphi(q)\varphi\left(q^2\right)X\left(q^4\right)=&\ \varphi\left(q^4\right)\varphi\left(q^8\right)X\left(q^4\right)+2q\varphi\left(q^8\right)\psi\left(q^8\right)X\left(q^4\right)\\&+2q^2\varphi\left(q^4\right)\psi\left(q^{16}\right)X\left(q^4\right)+4q^3\psi\left(q^8\right)\psi\left(q^{16}\right)X\left(q^4\right).\end{aligned} \quad (36)$$

Thus, (7) holds with

$$\begin{gathered}i=8,\ j=4,\ s=2,\ t=2,\ u=1,\ v=1,\ k=4,\\a_{12}=6,\ a_{22}=2,\ b_{12}=2,\ b_{22}=2,\ c_{12}=2,\ c_{22}=1,\\a_{14}=12,\ a_{24}=4,\ b_{14}=6,\ b_{24}=2,\ c_{14}=2,\ c_{24}=1.\end{gathered}$$

By Sun [23, Theorem 1.2(i)], the tuple $(12,4,4,0,2,0)$ is universal. Thus, by applying Theorem 1.1(i) on the identity (36), we see that the second term and the fourth term yield the universality of the tuples $(8,4,4,0,3,1)$ and $(16,8,8,4,3,1)$.

From (26), we have

$$\varphi\left(q^2\right)\psi\left(q^2\right)Y(q)=\varphi\left(q^2\right)\psi\left(q^2\right)X\left(q^8\right)+q\varphi\left(q^2\right)\psi\left(q^2\right)Y\left(q^4\right). \quad (37)$$

Since the tuples $(12,4,4,2,2,0)$ and $(12,8,4,2,2,0)$ are universal (cf. [23]), by applying Theorem 1.1(i) to the identity (37), we obtain the universality of the tuple $(8,4,6,4,4,0)$.

From (27), we obtain

$$\begin{aligned}\varphi\left(q^3\right)\psi\left(q^3\right)X(q)=&\ \varphi\left(q^3\right)\psi\left(q^3\right)f\left(q^{12},q^{15}\right)+q\varphi\left(q^3\right)\psi\left(q^3\right)f\left(q^6,q^{21}\right)\\&+q^2\varphi\left(q^3\right)\psi\left(q^3\right)f\left(q^3,q^{24}\right).\end{aligned} \quad (38)$$

Thus, (7) holds with

$$\begin{gathered}i=9,\ j=3,\ s=3,\ t=3,\ u=2,\ v=1,\ k=3,\\a_{12}=7,\ a_{22}=2,\ b_{12}=3,\ b_{22}=1,\ c_{12}=1,\ c_{22}=1.\end{gathered}$$

As the sum $3x^2+y(3y+1)+z(3z+2)$ (or the tuple $(6,4,6,2,6,0)$) is universal by Sun [22], with the aid of (11) the tuple $(12,6,6,0,3,1)$ is also universal. Thus, using the second term of (38) with the help of Theorem 1.1(i), we get the universality of the tuple $(9,5,4,2,2,0)$.

In a similar way, using (27) and Theorem 1.1(i), we determine the universality of the tuples listed in the last column of the following table:

The product	The tuple used	found in	term/s used	universal tuple/s
$X(q)X\left(q^3\right)f\left(q^9,q^6\right)$	$(15,3,9,3,3,1)$	[23]	all terms	$(9,1,5,1,3,1)$ $(9,5,5,1,3,1)$ $(9,7,5,1,3,1)$
$X(q)X\left(q^3\right)f\left(q^{12},q^3\right)$	$(15,9,9,3,3,1)$	[23]	all terms	$(9,1,5,3,3,1)$ $(9,5,5,3,3,1)$ $(9,7,5,3,3,1)$
$X(q)X\left(q^3\right)X\left(q^6\right)$	$(18,6,9,3,3,1)$	[12]	second term	$(9,5,6,2,3,1)$
$\psi\left(q^6\right)X(q)X\left(q^3\right)$	$(24,12,9,3,3,1)$	[23]	second term	$(9,5,8,4,3,1)$
$X(q)X\left(q^3\right)Y\left(q^3\right)$	$(18,12,9,3,3,1)$	[23]	third term	$(9,7,6,4,3,1)$
$\psi\left(q^3\right)\psi\left(q^6\right)X(q)$	$(24,12,12,6,3,1)$	[23]	third term	$(9,7,8,4,4,2)$
$\varphi\left(q^3\right)X(q)X\left(q^3\right)$	$(9,3,6,0,3,1)$	[23]	all terms	$(9,1,3,1,2,0)$ $(9,5,3,1,2,0)$ $(9,7,3,1,2,0)$
$\psi\left(q^9\right)X(q)X\left(q^3\right)$	$(36,18,9,3,3,1)$	[15]	all terms	$(12,6,9,1,3,1)$ $(12,6,9,5,3,1)$ $(12,6,9,7,3,1)$

