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Covering the integers by arithmetic sequences

by

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1. Introduction. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^+$ the set of positive reals. For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ we call

$$\alpha + \beta\mathbb{Z} = \{\ldots, \alpha - 2\beta, \alpha - \beta, \alpha, \alpha + \beta, \alpha + 2\beta, \ldots\}$$

an *arithmetic sequence with common difference* β. In the case $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^+$, $\alpha + \beta\mathbb{Z}$ is just the residue class α mod β with modulus β.

Let m be a positive integer. A finite system

$$(1) \qquad \mathcal{A} = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k \quad (\alpha_1, \ldots, \alpha_k \in \mathbb{R} \text{ and } \beta_1, \ldots, \beta_k \in \mathbb{R}^+)$$

of arithmetic sequences is said to be an (exact) *m-cover* of $\mathbb{Z}$ if it covers each integer at least (resp., exactly) m times. Instead of "1-cover" and "exact 1-cover" we use the terms "cover" and "exact cover" respectively.

Since they were introduced by P. Erdős ([5]) in the early 1930's, covers of $\mathbb{Z}$ by (finitely many) residue classes have been studied seriously and many nice applications have been found. (Cf. sections A19, B21, E23, F13 and F14 of R. K. Guy [9].) For problems and results in this area we refer the reader to surveys of Erdős [7, 8], Š. Porubský [13] and Š. Znám [21]. Recently further progress was made by various authors.

If a finite system

$$(2) \qquad A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k \quad (a_1, \ldots, a_k \in \mathbb{Z} \text{ and } n_1, \ldots, n_k \in \mathbb{Z}^+)$$

of residue classes forms an m-cover of $\mathbb{Z}$, then $\sum_{s=1}^k 1/n_s \geq m$, and the equality holds if and only if (2) is an exact m-cover of $\mathbb{Z}$. This becomes apparent if we calculate

$$\sum_{s=1}^k |\{0 \leq x < N : x \equiv a_s \pmod{n_s}\}| = \sum_{x=0}^{N-1} |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|$$

where N is the least common multiple of $n_1, \ldots, n_k$.

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In this paper we investigate properties of m-covers of $\mathbb{Z}$ in the form (1). In the next section we shall give three equivalent conditions for (1) to be an m-cover of $\mathbb{Z}$. One is that (1) covers W consecutive integers at least m times where

$$W = \left|\left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} : I \subseteq \{1,\ldots,k\}\right\}\right|$$

($[x]$ and $\{x\}$ stand for the integral and fractional parts of a real x respectively throughout the paper), the other two are finite systems of equalities (not inequalities) involving roots of unity. Our tools used to deduce them include Vandermonde determinants, Stirling numbers, a little analysis and linear algebra.

In Sections 3 and 4 we will derive a number of results including the following ones:

(I) Let (1) be an m-cover of $\mathbb{Z}$ and $J \subseteq \{1,\ldots,k\}$. Then

$$\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} = \left\{\sum_{s\in J}\frac{1}{\beta_s}\right\} \quad \text{for some } I \subseteq \{1,\ldots,k\} \text{ with } I \neq J,$$

provided $\sum_{s=1}^{k} 1/\beta_s = m$ (e.g. (1) is an exact m-cover of $\mathbb{Z}$ with $\alpha_s \in \mathbb{Z}$ and $\beta_s \in \mathbb{Z}^+$ for $s = 1,\ldots,k$) we have $\sum_{s\in I} 1/\beta_s = \sum_{s\in J} 1/\beta_s$ for some $I \subseteq \{1,\ldots,k\}$ with $I \neq J$ if $\emptyset \neq J \subset \{1,\ldots,k\}$, when $\sum_{s\in I} 1/\beta_s = \sum_{s\in J} 1/\beta_s$ for no $I \subseteq \{1,\ldots,k\}$ with $I \neq J$ there are at least m nonzero integers of the form $\sum_{s\in I} 1/\beta_s - \sum_{s\in J} 1/\beta_s$ where $I \subseteq \{1,\ldots,k\}$.

