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UNDECIDABILITY ON DIOPHANTINE EQUATIONS OVER $\mathbb{Z}[i]$ WITH 20 UNKNOWNS

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Abstract. It is known that Hilbert's Tenth Problem over the Gaussian ring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is undecidable. In this paper we obtain the following further result: There is no algorithm to decide whether an arbitrarily given polynomial equation $P(z_1, \ldots, z_{20}) = 0$ (with integer coefficients and 20 unknowns) is solvable over $\mathbb{Z}[i]$. This improves the previous record involving 52 unknowns.

1. Introduction

Hilbert's Tenth Problem (HTP in short) over a ring R asks for an algorithm to test whether an arbitrary polynomial Diophantine equation

$$P(x_1, \ldots, x_n) = 0$$

with coefficients in R has solutions over R. This was posed in 1900 by D. Hilbert when R is the ring $\mathbb{Z}$ of integers. In 1961, M. Davis, H. Putnam and J. Robinson [3] proved that the solvability of exponential Diophantine equations over $\mathbb{N}$ is undecidable. The original HTP over $\mathbb{Z}$ was finally solved by Y. Matiyasevich [7] negatively in 1970, see also the book [8]. In this direction, the second author [14] proved further that there is no algorithm to decide for any given $P(x_1, \ldots, x_{11}) \in \mathbb{Z}[x_1, \ldots, x_{11}]$ whether the equation $P(x_1, \ldots, x_{11}) = 0$ has integer solutions, which is usually called *the 11 unknowns theorem*.

Let K be any number field (which is a finite extension of the field $\mathbb{Q}$ of rational numbers), and let O_K be the ring of algebraic integers in K. What about HTP over O_K? If $\mathbb{Z}$ is Diophantine over O_K, i.e., there is a polynomial $P(x_0, x_1, \ldots, x_n)$ over O_K such that $a \in O_K$ lies in $\mathbb{Z}$ if and only if $P(a, x_1, \ldots, x_n) = 0$ for some $x_1, \ldots, x_n \in O_K$, then HTP over O_K is undecidable via Matiyasevich's theorem. J. Denef [4, 5] proved

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that if K is an imaginary quadratic field or a totally real field then $\mathbb{Z}$ is Diophantine over O_K and hence HTP over O_K is unsolvable (see also [12, pp. 98-100]). Recently, P. Koymans and C. Pagano [6], as well as L. Alpöge, M. Bhargava, W. Ho and A. Shnidman [1] successfully proved that O_K is always Diophantine over $\mathbb{Z}$ and hence HTP over O_K is unsolvable.

In view of the second author's 11 unknowns theorem for the ring $\mathbb{Z}$ of integers, it is natural to ask for similar undecidable results involving few unknowns over the ring O_K of algebraic integers in any number field K. To achieve this goal is a hard task which needs long-term study.

Surely, $K=\mathbb{Q}(i)=\{r+si:\ r,s\in\mathbb{Q}\}$ is the most famous number field different from $\mathbb{Q}$, and its ring O_K is the Gaussian ring $\mathbb{Z}[i]=\{a+bi:\ a,b\in\mathbb{Z}\}$. The authors [10] proved that there is no algorithm to decide whether for any $P(x_1,\ldots,x_{52})\in\mathbb{Z}[x_1,\ldots,x_{52}]$ the equation $P(x_1,\ldots,x_{52})=0$ is solvable over $\mathbb{Z}[i]$. In this paper we make further progress by establishing the following result.

Theorem 1.1. *There is no algorithm to decide for any $P(z_1,\ldots,z_{20})\in\mathbb{Z}[z_1,\ldots,z_{20}]$ whether the diophantine equation*

$$P(z_1,\ldots,z_{20})=0$$

with 20 unknowns has solutions with $z_1,\ldots,z_{20}\in\mathbb{Z}[i]$.

The following auxiliary theorem plays an important role in our proof of Theorem 1.1.