The universality of $(36,18,9,3,3,1)$ in the above table follows from the universality of the tuple $(36,18,4,2,3,1)$ (or the sum $p_3+9p_3+p_5$) (which was conjectured by [20, Conjecture 1.13] and confirmed by Ju, Oh, and Seo [15]), since $(36,18,4,2,3,1)\sim(36,18,9,3,3,1)$ by (9).

Using (28), we have

$$\varphi(q^3)X(q)X(q^2)=\varphi(q^3)\varphi(q^9)X(q^3)+q\varphi(q^3)X(q^3)Y(q^3)+2q^2\varphi(q^3)\psi(q^9)Y(q^3). \tag{39}$$

Sun [23, Section 3] showed the universality of the tuple $(6,2,6,0,3,1)$ over $\mathbb{Z}$. Combining this with the third term of (39) and Theorem 1.1(i), we get the universality of the tuple $(12,6,6,4,2,0)$.

Again, using (28), we find that

$$\psi(q^3)X(q)X(q^2)=\varphi(q^9)\psi(q^3)X(q^3)+q\psi(q^3)X(q^3)Y(q^3)+2q^2\psi(q^3)\psi(q^9)Y(q^3). \tag{40}$$

By [11], $x^2+p_3(y)+3p_3(z)$ is universal. So, the tuple $(12,6,6,2,3,1)$ is universal over $\mathbb{Z}$ with the aid of (12). Combining this with the third term of the identity (40) and Theorem 1.1(i), we obtain the universality of the tuple $(12,6,6,4,4,2)$.

From (28), we obtain

$$\psi(q^6)X(q)X(q^2)=\varphi(q^9)\psi(q^6)X(q^3)+q\psi(q^6)X(q^3)Y(q^3)+2q^2\psi(q^6)\psi(q^9)Y(q^3). \tag{41}$$

As $x^2+p_3(y)+6p_3(z)$ is universal by [11], the tuple $(24,12,6,2,3,1)$ is universal in view of (12). This, together with the third term of the identity (41) and Theorem 1.1(i), yields the universality of the tuple $(12,6,8,4,6,4)$.

Using (26), we find that

$$\psi\left(q^2\right)X\left(q^2\right)Y(q)=\psi\left(q^2\right)X\left(q^2\right)X\left(q^8\right)+q\psi\left(q^2\right)X\left(q^2\right)Y\left(q^4\right). \tag{42}$$

By [20, Theorem 1.11], the tuple $(12,6,8,4,3,1)$ is universal over $\mathbb{Z}$. In view of (11), this is equivalent to the universality of $(8,4,6,4,6,2)$. Combining this with the second term of (42) and Theorem 1.1(i), we see that the tuple $(12,8,4,2,3,1)$ is universal.

From (26), we obtain

$$X\left(q^4\right)Y(q)Y\left(q^2\right)=X\left(q^4\right)X\left(q^8\right)Y\left(q^2\right)+qX\left(q^4\right)Y\left(q^2\right)Y\left(q^4\right). \tag{43}$$

Since $(24,12,6,4,6,2)$ is universal by Sun [22], with the aid of (11) we see that the tuple $(12,8,12,4,6,4)$ is also universal. Combining this with the second term of the identity (43) and Theorem 1.1(i), we obtain the universality of the tuple $(12,8,6,4,6,2)$. Hence, $(16,8,8,4,6,4)$ is also universal in light of (13).

In view of (26), we have

$$X\left(q^{2}\right)X\left(q^{6}\right)Y\left(q\right)=X\left(q^{2}\right)X\left(q^{6}\right)X\left(q^{8}\right)+qX\left(q^{2}\right)X\left(q^{6}\right)Y\left(q^{4}\right). \tag{44}$$

Since the tuple $(18,6,6,4,6,2)$ is universal by Theorem 2.9, applying Theorem 1.1(i) to the second term of (44), we get the universality of the tuple $(12,8,9,3,3,1)$.