(II) Let $k \geq l \geq 0$ be integers. Then $2^{k-l}(l+1)$ is the smallest $n \in \mathbb{Z}^+$ such that any system of k arithmetic sequences with at least l equal common differences covers an arithmetic sequence at least m times if it covers n consecutive terms in the sequence at least m times.

The last section contains some unsolved problems related to possible extensions.

2. Characterizations of m-covers. Let us provide several technical lemmas the first of which serves as the starting point of our new approach.

LEMMA 1. *Let $m \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Then* (1) *covers x at least m times if and only if*

$$\prod_{s=1}^{k}(1 - r^{1/\beta_s}e^{2\pi i(\alpha_s - x)/\beta_s}) = o((1-r)^{m-1}) \quad (r \to 1). \tag{3}$$

Proof. Set $I = \{1 \leq s \leq k : x \in \alpha_s + \beta_s\mathbb{Z}\}$ and $I' = \{1,\ldots,k\} \setminus I$.

Clearly,

$$\lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{|I|}}$$

$$= \lim_{r\to 1}\prod_{s\in I'}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\lim_{r\to 1}\prod_{s\in I}\frac{1-r^{1/\beta_s}}{1-r}$$

$$= \prod_{s\in I'}(1-e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\prod_{s\in I}\frac{d}{dr}(r^{1/\beta_s})\Big|_{r=1}$$

$$= \prod_{s\in I'}(1-e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\prod_{s\in I}\beta_s^{-1}\neq 0,$$

and hence

$$\lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{m-1}}$$

$$= \lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{|I|}}(1-r)^{|I|-m+1}$$

$$= \begin{cases} 0 & \text{if } |I|>m-1,\\ \infty & \text{if } |I|<m-1.\end{cases}$$

Now it is apparent that $|I|\geq m$ if and only if (3) holds. We are done.

LEMMA 2. *Let $\theta_1,\ldots,\theta_n$ be real numbers with distinct fractional parts. For any $\varepsilon>0$ there exists a $\delta>0$ such that if*

$$\Big|\sum_{t=1}^{n} e^{2\pi is\theta_t}x_t\Big|<\delta$$

for every $s=1,\ldots,n$ then $|x_t|<\varepsilon$ for all $t=1,\ldots,n$.

Proof. Let A be the matrix $(e^{2\pi is\theta_t})_{\substack{1\leq s\leq n\\ 1\leq t\leq n}}$. Then

$$\frac{|A|}{e^{2\pi i\theta_1}e^{2\pi i\theta_2}\ldots e^{2\pi i\theta_n}} = \begin{vmatrix} 1 & 1 & \ldots & 1\\ e^{2\pi i\theta_1} & e^{2\pi i\theta_2} & \ldots & e^{2\pi i\theta_n}\\ (e^{2\pi i\theta_1})^2 & (e^{2\pi i\theta_2})^2 & \ldots & (e^{2\pi i\theta_n})^2\\ \ldots & \ldots & \ldots & \ldots\\ (e^{2\pi i\theta_1})^{n-1} & (e^{2\pi i\theta_2})^{n-1} & \ldots & (e^{2\pi i\theta_n})^{n-1}\end{vmatrix}$$

is a determinant of Vandermonde's type. As $|A|\neq 0$ the inverse matrix of A exists; we denote it by $B=(b_{st})_{\substack{1\leq s\leq n\\ 1\leq t\leq n}}$.

Let $b=\max\{|b_{st}|:s,t=1,\ldots,n\}>0$ and $\delta=\varepsilon/(bn)$. Let $x_1,\ldots,x_n$

be any complex numbers, and set

$$y_s = \sum_{t=1}^{n} e^{2\pi i s\theta_t} x_t \quad \text{for } s = 1, \ldots, n.$$

Let

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then $\vec{x} = BA\vec{x} = B\vec{y}$. If $|y_s| < \delta$ for every $s = 1, \ldots, n$, then

$$|x_s| = \Big|\sum_{t=1}^{n} b_{st} y_t\Big| \le \sum_{t=1}^{n} b|y_t| < bn\delta = \varepsilon \quad \text{for all } s = 1, \ldots, n.$$

This concludes the proof.

