J. Number Theory, in press.

UNDECIDABILITY ON DIOPHANTINE EQUATIONS OVER $\mathbb{Z}[i]$ WITH 20 UNKNOWNS

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Abstract. It is known that Hilbert's Tenth Problem over the Gaussian ring $\mathbb{Z}[i] = \{a + bi : \ a, b \in \mathbb{Z}\}$ is undecidable. In this paper we obtain the following further result: There is no algorithm to decide whether an arbitrarily given polynomial equation $P(z_1, \ldots, z_{20}) = 0$ (with integer coefficients and 20 unknowns) is solvable over $\mathbb{Z}[i]$. This improves the previous record involving 52 unknowns.

1. Introduction

Hilbert's Tenth Problem (HTP in short) over a ring R asks for an algorithm to test whether an arbitrary polynomial Diophantine equation

$$P(x_1, \ldots, x_n) = 0$$

with coefficients in R has solutions over R. This was posed in 1900 by D. Hilbert when R is the ring $\mathbb{Z}$ of integers. In 1961, M. Davis, H. Putnam and J. Robinson [3] proved that the solvability of exponential Diophantine equations over $\mathbb{N}$ is undecidable. The original HTP over $\mathbb{Z}$ was finally solved by Y. Matiyasevich [7] negatively in 1970, see also the book [8]. In this direction, the second author [14] proved further that there is no algorithm to decide for any given $P(x_1, \ldots, x_{11}) \in \mathbb{Z}[x_1, \ldots, x_{11}]$ whether the equation $P(x_1, \ldots, x_{11}) = 0$ has integer solutions, which is usually called *the 11 unknowns theorem*.

Let K be any number field (which is a finite extension of the field $\mathbb{Q}$ of rational numbers), and let O_K be the ring of algebraic integers in K. What about HTP over O_K? If $\mathbb{Z}$ is Diophantine over O_K, i.e., there is a polynomial $P(x_0, x_1, \ldots, x_n)$ over O_K such that $a \in O_K$ lies in $\mathbb{Z}$ if and only if $P(a, x_1, \ldots, x_n) = 0$ for some $x_1, \ldots, x_n \in O_K$, then HTP over O_K is undecidable via Matiyasevich's theorem. J. Denef [4, 5] proved

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2010 *Mathematics Subject Classification.* Primary 11U05, 03D35; Secondary 03D25, 11D99, 11R11.

Key words and phrases. Hilbert's Tenth Problem, Diophantine equation, Gaussian ring, imaginary quadratic field, undecidability.

The second author is supported by the Natural Science Foundation of China (grant no. 12371004).

that if K is an imaginary quadratic field or a totally real field then $\mathbb{Z}$ is Diophantine over O_K and hence HTP over O_K is unsolvable (see also [12, pp. 98-100]). Recently, P. Koymans and C. Pagano [6], as well as L. Alpöge, M. Bhargava, W. Ho and A. Shnidman [1] successfully proved that O_K is always Diophantine over $\mathbb{Z}$ and hence HTP over O_K is unsolvable.

In view of the second author's 11 unknowns theorem for the ring $\mathbb{Z}$ of integers, it is natural to ask for similar undecidable results involving few unknowns over the ring O_K of algebraic integers in any number field K. To achieve this goal is a hard task which needs long-term study.

Surely, $K = \mathbb{Q}(i) = \{r + si : \ r, s \in \mathbb{Q}\}$ is the most famous number field different from $\mathbb{Q}$, and its ring O_K is the Gaussian ring $\mathbb{Z}[i] = \{a + bi : \ a, b \in \mathbb{Z}\}$. The authors [10] proved that there is no algorithm to decide whether for any $P(x_1, \ldots, x_{52}) \in \mathbb{Z}[x_1, \ldots, x_{52}]$ the equation $P(x_1, \ldots, x_{52}) = 0$ is solvable over $\mathbb{Z}[i]$. In this paper we make further progress by establishing the following result.