To determine the universality of the tuple $(12,8,12,6,3,1)$, using (26) we have

$$\psi(q^{6})X(q^{2})Y(q)=\psi(q^{6})X(q^{2})X(q^{8})+q\psi(q^{6})X(q^{2})Y(q^{4}). \tag{45}$$

Sun [22] proved that the sum $x(3x+1)+y(3y+2)+z(3z+3)$ (or $(24,12,6,4,6,2)$) is universal. Thus, in light of the second term of (45) and Theorem 1.1(i), we obtain the universality of $(12,8,12,6,3,1)$.

Using (24), we have

$$\varphi(q)\psi\left(q^{4}\right)X\left(q^{2}\right)=\varphi\left(q^{4}\right)\psi\left(q^{4}\right)X\left(q^{2}\right)+2q\psi\left(q^{4}\right)\psi\left(q^{8}\right)X\left(q^{2}\right). \tag{46}$$

By the paragraph containing (36), the tuples $(8,4,4,0,3,1)$ and $(16,8,8,4,3,1)$ are both universal. Thus, applying Theorem 1.1(i) to the identity (46), we get the universality of the tuple $(16,8,6,2,2,0)$.

In view of the above, we have completed the proof of Theorem 1.2. □

The tuples $(20,10,5,1,3,1)$ and $(20,10,5,3,3,1)$ were proved to be universal in [24, Theorem 1.1(iv)] via the theory of ternary quadratic forms. Now, we give a new proof by using our approach via theta functions. Let $|q|<1$. Using [4, Corollary 2.11] with $a=q$, $b=q^4$, $c=q^2$, and $d=q^3$, we obtain

$$\begin{aligned}f(q,q^{4})f(q^{2},q^{3})=&f^{2}(q^{10},q^{15})+qf(q^{15},q^{10})f(q^{20},q^{5})+q^{2}f^{2}(q^{20},q^{5})\\&+2q^{3}\psi(q^{25})f(q^{15},q^{10})+2q^{4}\psi(q^{25})f(q^{20},q^{5}),\end{aligned}$$

and hence

$$\begin{aligned}X(q^{5})f(q,q^{4})f(q^{2},q^{3})=&\ X(q^{5})f^{2}(q^{10},q^{15})+qX(q^{5})f(q^{15},q^{10})f(q^{20},q^{5})\\&+q^{2}X(q^{5})f^{2}(q^{20},q^{5})+2q^{3}\psi(q^{5})X(q^{25})f(q^{15},q^{10})\\&+2q^{4}X(q^{5})\psi(q^{25})f(q^{20},q^{5}),\end{aligned} \tag{47}$$

Wu and Sun [25] showed that the tuple $(15,5,4,2,4,2)$ is universal. Thus, the tuple $(15,5,5,3,5,1)$ is also universal with the aid of (15). Therefore, in light of the last two terms of the identity (47) and Theorem 1.1(i), we see that the tuples $(20,10,5,1,3,1)$ and $(20,10,5,3,3,1)$ are universal.

5. **Proof of Theorem 1.3.**

(i) Kane and Sun [16] showed that the sum

$$x^{2}+\frac{y(y+1)}{2}+9\frac{z(z+1)}{2}$$

is almost universal. Note that

$$x^{2}+\frac{y(y+1)}{2}+9\frac{z(z+1)}{2}\sim 9x(2x+1)+y(3y+1)+\frac{z(3z+1)}{2}$$

by (12) and (2). So, the tuple $(36,18,6,2,3,1)$ is almost universal.

Let $|q|<1$. Multiplying both sides of (28) by $\psi(q^9)=f(q^9,q^{27})$, we find that

$$\begin{aligned}&f(q^9,q^{27})f(q,q^2)f(q^2,q^4)\\ &=f(q^9,q^9)f(q^9,q^{27})f(q^3,q^6)+qf(q^9,q^{27})f(q^3,q^6)f(q^3,q^{15})\\ &\quad+2q^2f(q^9,q^{27})^2f(q^3,q^{15}).\end{aligned}\tag{48}$$

As the tuple $(36,18,6,2,3,1)$ is almost universal, applying Theorem 1.1(ii) to the identity (48) and recalling (2), we find that the sums

$$3x^2+3\frac{y(y+1)}{2}+\frac{z(3z+1)}{2}\quad\text{and}\quad 3\frac{x(x+1)}{2}+\frac{y(3y+1)}{2}+z(3z+2),$$

are almost universal in view of the first and second terms of the right-hand side of (48).