LEMMA 3. *Let* $m \in \mathbb{Z}^+$. *Then*

$$\sum_{n=0}^{m-1} a_n t^{n-m+1} = o(1) \quad (t \to 0) \tag{4}$$

if and only if $a_0 = \ldots = a_{m-1} = 0$.

Proof. The "if" direction is trivial. When $a_0, \ldots, a_{m-1}$ are not all zero, for the least k such that $a_k \ne 0$ we have

$$\sum_{n=0}^{m-1} a_n (x^{-1})^{n-m+1} = \sum_{n=k}^{m-1} a_n x^{m-1-n} \sim a_k x^{m-1-k} \quad (x \to \infty),$$

which contradicts (4). This ends the proof.

LEMMA 4. *Let* $n \ge m > 0$ *be integers and* $a_1, \ldots, a_n$ *distinct numbers. Then the system*

$$\begin{cases} x_1 + \ldots + x_n = 0, \\ a_1 x_1 + \ldots + a_n x_n = 0, \\ a_1^2 x_1 + \ldots + a_n^2 x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{m-1} x_1 + \ldots + a_n^{m-1} x_n = 0, \end{cases} \tag{5}$$

is equivalent to

$$\begin{cases} a_{11} x_1 + \ldots + a_{1n} x_n = 0, \\ a_{21} x_1 + \ldots + a_{2n} x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} x_1 + \ldots + a_{mn} x_n = 0, \end{cases} \tag{6}$$

where

$$a_{st} = \prod_{\substack{i=1\\ i\neq s}}^{m} \frac{a_i - a_t}{a_i - a_s} \quad \textit{for } s = 1, \ldots, m \textit{ and } t = 1, \ldots, n.$$

Proof. Rewrite (5) in the form

$$\begin{cases} x_1 + \ldots + x_m = -\sum_{m<t\le n} x_t, \\ a_1 x_1 + \ldots + a_m x_m = -\sum_{m<t\le n} a_t x_t, \\ a_1^2 x_1 + \ldots + a_m^2 x_m = -\sum_{m<t\le n} a_t^2 x_t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{m-1} x_1 + \ldots + a_m^{m-1} x_m = -\sum_{m<t\le n} a_t^{m-1} x_t. \end{cases}$$

By Cramer's rule, this says that

$$x_s = \begin{vmatrix} 1 & \ldots & 1 & -\sum_{m<t\le n} x_t & 1 & \ldots & 1 \\ a_1 & \ldots & a_{s-1} & -\sum_{m<t\le n} a_t x_t & a_{s+1} & \ldots & a_m \\ a_1^2 & \ldots & a_{s-1}^2 & -\sum_{m<t\le n} a_t^2 x_t & a_{s+1}^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_{s-1}^{m-1} & -\sum_{m<t\le n} a_t^{m-1} x_t & a_{s+1}^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_m \\ a_1^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}^{-1}$$

$$= -\sum_{m<t\le n} x_t \begin{vmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ a_1 & \ldots & a_{s-1} & a_t & a_{s+1} & \ldots & a_m \\ a_1^2 & \ldots & a_{s-1}^2 & a_t^2 & a_{s+1}^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_{s-1}^{m-1} & a_t^{m-1} & a_{s+1}^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_m \\ a_1^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}^{-1}$$

$$= -\sum_{m<t\le n} x_t \frac{\prod_{1\le i<s}(a_t-a_i)\cdot\prod_{s<i\le m}(a_i-a_t)\cdot\prod_{\substack{1\le i<j\le m\\ i,j\ne s}}(a_j-a_i)}{\prod_{1\le i<s}(a_s-a_i)\cdot\prod_{s<i\le m}(a_i-a_s)\cdot\prod_{\substack{1\le i<j\le m\\ i,j\ne s}}(a_j-a_i)}$$

$$= -\sum_{m<t\le n} a_{st}x_t \qquad \text{(Vandermonde)}$$

for every $s=1,\ldots,m$, i.e.