Theorem 1.2. *Let d be a squarefree positive integer. Let K be the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and let O_K be the ring of algebraic integers in K. Let $x_1,\ldots,x_n\in O_K$ and set $y=2\prod_{k=1}^n(3x_k+1)$. Then we have*

$$y+\sum_{k=1}^n\frac{x_k}{y^k}\in\mathbb{Q}\iff x_1,\ldots,x_n\in\mathbb{Z}.$$

We are going to prove Theorem 1.2 and Theorem 1.1 in Sections 2 and 3, respectively. Although we have Theorem 1.2 for a general imaginary quadratic field, our proof of Theorem 1.1 depends heavily on a particular result for $\mathbb{Z}[i]$ (cf. Lemma 3.3 in Section 3) established in [10] via using some special properties of $\mathbb{Z}[i]$. To obtain undecidable results similar to Theorem 1.1 for all other quadratic fields is our goal in the future, which needs further studies and many technical things.

In 1934 T. Skolem [13] used a simple trick to reduce any polynomial diophantine equation over $\mathbb{Z}$ to one with degree 4. The basic idea is to introduce more unknowns to lower the degree of the equation. For

example, $y = x^5$ for some $x \in \mathbb{Z}$ if and only if there are $u, v, w, x \in \mathbb{Z}$ satisfying the fourth equation

$$(y - ux)^2 + (u - vx)^2 + (v - wx)^2 + (w - x^2)^2 = 0.$$

By the same trick and Lemma 3.2, we can reduce any polynomial diophantine equation over $\mathbb{Z}[i]$ to one with degree 4. Thus the solvability of a general polynomial equation of degree four over $\mathbb{Z}[i]$ is undecidable.

In 1972 C. L. Siegel [11] proved that the solvability of a general quadratic equation (with integer coefficients) over $\mathbb{Z}$ is decidable! Now it is natural to ask the following question which might have a negative answer.

Question 1.1. *Whether the solvability of a general quadratic equation over $\mathbb{Z}[i]$ is decidable?*

2. Proof of Theorem 1.2

Let d be a squarefree positive integer, and set $K = \mathbb{Q}(\sqrt{-d})$. For $\alpha \in K$, we write $\alpha = r(\alpha) + s(\alpha)\sqrt{-d}$ with $r(\alpha), s(\alpha) \in \mathbb{Q}$, and the norm of α is given by $N(\alpha) = r(\alpha)^2 + d\, s(\alpha)^2$. It is well known that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in K$. Note that

$$|s(\alpha)| \leqslant \frac{N(\alpha)}{\sqrt{d}} \qquad \text{for any } \alpha \in K.$$

It is well known that

$$O_K = \begin{cases} \{a + b\sqrt{-d} : \ a, b \in \mathbb{Z}\} & \text{if } d \not\equiv -1 \pmod 4, \\ \{\frac{a+b\sqrt{-d}}{2} : \ a, b \in \mathbb{Z} \text{ and } 2 \mid a - b\} & \text{if } d \equiv -1 \pmod 4. \end{cases}$$

For $x \in O_K$, we clearly have $N(x) \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and $N(x) = 0$ if and only if $x = 0$.

Lemma 2.1. *For any $x \in O_K$ we have $N(3x + 1) \geqslant N(x)$.*

Proof. Write $x = (a+b\sqrt{-d})/2$ with $a, b \in \mathbb{Z}$ and $a \equiv b \pmod 2$. Then

$$N(3x+1) = N\left(\frac{3a + 2 + 3b\sqrt{-d}}{2}\right) = \frac{(3a + 2)^2 + d(9b^2)}{4} \geqslant \frac{(3a + 2)^2 + db^2}{4}.$$