Theorem 1.1. *There is no algorithm to decide for any* $P(z_1, \ldots, z_{20}) \in \mathbb{Z}[z_1, \ldots, z_{20}]$ *whether the diophantine equation*

$$P(z_1, \ldots, z_{20}) = 0$$

with 20 *unknowns has solutions with* $z_1, \ldots, z_{20} \in \mathbb{Z}[i]$.

The following auxiliary theorem plays an important role in our proof of Theorem 1.1.

Theorem 1.2. *Let d be a squarefree positive integer. Let K be the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and let O_K be the ring of algebraic integers in K. Let $x_1, \ldots, x_n \in O_K$ and set $y = 2\prod_{k=1}^{n}(3x_k + 1)$. Then we have*

$$y + \sum_{k=1}^{n} \frac{x_k}{y^k} \in \mathbb{Q} \iff x_1, \ldots, x_n \in \mathbb{Z}.$$

We are going to prove Theorem 1.2 and Theorem 1.1 in Sections 2 and 3, respectively. Although we have Theorem 1.2 for a general imaginary quadratic field, our proof of Theorem 1.1 depends heavily on a particular result for $\mathbb{Z}[i]$ (cf. Lemma 3.3 in Section 3) established in [10] via using some special properties of $\mathbb{Z}[i]$. To obtain undecidable results similar to Theorem 1.1 for all other quadratic fields is our goal in the future, which needs further studies and many technical things.

In 1934 T. Skolem [13] used a simple trick to reduce any polynomial diophantine equation over $\mathbb{Z}$ to one with degree 4. The basic idea is to introduce more unknowns to lower the degree of the equation. For example, $y = x^5$ for some $x \in \mathbb{Z}$ if and only if there are $u, v, w, x \in \mathbb{Z}$ satisfying the fourth equation

$$(y - ux)^2 + (u - vx)^2 + (v - wx)^2 + (w - x^2)^2 = 0.$$

By the same trick and Lemma 3.2, we can reduce any polynomial diophantine equation over $\mathbb{Z}[i]$ to one with degree 4. Thus the solvability of a general polynomial equation of degree four over $\mathbb{Z}[i]$ is undecidable.

In 1972 C. L. Siegel [11] proved that the solvability of a general quadratic equation (with integer coefficients) over $\mathbb{Z}$ is decidable! Now it is natural to ask the following question which might have a negative answer.

Question 1.1. *Whether the solvability of a general quadratic equation over $\mathbb{Z}[i]$ is decidable?*

2. Proof of Theorem 1.2

Let d be a squarefree positive integer, and set $K = \mathbb{Q}(\sqrt{-d})$. For $\alpha \in K$, we write $\alpha = r(\alpha) + s(\alpha)\sqrt{-d}$ with $r(\alpha), s(\alpha) \in \mathbb{Q}$, and the norm of α is given by $N(\alpha) = r(\alpha)^2 + d\,s(\alpha)^2$. It is well known that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in K$. Note that

$$|s(\alpha)| \leqslant \frac{N(\alpha)}{\sqrt{d}} \qquad \text{for any } \alpha \in K.$$

It is well known that

$$O_K = \begin{cases} \{a + b\sqrt{-d} :\ a, b \in \mathbb{Z}\} & \text{if } d \not\equiv -1 \pmod 4, \\ \{\frac{a+b\sqrt{-d}}{2} :\ a, b \in \mathbb{Z} \text{ and } 2 \mid a - b\} & \text{if } d \equiv -1 \pmod 4. \end{cases}$$

For $x \in O_K$, we clearly have $N(x) \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and $N(x) = 0$ if and only if $x = 0$.