$$\sum_{t=1}^{m}\delta_{st}x_t + \sum_{m<t\le n} a_{st}x_t = 0 \quad \text{for } s=1,\ldots,m$$

where δ_{st} is the Kronecker delta. Since $a_{st}=\delta_{st}$ for $s,t=1,\ldots,m$, we have finished the proof.

Now we are ready to present

THEOREM 1. *Let $\mathcal{A}=\{\alpha_s+\beta_s\mathbb{Z}\}_{s=1}^k$, where $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$ and $\beta_1,\ldots,\beta_k\in\mathbb{R}^+$. Let $m\in\mathbb{Z}^+$ and*

$$S=\left\{0\le\theta<1:\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}=\theta \text{ for some } I\subseteq\{1,\ldots,k\}\right\}.$$

Let

$$V(\theta)=\left\{\sum_{s\in I}\frac{1}{\beta_s}: I\subseteq\{1,\ldots,k\} \text{ and } \sum_{s\in I}\frac{1}{\beta_s}-\theta\in\mathbb{Z}\right\}$$

and $U(\theta)$ be a set of m distinct numbers comparable with $V(\theta)$ (i.e. $|U(\theta)| = m$, and either $U(\theta)\subseteq V(\theta)$ or $U(\theta)\supseteq V(\theta)$). Then the following statements are equivalent:

(a) *$\mathcal{A}$ is an m-cover of $\mathbb{Z}$.*

(b) *$\mathcal{A}$ covers $|S|$ consecutive integers at least m times.*

(c) *For each $\theta\in S$,*

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\sum_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\sum_{s\in I}\alpha_s/\beta_s}=0 \tag{7}$$

holds for every $n=0,1,\ldots,m-1$. (As usual $\binom{x}{n}$ denotes $\frac{x(x-1)\ldots(x-n+1)}{1\cdot2\cdot\ldots\cdot(n-1)n}$.)

(d) *For any $\theta\in S$,*

$$\sum_{v\in V(\theta)} a_{uv}f(v)=0 \quad \text{for all } u\in U(\theta), \tag{8}$$

where

$$a_{uv} = \prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{x-v}{x-u} \quad \text{and} \quad f(v) = \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \Sigma_{s\in I} 1/\beta_s = v}} (-1)^{|I|} e^{2\pi i \Sigma_{s\in I} \alpha_s/\beta_s}.$$

Proof. (a)⇒(b). This is obvious.

(b)⇒(c). Suppose that each of $x+1,\ldots,x+|S|$ is covered by $\mathcal{A}$ at least m times, where x is an integer. By Lemma 1 for every $n = 1,\ldots,|S|$ we have

$$\begin{aligned}
0 &= \lim_{r\to 1} \frac{\prod_{s=1}^{k}(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x - n)/\beta_s})}{(1-r)^{m-1}} \\
&= \lim_{r\to 1} \Big((1-r)^{1-m} \\
&\quad \times \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} r^{\Sigma_{s\in I} 1/\beta_s} e^{2\pi i \Sigma_{s\in I}(\alpha_s - x)/\beta_s} e^{-2\pi i n \Sigma_{s\in I} 1/\beta_s}\Big) \\
&= \lim_{r\to 1} \sum_{\theta\in S} F(r,\theta) e^{-2\pi i n\theta},
\end{aligned}$$

where

$$F(r,\theta) = \sum_{\substack{I\subseteq\{1,\ldots,k\} \\ \{\Sigma_{s\in I} 1/\beta_s\} = \theta}} (-1)^{|I|} r^{\Sigma_{s\in I} 1/\beta_s} e^{2\pi i \Sigma_{s\in I} \alpha_s/\beta_s} e^{-2\pi i x\theta}/(1-r)^{m-1}.$$