So it suffices to prove the inequality $(3a + 2)^2 \geqslant a^2$. Note that

$$(3a + 2)^2 - a^2 = 4(a + 1)(2a + 1) \geqslant 0$$

since $a \notin (-1/2, -1)$. This ends our proof. □

Lemma 2.2. *Let $x_1, \ldots, x_n \in O_K$ and $y = 2x_0 \prod_{k=1}^n (3x_k + 1)$ with $x_0 \in O_K \setminus \{0\}$. Let $z \in O_K$ with $z + \sum_{k=1}^n x_k/y^k \in \mathbb{Q}$. Then $z \in \mathbb{Z}$.*

Proof. Let $Z=\sum_{k=1}^n x_k/y^k$. As $z+Z\in\mathbb{Q}$, we have $s(z)=-s(Z)$. Note that $N(x_0)\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. For each $k=1,\ldots,n$, clearly $N(3x_k+1)\in\mathbb{Z}^+$ and

$$1\leqslant N(3x_k+1)\leqslant N(x_0)\prod_{j=1}^n N(3x_j+1)=N\left(\frac{y}{2}\right)=\frac{N(y)}{4},$$

hence $N(x_k)\leqslant N(y)/4$ by Lemma 2.1. Observe that

$$|s(z)|=|s(Z)|\leqslant\sum_{k=1}^n\left|s\left(\frac{x_k}{y^k}\right)\right|\leqslant\sum_{k=1}^n\sqrt{\frac{N(x_k/y^k)}{d}}=\frac{1}{\sqrt{d}}\sum_{k=1}^n\sqrt{\frac{N(x_k)}{N(y)^k}}\tag{2.1}$$

and thus

$$\sqrt{d}\,|s(z)|\leqslant\sum_{k=1}^n\frac{1}{2\sqrt{N(y)}^{k-1}}\leqslant\sum_{k=1}^n\frac{1}{2^k}<\sum_{k=1}^\infty\frac{1}{2^k}=1.\tag{2.2}$$

Note that $z\in O_K$ and $2s(z)\in\mathbb{Z}$. We want to show $z\in\mathbb{Z}=\mathbb{Q}\cap O_K$ (i.e., $s(z)=0$). If $d\geqslant 5$, then by (2.2) we have

$$2|s(z)|\leqslant\sqrt{d}\,|s(z)|<1$$

and hence $s(z)=0$. For $d\in\{1,2\}$, by (2.2) we have

$$|s(z)|\leqslant\sqrt{d}\,|s(z)|<1$$

and hence $s(z)=0$ since $s(z)\in\mathbb{Z}$.

Now we handle the case $d=3$. Without loss of generality, we may simply assume that $x_1,\ldots,x_n$ are nonzero. Let ω denote the cubic root $(-1+\sqrt{-3})/2$ of unity. Then $\pm1,\pm\omega,\pm\omega^2$ are the only units in $O_K=\{a+b\omega:\ a,b\in\mathbb{Z}\}$. For $\alpha=a+b\omega\neq0$ with $a,b\in\mathbb{Z}$, clearly $3\alpha+1=3a+1+3b\omega\notin\{\pm1,\pm\omega,\pm\omega^2\}$ and hence $N(3\alpha+1)\neq1$. Since $x_1,\ldots,x_n\in O_K\setminus\{0\}$, we have $N(3x_k+1)\geqslant 2$ for all $k=1,\ldots,n$. Thus, for each $1\leqslant k\leqslant n$, we have

$$N\left(\frac{y}{2}\right)=N(x_0)N(3x_k+1)\prod_{\substack{j=1\\j\neq k}}^n N(3x_j+1)\geqslant 2^{k-1}N(3x_k+1)\geqslant 2^{k-1}N(x_k)$$

and hence $N(y)\geqslant 2^{k+1}N(x_k)\geqslant 2^{k+1}$. Combining this with (2.1), we obtain that