(ii) Kane and Sun [16] showed that the sums

$$x^2+6y(y+1)+\frac{z(z+1)}{2}\ \text{ and }\ x^2+6y^2+\frac{z(z+1)}{2}$$

are almost universal. By (12) and (2),

$$x^2+6y(y+1)+\frac{z(z+1)}{2}\sim 12x(2x+1)+y(3y+1)+\frac{z(3z+1)}{2}$$

and

$$x^2+6y^2+\frac{z(z+1)}{2}\sim 6x^2+y(3y+1)+\frac{z(3z+1)}{2}.$$

So, the tuples $(24,12,6,2,3,1)$ and $(12,0,6,2,3,1)$ are almost universal.

Let $|q|<1$. Multiplying both sides of (28) by $\psi(q^{12})=f(q^{12},q^{36})$, we obtain the identity

$$\begin{aligned}&f(q^{12},q^{36})f(q,q^2)f(q^2,q^4)\\ &=f(q^9,q^9)f(q^{12},q^{36})f(q^3,q^6)+qf(q^{12},q^{36})f(q^3,q^6)f(q^3,q^{15})\\ &\quad+2q^2f(q^9,q^{27})f(q^{12},q^{36})f(q^3,q^{15}).\end{aligned}\tag{49}$$

As the tuple $(24,12,6,2,3,1)$ is almost universal, applying Theorem 1.1(ii) to the identity (49) and recalling (2), we see that the sums

$$3x^2+2y(y+1)+\frac{z(3z+1)}{2}\quad\text{and}\quad 2x(x+1)+\frac{y(3y+1)}{2}+z(3z+2)$$

are almost universal in light of the first and second terms of the right-hand side of (49). Similarly, as the tuple $(12,0,6,2,3,1)$ is almost universal, applying Theorem 1.1(ii) to the identity (28) multiplied by $\varphi(q^6)=f(q^6,q^6)$, we deduce that the sums

$$2x^2+3y^2+\frac{z(3z+1)}{2}\quad\text{and}\quad 2x^2+\frac{y(3y+1)}{2}+z(3z+2)$$

are both almost universal.

(iii) Kane and Sun [16] proved that the sums

$$3x(x+1)+y(y+1)+\frac{z(z+1)}{2}\ \text{ and }\ 6x^2+y(y+1)+\frac{z(z+1)}{2}$$

are almost universal. By (13) and (2), we have

$$3x(x+1)+y(y+1)+\frac{z(z+1)}{2}\sim 6x(2x+1)+\frac{y(3y+1)}{2}+z(3z+2)$$

and

$$6x^2+y(y+1)+\frac{z(z+1)}{2}\sim 6x^2+\frac{y(3y+1)}{2}+z(3z+2).$$

Thus, the tuples $(24, 12, 6, 4, 3, 1)$ and $(12, 0, 6, 4, 3, 1)$ are almost universal.

Let $|q| < 1$. Multiplying both sides of (29) by $\psi(q^6) = f(q^6, q^{18})$, we get the identity

$$\begin{aligned} &f(q^6, q^{18}) f(q, q^2) f(q, q^5) \\ &= f(q^6, q^{18}) f(q^3, q^6) f(q^6, q^{12}) + 2q f(q^6, q^{18}) f(q^9, q^{27}) f(q^6, q^{12}) \\ &\quad + 2q^2 f(q^6, q^{18}) f(q^{18}, q^{54}) f(q^3, q^6). \end{aligned} \tag{50}$$

As the tuple $(24, 12, 6, 4, 3, 1)$ is universal, applying Theorem 1.1(ii) to the identity (50) and recalling (2), we find that the sums

$$x(x+1) + y(3y+1) + \frac{z(3z+1)}{2} \quad \text{and} \quad x(x+1) + 3y(y+1) + \frac{z(3z+1)}{2}$$

are almost universal in view of the first and third terms of the right-hand side of (50). Similarly, as the tuple $(12, 0, 6, 4, 3, 1)$ is almost universal, applying Theorem 1.1(ii) to the identity (29) multiplied by $\varphi(q^6) = f(q^6, q^6)$ we deduce that the sums

$$2x^2 + y(3y+1) + \frac{z(3z+1)}{2} \quad \text{and} \quad 2x^2 + 3y(y+1) + \frac{z(3z+1)}{2}$$

are both almost universal.