Let ε be an arbitrary positive number. By Lemma 2 there is an $\eta > 0$ such that if

$$\Big|\sum_{\theta\in S} e^{-2\pi i n\theta} x_\theta\Big| < \eta$$

for every $n = 1,\ldots,|S|$ then $|x_\theta| < \varepsilon$ for all $\theta \in S$. Since

$$\sum_{\theta\in S} F(r,\theta) e^{-2\pi i n\theta} = o(1) \quad (r \to 1) \qquad \text{for } n = 1,\ldots,|S|,$$

there exists a $\delta > 0$ such that whenever $|r-1| < \delta$,

$$\Big|\sum_{\theta\in S} F(r,\theta) e^{-2\pi i n\theta}\Big| < \eta \qquad \text{for all } n = 1,\ldots,|S|$$

and hence by the above $|F(r,\theta)| < \varepsilon$ for each $\theta \in S$. This shows that $\lim_{r\to 1} F(r,\theta) = 0$ for every $\theta \in S$.

For any $\theta \in S$ we have

$$\begin{aligned}
0 &= \lim_{r\to 1} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|} r^{\Sigma_{s\in I}1/\beta_s} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}/(1-r)^{m-1}\\
&= \lim_{t\to 0} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}(1-t)^{[\Sigma_{s\in I}1/\beta_s]+\theta} t^{1-m} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&= \lim_{t\to 0} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|} \sum_{n=0}^{[\Sigma_{s\in I}1/\beta_s]} \binom{[\sum_{s\in I}1/\beta_s]}{n}(-t)^n t^{1-m} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&= \lim_{t\to 0} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} \Bigg((-1)^{|I|} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&\qquad \times \sum_{\substack{n=0\\ n\le[\Sigma_{s\in I}1/\beta_s]}}^{m-1} \binom{[\sum_{s\in I}1/\beta_s]}{n}(-1)^n t^{n-m+1}\Bigg)\\
&= \lim_{t\to 0}\sum_{n=0}^{m-1}(-1)^n\Bigg(\sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Bigg)t^{n-m+1}.
\end{aligned}$$

In view of Lemma 3, (7) holds for every $n = 0, 1, \dots, m-1$. Therefore part (c) follows.

(c)$\Rightarrow$(d). Fix $\theta \in S$. For each $n = 0, 1, \dots, m-1$,

$$x^n = \sum_{j=0}^{n} S(n,j)x(x-1)\dots(x-j+1)$$

where $S(n,j)$ $(0 \le j \le n)$ are Stirling numbers of the second kind, so by (c) we have

$$\begin{aligned}
&\sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\Big[\sum_{s\in I}1/\beta_s\Big]^n e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&\qquad = \sum_{j=0}^{n} j!S(n,j) \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{j} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s} = 0,
\end{aligned}$$

i.e.

$$\sum_{v\in V(\theta)} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ \Sigma_{s\in I}1/\beta_s=v}} (-1)^{|I|} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}[v]^n = 0.$$

Case 1: $|V(\theta)| \le m$. In this case

$$\sum_{v\in V(\theta)} [v]^n f(v) = 0 \quad \text{for every } n = 0, 1, \dots, |V(\theta)| - 1.$$

Hence (8) holds since $f(v) = 0$ for all $v \in V(\theta)$ (Vandermonde).

Case 2: $|V(\theta)| > m$. In this case, $U(\theta) \subset V(\theta)$ and

$$\sum_{v\in U(\theta)} [v]^n f(v) + \sum_{v\in V(\theta)\setminus U(\theta)} [v]^n f(v) = 0$$

for each $n = 0, 1, \dots, m-1$. According to Lemma 4,

$$\sum_{v\in V(\theta)} a_{uv} f(v) = \sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{[x]-[v]}{[x]-[u]} \Bigg) f(v) = 0 \quad \text{for all } u \in U(\theta).$$

So in either case we have (8).