$$\begin{aligned}\sqrt{3}\,|s(z)|&\leqslant\sum_{k=1}^n\sqrt{\frac{N(x_k)}{N(y)^k}}\leqslant\sum_{k=1}^n\sqrt{\frac{N(y)/2^{k+1}}{N(y)^k}}=\frac{1}{2}\sum_{k=1}^n\frac{1}{\sqrt{2N(y)}^{k-1}}\\&\leqslant\frac{1}{2}\sum_{k=1}^n\frac{1}{2^{(k+2)(k-1)/2}}<\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{2^5}+\sum_{k=4}^\infty\frac{1}{2^{k-1}}\right)=\frac{49}{64}\end{aligned}$$

and hence

$$|s(z)| < \frac{49}{64\sqrt{3}} < \frac{1}{2}.$$

Thus $s(z) = 0$ since $2s(z) \in \mathbb{Z}$.

In view of the above, we have completed the proof of Lemma 2.2. □

Proof of Theorem 1.2. Set $Y = \sum_{k=1}^{n} x_k/y^k$.

If $x_1, \ldots, x_n \in \mathbb{Z}$, then $y = 2\prod_{k=1}^{n}(3x_k+1) \in \mathbb{Z}$ and hence $y+Y \in \mathbb{Q}$.

Now suppose $y + Y \in \mathbb{Q}$. By Lemma 2.2, $y \in \mathbb{Z}$ and thus $Y \in \mathbb{Q}$. We want to use induction to show that $x_m \in \mathbb{Z}$ for all $m = 1, \ldots, n$. Fix $m \in \{1, \ldots, n\}$, and assume that $x_k \in \mathbb{Z}$ for all $0 < k < m$. Then

$$\sum_{k=m}^{n} \frac{x_k}{y^k} = Y - \sum_{0<k<m} \frac{x_k}{y^k} \in \mathbb{Q}$$

and hence

$$x_m + \sum_{m<k\leqslant n} \frac{x_k}{y^{k-m}} = y^m \sum_{k=m}^{n} \frac{x_k}{y^k} \in \mathbb{Q}. \tag{2.3}$$

Observe that $y = 2x_0 \prod_{m<k\leqslant n}(3x_k + 1)$ with $x_0 = \prod_{1\leqslant k\leqslant m}(3x_k + 1) \in O_K \setminus \{0\}$. In view of (2.3), by applying Lemma 2.2 we obtain $x_m \in \mathbb{Z}$. This concludes the induction step.

By the above, we have completed the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

Motivated by the Matiyasevich–Robinson Relation-Combining Theorem (cf. [9]), the second author [15, p. 69] deduced the following lemma for the rational field $\mathbb{Q}$. We now extend it to any number field.

Lemma 3.1. *Let K be a number field, and let $A_1, A_2, S, T \in O_K$ with $A_1 \neq A_2$ and $S \neq 0$. Then*

$$A_1 \in \square \wedge A_2 \in \square \wedge S \mid T$$

if and only if for some $m \in O_K$ we have

$$f(A_1, A_2, S, T, m) = 0 \tag{3.1}$$

where $\square = \{\alpha^2 : \alpha \in O_K\}$ and

$$f(A_1, A_2, S, T, m) := (T-mS)^4 - 2(A_1+A_2)S^2(T-mS)^2 + (A_1-A_2)^2S^4.$$

Proof. Observe that

$$\begin{aligned} & S^{-4}f(A_1,A_2,S,T,m)\\ =&\left(\left(\frac{T}{S}-m\right)^2-(A_1+A_2)\right)^2-4A_1A_2\\ =&\left(\left(\frac{T}{S}-m\right)^2+A_1-A_2\right)^2-4A_1\left(\frac{T}{S}-m\right)^2.\end{aligned}$$

If $A_1=\alpha_1^2$ and $A_2=\alpha_2^2$ with $\alpha_1,\alpha_2\in O_K$, and $S\mid T$, then $m=T/S-\alpha_1-\alpha_2\in O_K$ and

$$\begin{aligned}&\left(\left(\frac{T}{S}-m\right)^2-(A_1+A_2)\right)^2\\ =&((\alpha_1+\alpha_2)^2-\alpha_1^2-\alpha_2^2)^2=(2\alpha_1\alpha_2)^2=4A_1A_2,\end{aligned}$$

thus $f(A_1,A_2,S,T,m)=0$.