Lemma 2.1. *For any $x \in O_K$ we have $N(3x + 1) \geqslant N(x)$.*

Proof. Write $x = (a + b\sqrt{-d})/2$ with $a, b \in \mathbb{Z}$ and $a \equiv b \pmod 2$. Then

$$N(3x+1) = N\left(\frac{3a + 2 + 3b\sqrt{-d}}{2}\right) = \frac{(3a + 2)^2 + d(9b^2)}{4} \geqslant \frac{(3a + 2)^2 + db^2}{4}.$$

So it suffices to prove the inequality $(3a + 2)^2 \geqslant a^2$. Note that

$$(3a + 2)^2 - a^2 = 4(a + 1)(2a + 1) \geqslant 0$$

since $a \notin (-1/2, -1)$. This ends our proof. $\square$

Lemma 2.2. *Let $x_1, \ldots, x_n \in O_K$ and $y = 2x_0 \prod_{k=1}^{n}(3x_k + 1)$ with $x_0 \in O_K \setminus \{0\}$. Let $z \in O_K$ with $z + \sum_{k=1}^{n} x_k/y^k \in \mathbb{Q}$. Then $z \in \mathbb{Z}$.*

Proof. Let $Z=\sum_{k=1}^n x_k/y^k$. As $z+Z\in\mathbb{Q}$, we have $s(z)=-s(Z)$. Note that $N(x_0)\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. For each $k=1,\ldots,n$, clearly $N(3x_k+1)\in\mathbb{Z}^+$ and

$$1\leqslant N(3x_k+1)\leqslant N(x_0)\prod_{j=1}^n N(3x_j+1)=N\left(\frac{y}{2}\right)=\frac{N(y)}{4},$$

hence $N(x_k)\leqslant N(y)/4$ by Lemma 2.1. Observe that

$$|s(z)|=|s(Z)|\leqslant\sum_{k=1}^n\left|s\left(\frac{x_k}{y^k}\right)\right|\leqslant\sum_{k=1}^n\sqrt{\frac{N(x_k/y^k)}{d}}=\frac{1}{\sqrt{d}}\sum_{k=1}^n\sqrt{\frac{N(x_k)}{N(y)^k}}\tag{2.1}$$

and thus

$$\sqrt{d}\,|s(z)|\leqslant\sum_{k=1}^n\frac{1}{2\sqrt{N(y)}^{k-1}}\leqslant\sum_{k=1}^n\frac{1}{2^k}<\sum_{k=1}^\infty\frac{1}{2^k}=1.\tag{2.2}$$

Note that $z\in O_K$ and $2s(z)\in\mathbb{Z}$. We want to show $z\in\mathbb{Z}=\mathbb{Q}\cap O_K$ (i.e., $s(z)=0$). If $d\geqslant 5$, then by (2.2) we have

$$2|s(z)|\leqslant\sqrt{d}\,|s(z)|<1$$

and hence $s(z)=0$. For $d\in\{1,2\}$, by (2.2) we have

$$|s(z)|\leqslant\sqrt{d}\,|s(z)|<1$$

and hence $s(z)=0$ since $s(z)\in\mathbb{Z}$.

Now we handle the case $d=3$. Without loss of generality, we may simply assume that $x_1,\ldots,x_n$ are nonzero. Let ω denote the cubic root $(-1+\sqrt{-3})/2$ of unity. Then $\pm1,\pm\omega,\pm\omega^2$ are the only units in $O_K=\{a+b\omega:\ a,b\in\mathbb{Z}\}$. For $\alpha=a+b\omega\neq0$ with $a,b\in\mathbb{Z}$, clearly $3\alpha+1=3a+1+3b\omega\notin\{\pm1,\pm\omega,\pm\omega^2\}$ and hence $N(3\alpha+1)\neq1$. Since $x_1,\ldots,x_n\in O_K\setminus\{0\}$, we have $N(3x_k+1)\geqslant2$ for all $k=1,\ldots,n$. Thus, for each $1\leqslant k\leqslant n$, we have

$$N\left(\frac{y}{2}\right)=N(x_0)N(3x_k+1)\prod_{\substack{j=1\\ j\neq k}}^n N(3x_j+1)\geqslant2^{k-1}N(3x_k+1)\geqslant2^{k-1}N(x_k)$$