In view of the above, we have completed the proof of Theorem 1.3. □

Acknowledgments. The authors would like to thank the anonymous referee for helpful comments.

REFERENCES

[1] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, Chapter 16 of Ramanujan's second notebook: Theta functions and q-series, *Mem. Amer. Math. Soc.*, **53** (1985), 85 pp.

[2] C. Adiga and N. A. S. Bulkhali, Identities for certain products of theta functions with applications to modular relations, *J. Anal. Number Theory*, **2** (2014), 1-15.

[3] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Stud. Math. Libr., 34, American Mathematical Society, Providence, RI, 2006.

[4] Z. Cao, Integer matrix exact covering systems and product identities for theta functions, *Int. Math. Res. Notices*, **2011** (2011), 4471-4514.

[5] W. K. Chan and B.-K. Oh, Almost universal ternary sums of triangular numbers, *Proc. Amer. Math. Soc.*, **137** (2009), 3553-3562.

[6] J. H. Conway, R. K. Guy, W. A. Schneeberger and N. J. A. Sloane, The primary pretenders, *Acta Arith.*, **78** (1997), 307-313.

[7] L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939.

[8] L. E. Dickson, *History of the Theory of Numbers, Vol. II: Diophantine Analysis*, Chelsea Publishing Co., New York, 1966.

[9] W. Duke and R. Schulze-Pillot, Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids, *Invent. Math.*, **99** (1990), 49-57.

[10] F. Ge and Z.-W. Sun, On some universal sums of generalized polygonal numbers, *Colloq. Math.*, **145** (2016), 149-155.

[11] S. Guo, H. Pan and Z.-W. Sun, Mixed sums of squares and triangular numbers, (II), *Integers*, **7** (2007), A56, 5pp (electronic).

[12] R. K. Guy, Every number is expressible as the sum of how many polygonal numbers? *Amer. Math. Monthly*, **101** (1994), 169-172.

[13] J. Ju, Almost universal sums of triangular numbers with one exception, *J. Korean Math. Soc.*, **60** (2023), 931-957.

[14] J. Ju and B.-K. Oh, A generalization of Gauss' triangular theorem, *Bull. Korean Math. Soc.*, **55** (2018), 1149-1159.

[15] J. Ju, B.-K. Oh and B. Seo, Ternary universal sums of generalized polygonal numbers, *Int. J. Number Theory*, **15** (2019), 655-675.

[16] B. Kane and Z.-W. Sun, On almost universal mixed sums of squares and triangular numbers, *Trans. Amer. Math. Soc.*, **362** (2010), 6425-6455.

[17] B.-K. Oh, Ternary universal sums of generalized pentagonal numbers, *J. Korean Math. Soc.*, **48** (2011), 837-847.

[18] B.-K. Oh and Z.-W. Sun, Mixed sums of squares and triangular numbers (III), *J. Number Theory*, **129** (2009), 964-969.

[19] Z.-W. Sun, Mixed sums of squares and triangular numbers, *Acta Arith.*, **127** (2007), 103-113.

[20] Z.-W. Sun, On universal sums of polygonal numbers, *Sci. China Math.*, **58** (2015), 1367-1396.

[21] Z.-W. Sun, A result similar to Lagrange's theorem, *J. Number Theory*, **162** (2016), 190-211.

[22] Z.-W. Sun, On $x(ax+1)+y(by+1)+z(cz+1)$ and $x(ax+b)+y(ay+c)+z(az+d)$, *J. Number Theory*, **171** (2017), 275-283.

[23] Z.-W. Sun, On universal sums $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$, *Nanjing Univ. J. Math. Biquarterly*, **35** (2018), 85-199.

[24] Z.-W. Sun, Universal sums of three quadratic polynomials, *Sci. China Math.*, **63** (2020), 501-520.

[25] H.-L. Wu and Z.-W. Sun, Some universal quadratic sums over the integers, *Electron. Res. Arch.*, **27** (2019), 69-87.

[26] H.-L. Wu and Z.-W. Sun, Arithmetic progressions represented by diagonal ternary quadratic forms, *Nanjing Univ. J. Math. Biquarterly*, **4**0 (2023), 54-71.

Received February 22, 2026; revised July 14, 2026.