Now assume that (3.1) holds for some $m\in O_K$. Set $x=T/S-m$. As $(x^2+A_1-A_2)^2-4A_1x^2=0$, x is an algebraic integer and hence $x\in O_K$. Thus $S\mid T$. Since $A_1\neq A_2$, we have $x\neq 0$. For $y=(A_1-A_2)/x\in K$, we have $(x+y)^2=4A_1$ and hence y is an algebraic integer. As $x,y\in O_K$ and $(\frac{x+y}{2})^2=A_1\in O_K$, $\frac{x+y}{2}$ must be an algebraic integer. Note that

$$A_2=A_1-xy=\left(\frac{x+y}{2}\right)^2-xy=\left(\frac{x-y}{2}\right)^2=\left(\frac{x+y}{2}-y\right)^2.$$

So $A_1,A_2\in\square$.

By the above, we have finished the proof of Lemma 3.1. □

The following lemma is an easy fact, see, e.g., [10, Lemma 2.4].

Lemma 3.2. *For any $x,y\in\mathbb{Z}[i]$, we have*

$$x=0\wedge y=0\iff x^2+2y^2=0.$$

Lemma 3.3. *A number $t\in\mathbb{Z}[i]$ is a rational integer if and only if there are $v,x,y\in\mathbb{Z}[i]$ with $v\neq 0$ such that*

$$4(2v(2(2t+1)^2+1)-y)^2-3y^2-1=0 \tag{3.2}$$

and

$$3y^2(2t+1-xy)^2+1\in\square, \tag{3.3}$$

where $\square=\{\alpha^2:\ \alpha\in\mathbb{Z}[i]\}$. When $t\in\mathbb{Z}$, we can actually require further that $v,x,y\in\mathbb{Z}$.

Remark 3.1. Lemma 3.3 follows from [10, Theorem 1.1] and its proof in view of Lemma 3.2.

We also need the following result observed by S. P. Tung [16],

Lemma 3.4. *An integer m is nonzero if and only if $m = (2r+1)(3s+1)$ for some $r, s \in \mathbb{Z}$.*

`Proof of Theorem 1.1`. Let $\mathcal{A}$ be a subset of $\mathbb{N}$ which is recursively enumerable but not recursive. (The existence of such a set is well known, see, e.g., N. Cutland [2, pp. 140-141].) By Sun [14, Theorem 1.1(ii)], there is a polynomial $Q(z_0, \ldots, z_{10}) \in \mathbb{Z}[z_0, \ldots, z_{10}]$ such that $a \in \mathbb{N}$ belongs to $\mathcal{A}$ if and only if

$$Q(a, z_1, \ldots, z_{10}) = 0 \tag{3.4}$$

for some $z_1, \ldots, z_{10} \in \mathbb{Z}$ with $z_{10} \neq 0$.

Let $y = 2\prod_{k=1}^{10}(3z_k + 1)$. By Theorem 1.1, when $z_1, \ldots, z_{10} \in \mathbb{Z}[i]$, we have

$$\begin{aligned}
& z_1, \ldots, z_{10} \in \mathbb{Z} \\
\iff\ & y + \sum_{k=1}^{10} \frac{z_k}{y^k} \in \mathbb{Q} \\
\iff\ & \exists t \in \mathbb{Z}\ \left[t \neq 0 \wedge t\left(y + \sum_{k=1}^{10} \frac{z_k}{y^k} \right) \in \mathbb{Z} \right] \\
\iff\ & \exists t \in \mathbb{Z}\ \left[t \neq 0 \wedge ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k} \wedge t\left(y + \sum_{k=1}^{10} \frac{z_k}{y^k} \right) \in \mathbb{Z} \right].
\end{aligned}$$