and hence $N(y)\geqslant2^{k+1}N(x_k)\geqslant2^{k+1}$. Combining this with (2.1), we obtain that

$$\begin{aligned}\sqrt{3}\,|s(z)|&\leqslant\sum_{k=1}^n\sqrt{\frac{N(x_k)}{N(y)^k}}\leqslant\sum_{k=1}^n\sqrt{\frac{N(y)/2^{k+1}}{N(y)^k}}=\frac12\sum_{k=1}^n\frac{1}{\sqrt{2N(y)}^{k-1}}\\&\leqslant\frac12\sum_{k=1}^n\frac{1}{2^{(k+2)(k-1)/2}}<\frac12\left(1+\frac{1}{2^2}+\frac{1}{2^5}+\sum_{k=4}^\infty\frac{1}{2^{k-1}}\right)=\frac{49}{64}\end{aligned}$$

and hence

$$|s(z)| < \frac{49}{64\sqrt{3}} < \frac{1}{2}.$$

Thus $s(z) = 0$ since $2s(z) \in \mathbb{Z}$.

In view of the above, we have completed the proof of Lemma 2.2. $\square$

`Proof of Theorem 1.2.` Set $Y = \sum_{k=1}^{n} x_k/y^k$.

If $x_1, \ldots, x_n \in \mathbb{Z}$, then $y = 2\prod_{k=1}^{n}(3x_k+1) \in \mathbb{Z}$ and hence $y+Y \in \mathbb{Q}$.

Now suppose $y + Y \in \mathbb{Q}$. By Lemma 2.2, $y \in \mathbb{Z}$ and thus $Y \in \mathbb{Q}$. We want to use induction to show that $x_m \in \mathbb{Z}$ for all $m = 1, \ldots, n$. Fix $m \in \{1, \ldots, n\}$, and assume that $x_k \in \mathbb{Z}$ for all $0 < k < m$. Then

$$\sum_{k=m}^{n} \frac{x_k}{y^k} = Y - \sum_{0<k<m} \frac{x_k}{y^k} \in \mathbb{Q}$$

and hence

$$x_m + \sum_{m<k\leqslant n} \frac{x_k}{y^{k-m}} = y^m \sum_{k=m}^{n} \frac{x_k}{y^k} \in \mathbb{Q}. \tag{2.3}$$

Observe that $y = 2x_0 \prod_{m<k\leqslant n}(3x_k + 1)$ with $x_0 = \prod_{1\leqslant k\leqslant m}(3x_k + 1) \in O_K \setminus \{0\}$. In view of (2.3), by applying Lemma 2.2 we obtain $x_m \in \mathbb{Z}$. This concludes the induction step.

By the above, we have completed the proof of Theorem 1.2. $\square$

3. Proof of Theorem 1.1

Motivated by the Matiyasevich–Robinson Relation-Combining Theorem (cf. [9]), the second author [15, p. 69] deduced the following lemma for the rational field $\mathbb{Q}$. We now extend it to any number field.

Lemma 3.1. *Let K be a number field, and let $A_1, A_2, S, T \in O_K$ with $A_1 \neq A_2$ and $S \neq 0$. Then*

$$A_1 \in \square \wedge A_2 \in \square \wedge S \mid T$$

if and only if for some $m \in O_K$ we have

$$f(A_1, A_2, S, T, m) = 0 \tag{3.1}$$

where $\square = \{\alpha^2 : \ \alpha \in O_K\}$ and

$$f(A_1, A_2, S, T, m) := (T-mS)^4-2(A_1+A_2)S^2(T-mS)^2+(A_1-A_2)^2S^4.$$

Proof. Observe that

$$\begin{aligned} &S^{-4}f(A_1,A_2,S,T,m)\\ =&\left(\left(\frac{T}{S}-m\right)^2-(A_1+A_2)\right)^2-4A_1A_2\\ =&\left(\left(\frac{T}{S}-m\right)^2+A_1-A_2\right)^2-4A_1\left(\frac{T}{S}-m\right)^2. \end{aligned}$$