(d)⇒(a). Assume that (d) holds. Let $\theta \in S$. For $u, v \in U(\theta)$,

$$a_{uv} = \prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x-v}{x-u} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Case 1: $|V(\theta)| \le m$. In this case $V(\theta) \subseteq U(\theta)$. As

$$f(u) = \sum_{v\in V(\theta)} a_{uv} f(v) = 0 \quad \text{for each } u \in V(\theta),$$

we get

$$\sum_{v\in V(\theta)} f(v)[v]^n = 0 \quad \text{for all } n = 0, 1, 2, \dots$$

Case 2: $|V(\theta)| > m$. In this case $U(\theta) \subset V(\theta)$, so for any $u \in U(\theta)$ and $v \in V(\theta)$ we have $\{u\} = \{v\} = \theta$ and hence $[u] - [v] = u - v$. Since

$$\sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{[x]-[v]}{[x]-[u]} \Bigg) f(v) = \sum_{v\in V(\theta)} a_{uv} f(v) = 0$$

for every $u \in U(\theta)$, it follows from Lemma 4 that

$$\sum_{v\in V(\theta)} f(v)[v]^n = \sum_{v\in U(\theta)} [v]^n f(v) + \sum_{v\in V(\theta)\setminus U(\theta)} [v]^n f(v) = 0$$

for all $n = 0, 1, \dots, m-1$.

In both cases,

$$\sum_{v\in V(\theta)} f(v)[v]^n = 0 \quad \text{for } n = 0, 1, \dots, m-1.$$

Thus for each nonnegative integer $n < m$,

$$\sum_{v\in V(\theta)} f(v)\binom{[v]}{n} = \sum_{v\in V(\theta)} f(v)\sum_{j=0}^{n}(-1)^{n-j}s(n,j)[v]^j/n!$$
$$= \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}s(n,j)\sum_{v\in V(\theta)} f(v)[v]^j = 0,$$

where $s(n,j)$ $(0 \le j \le n)$ are Stirling numbers of the first kind, i.e.

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$
$$= \sum_{v\in V(\theta)} \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=v}} \binom{[v]}{n}(-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s} = 0.$$

Therefore by the proof of (b)$\Rightarrow$(c),

$$\lim_{r\to 1} \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}/(1-r)^{m-1}$$
$$= \lim_{t\to 0}\sum_{n=0}^{m-1}(-1)^n\Bigg(\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Bigg)t^{n-m+1}$$
$$= 0.$$

Now for every integer x,

$$\prod_{s=1}^{k}(1 - r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})$$
$$= \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}(\alpha_s-x)/\beta_s}$$
$$= \sum_{\theta\in S} e^{-2\pi ix\theta} \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$
$$= \sum_{\theta\in S} e^{-2\pi ix\theta}o((1-r)^{m-1}) = o((1-r)^{m-1}) \quad (r\to 1).$$

Applying Lemma 1 we then obtain part (a).

The proof of Theorem 1 is now complete.

3. Reciprocals of common differences. In 1989 M. Z. Zhang [19] showed the following surprising result analytically: Provided that (2) is a cover of $\mathbb{Z}$, $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq \{1,\ldots,k\}$. Here we give

THEOREM 2. *Let* (1) *be a cover of* $\mathbb{Z}$. *Then for any* $J \subseteq \{1,\ldots,k\}$ *there is an* $I \subseteq \{1,\ldots,k\}$ *with* $I \neq J$ *such that*

$$\sum_{s\in I}\frac{1}{\beta_s} - \sum_{s\in J}\frac{1}{\beta_s} \in \mathbb{Z}. \tag{9}$$

Proof. Set $\theta = \{\sum_{s\in J} 1/\beta_s\}$. By Theorem 1,

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\sum_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{0} e^{2\pi i\sum_{s\in I}\alpha_s/\beta_s} = 0,$$

that is,

$$\sum_{\substack{J\neq I\subseteq\{1,\ldots,k\}\\ \{\sum_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|} e^{2\pi i\sum_{s\in I}\alpha_s/\beta_s} = -(-1)^{|J|} e^{2\pi i\sum_{s\in J}\alpha_s/\beta_s}.$$

Therefore

$$\left\{I \subseteq \{1,\ldots,k\} : I \neq J \text{ and } \left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} = \theta\right\} \neq \emptyset.$$

We are done.