Let $\square = \{\alpha^2 : \alpha \in \mathbb{Z}[i]\}$. Suppose that $t \in \mathbb{Z}[i]$ and $ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k}$. By Lemma 3.3, $t \in \mathbb{Z}$ if and only there are $v_0, x_0, y_0 \in \mathbb{Z}[i]$ with $v_0 \neq 0$ such that

$$4(2v_0(2(2t+1)^2+1) - y_0)^2 - 3y_0^2 - 1 = 0 \tag{3.5}$$

and

$$A_1 := 9(3y_0^2(2t + 1 - x_0y_0)^2 + 1) \in \square, \tag{3.6}$$

and $t(y + \sum_{k=1}^{10} z_k/y^k) \in \mathbb{Z}$ if and only if there are $v_1, x_1, y_1 \in \mathbb{Z}[i]$ with $v_1 \neq 0$ such that

$$4(2v_1(2(2t(y^{11} + z_1y^9 + \cdots + z_{10}y^0) + y^{10})^2 + y^{20}) - y_1y^{20})^2 = (3y_1^2+1)y^{40} \tag{3.7}$$

and

$$A_2 := 3y_1^2(2t(y^{11} + z_1y^9 + \ldots + z_{10}y^0) + y^{10} - x_1y_1y^{10})^2 + y^{20} \in \square. \tag{3.8}$$

Moreover, when $z_1, \ldots, z_{10}, t \in \mathbb{Z}$, we can require further that $v_0, x_0, y_0 \in \mathbb{Z}$, and also that $v_1, x_1, y_1 \in \mathbb{Z}$ if $t(y + \sum_{k=1}^{10} z_k/y^k) \in \mathbb{Z}$. Note that $A_1 \neq A_2$ since $A_1 \equiv 0 \pmod 3$ and $A_2 \equiv y^{20} \equiv 2^{20} \equiv 1 \pmod 3$.

Let $a \in \mathbb{N}$. By the above, $a \in \mathcal{A}$ if and only if there are

$$z_1, \ldots, z_{10}, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i]$$

with $z_{10}tv_0v_1 \neq 0$ such that (3.4), (3.5) and (3.7) all hold and also

$$A_1 \in \square \wedge A_2 \in \square \wedge ty^{10} \mid \sum_{k=1}^{10} z_k y^{10-k}.$$

Note that when $a \in A$ we can actually find

$$z_1, \ldots, z_{10}, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}$$

to meet the requirement. When $z_{10}tv_0v_1 \in \mathbb{Z} \setminus \{0\}$, there are $r, s \in \mathbb{Z}$ such that $z_{10}tv_0v_1 = (2r+1)(3s+1)$. If the last equality holds for some $r, s \in \mathbb{Z}[i]$, then we obviously have $z_{10}tv_0v_1 \neq 0$. Thus, in view of Lemma 3.1, $a \in \mathcal{A}$ if and only if there are

$$z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i]$$

such that (3.4), (3.5), (3.7), and the equalities

$$z_{10}tv_0v_1 = (2r+1)(3s+1)$$

and

$$f\left(A_1, A_2, ty^{10}, \sum_{k=1}^{10} z_k y^{10-k}, m\right) = 0$$

all hold. Therefore, by using Lemma 3.2 we see that $a \in \mathcal{A}$ if and only if

$$F(a, z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1) = 0$$

for some

$$z_1, \ldots, z_{10}, m, r, s, t, v_0, x_0, y_0, v_1, x_1, y_1 \in \mathbb{Z}[i],$$

where F is a suitable polynomial with integer coefficients in 21 variables.

As $\mathcal{A}$ is not recursive, by the conclusion in the last paragraph we do have the desired result. This completes our proof of Theorem 1.2. □

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