If $A_1=\alpha_1^2$ and $A_2=\alpha_2^2$ with $\alpha_1,\alpha_2\in O_K$, and $S\mid T$, then $m=T/S-\alpha_1-\alpha_2\in O_K$ and

$$\begin{aligned} &\left(\left(\frac{T}{S}-m\right)^2-(A_1+A_2)\right)^2\\ =&((\alpha_1+\alpha_2)^2-\alpha_1^2-\alpha_2^2)^2=(2\alpha_1\alpha_2)^2=4A_1A_2, \end{aligned}$$

thus $f(A_1,A_2,S,T,m)=0$.

Now assume that (3.1) holds for some $m\in O_K$. Set $x=T/S-m$. As $(x^2+A_1-A_2)^2-4A_1x^2=0$, x is an algebraic integer and hence $x\in O_K$. Thus $S\mid T$. Since $A_1\neq A_2$, we have $x\neq 0$. For $y=(A_1-A_2)/x\in K$, we have $(x+y)^2=4A_1$ and hence y is an algebraic integer. As $x,y\in O_K$ and $(\frac{x+y}{2})^2=A_1\in O_K$, $\frac{x+y}{2}$ must be an algebraic integer. Note that

$$A_2=A_1-xy=\left(\frac{x+y}{2}\right)^2-xy=\left(\frac{x-y}{2}\right)^2=\left(\frac{x+y}{2}-y\right)^2.$$

So $A_1,A_2\in\square$.

By the above, we have finished the proof of Lemma 3.1. $\square$

The following lemma is an easy fact, see, e.g., [10, Lemma 2.4].

Lemma 3.2. *For any $x,y\in\mathbb{Z}[i]$, we have*

$$x=0\wedge y=0\iff x^2+2y^2=0.$$

Lemma 3.3. *A number $t\in\mathbb{Z}[i]$ is a rational integer if and only if there are $v,x,y\in\mathbb{Z}[i]$ with $v\neq 0$ such that*

$$4(2v(2(2t+1)^2+1)-y)^2-3y^2-1=0 \tag{3.2}$$

and

$$3y^2(2t+1-xy)^2+1\in\square, \tag{3.3}$$

where $\square=\{\alpha^2:\ \alpha\in\mathbb{Z}[i]\}$. When $t\in\mathbb{Z}$, we can actually require further that $v,x,y\in\mathbb{Z}$.

Remark 3.1. Lemma 3.3 follows from [10, Theorem 1.1] and its proof in view of Lemma 3.2.

We also need the following result observed by S. P. Tung [16],

Lemma 3.4. *An integer m is nonzero if and only if $m = (2r+1)(3s+1)$ for some $r, s \in \mathbb{Z}$.*

`Proof of Theorem 1.1`. Let $\mathcal{A}$ be a subset of $\mathbb{N}$ which is recursively enumerable but not recursive. (The existence of such a set is well known, see, e.g., N. Cutland [2, pp. 140-141].) By Sun [14, Theorem 1.1(ii)], there is a polynomial $Q(z_0, \dots, z_{10}) \in \mathbb{Z}[z_0, \dots, z_{10}]$ such that $a \in \mathbb{N}$ belongs to $\mathcal{A}$ if and only if

$$Q(a, z_1, \dots, z_{10}) = 0 \tag{3.4}$$

for some $z_1, \dots, z_{10} \in \mathbb{Z}$ with $z_{10} \neq 0$.

Let $y = 2\prod_{k=1}^{10}(3z_k + 1)$. By Theorem 1.1, when $z_1, \dots, z_{10} \in \mathbb{Z}[i]$, we have

$$\begin{aligned}
& z_1, \dots, z_{10} \in \mathbb{Z} \\
\iff\ & y + \sum_{k=1}^{10} \frac{z_k}{y^k} \in \mathbb{Q} \\
\iff\ & \exists t \in \mathbb{Z} \left[t \neq 0 \wedge t\left(y + \sum_{k=1}^{10} \frac{z_k}{y^k} \right) \in \mathbb{Z} \right] \\
\iff\ & \exists t \in \mathbb{Z} \left[t \neq 0 \wedge ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k} \wedge t\left(y + \sum_{k=1}^{10} \frac{z_k}{y^k} \right) \in \mathbb{Z} \right].
\end{aligned}$$

Let $\square = \{\alpha^2 : \alpha \in \mathbb{Z}[i]\}$. Suppose that $t \in \mathbb{Z}[i]$ and $ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k}$. By Lemma 3.3, $t \in \mathbb{Z}$ if and only there are $v_0, x_0, y_0 \in \mathbb{Z}[i]$ with $v_0 \neq 0$ such that

$$4(2v_0(2(2t+1)^2 + 1) - y_0)^2 - 3y_0^2 - 1 = 0 \tag{3.5}$$

and

$$A_1 := 9(3y_0^2(2t + 1 - x_0 y_0)^2 + 1) \in \square, \tag{3.6}$$

and $t(y + \sum_{k=1}^{10} z_k/y^k) \in \mathbb{Z}$ if and only if there are $v_1, x_1, y_1 \in \mathbb{Z}[i]$ with $v_1 \neq 0$ such that

$$4(2v_1(2(2t(y^{11} + z_1 y^9 + \cdots + z_{10} y^0) + y^{10})^2 + y^{20}) - y_1 y^{20})^2 = (3y_1^2 + 1)y^{40} \tag{3.7}$$

and

$$A_2 := 3y_1^2(2t(y^{11} + z_1 y^9 + \ldots + z_{10} y^0) + y^{10} - x_1 y_1 y^{10})^2 + y^{20} \in \square. \tag{3.8}$$

Moreover, when $z_1, \dots, z_{10}, t \in \mathbb{Z}$, we can require further that $v_0, x_0, y_0 \in \mathbb{Z}$, and also that $v_1, x_1, y_1 \in \mathbb{Z}$ if $t(y + \sum_{k=1}^{10} z_k/y^k) \in \mathbb{Z}$. Note that $A_1 \neq A_2$ since $A_1 \equiv 0 \pmod 3$ and $A_2 \equiv y^{20} \equiv 2^{20} \equiv 1 \pmod 3$.

Let $a \in \mathbb{N}$. By the above, $a \in \mathcal{A}$ if and only if there are

$$z_1, \ldots, z_{10}, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i]$$

with $z_{10}tv_0v_1 \neq 0$ such that (3.4), (3.5) and (3.7) all hold and also

$$A_1 \in \square \wedge A_2 \in \square \wedge ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k}.$$

Note that when $a \in A$ we can actually find

$$z_1, \ldots, z_{10}, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}$$

to meet the requirement. When $z_{10}tv_0v_1 \in \mathbb{Z} \setminus \{0\}$, there are $r, s \in \mathbb{Z}$ such that $z_{10}tv_0v_1 = (2r+1)(3s+1)$. If the last equality holds for some $r, s \in \mathbb{Z}[i]$, then we obviously have $z_{10}tv_0v_1 \neq 0$. Thus, in view of Lemma 3.1, $a \in \mathcal{A}$ if and only if there are

$$z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i]$$

such that (3.4), (3.5), (3.7), and the equalities

$$z_{10}tv_0v_1 = (2r+1)(3s+1)$$

and

$$f\left(A_1, A_2, ty^{10}, \sum_{k=1}^{10} z_k y^{10-k}, m\right) = 0$$

all hold. Therefore, by using Lemma 3.2 we see that $a \in \mathcal{A}$ if and only if

$$F(a, z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1) = 0$$

for some

$$z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i],$$

where F is a suitable polynomial with integer coefficients in 21 variables.

As $\mathcal{A}$ is not recursive, by the conclusion in the last paragraph we do have the desired result. This completes our proof of Theorem 1.2. $\square$

Acknowledgment. The authors are indebted to the anonymous referee for helpful comments.

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