In the case $J = \emptyset$, Theorem 2 yields a generalization of Zhang's result ([19]).

Provided that (1) is an m-cover of $\mathbb{Z}$ with $m \in \mathbb{Z}^+$, Theorem 2 asserts that for any $J \subseteq \{1,\ldots,k\}$,

$$S(J) = \left\{I \subseteq \{1,\ldots,k\} : I \neq J \text{ and } \sum_{s\in I}\frac{1}{\beta_s} - \sum_{s\in J}\frac{1}{\beta_s} \in \mathbb{Z}\right\} \tag{10}$$

is nonempty. This becomes trivial if

$$\sum_{s\in I}\frac{1}{\beta_s} = \sum_{s\in J}\frac{1}{\beta_s} \quad \text{for some } I \subseteq \{1,\ldots,k\} \text{ with } I \neq J. \tag{11}$$

What can we say about

$$Z(J) = \left\{\sum_{s\in I}\frac{1}{\beta_s} - \sum_{s\in J}\frac{1}{\beta_s} : I \in S(J)\right\} \tag{12}$$

if it does not contain zero? The following theorem gives us more information.

THEOREM 3. *Assume that* (1) *is an* m*-cover of* $\mathbb{Z}$. *Let* J *be a subset of* $\{1,\ldots,k\}$ *such that* (11) *fails, i.e.* $0 \notin Z(J)$ *where* $S(J)$ *and* $Z(J)$ *are given by* (10) *and* (12). *Then*

(i) $|Z(J)| \geq m$ *and hence*

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq md(J) + \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} \geq m, \tag{13}$$

where $d(J)$ *is the least positive integer that can be written as the difference of two (distinct) numbers of the form*

$$\sum_{s \in I} \frac{1}{\beta_s} \in \mathbb{Z} + \sum_{s \in J} \frac{1}{\beta_s} \quad \text{where } I \subseteq \{1, \ldots, k\}.$$

(ii) *When* $d(J) \geq [\sum_{s=1}^{k} 1/\beta_s]/m$, $d(J)$ *equals* $[\sum_{s=1}^{k} 1/\beta_s]/m$ *and divides* $[\sum_{s \in J} 1/\beta_s]$, *and for every* $n = 0, 1, \ldots, m$ *there exist at least*

$$\binom{m}{n} \Big/ \binom{m}{m[\sum_{s \in J} 1/\beta_s]/[\sum_{s=1}^{k} 1/\beta_s]}$$

subsets I *of* $\{1, \ldots, k\}$ *such that*

$$\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[\sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\}, \tag{14}$$

hence

$$|S(J)| \geq 2^m \Big/ \binom{m}{m[\sum_{s \in J} 1/\beta_s]/[\sum_{s=1}^{k} 1/\beta_s]} - 1 \quad \text{and} \quad |Z(J)| = m.$$

Proof. Let $\theta = \{\sum_{s \in J} 1/\beta_s\}$, $V(\theta)$, $U(\theta)$ and $f(x)$ be as in Theorem 1. If $|V(\theta)| \leq m$, then $V(\theta) \subseteq U(\theta)$, hence by Theorem 1 for all $u \in V(\theta) \subseteq U(\theta)$,

$$f(u) = \sum_{v \in V(\theta)} \left(\prod_{\substack{x \in U(\theta) \\ x \neq u}} \frac{x - v}{x - u} \right) f(v) = 0,$$

which is impossible since $0 \notin Z(J)$ and

$$f\left(\sum_{s \in J} \frac{1}{\beta_s} \right) = (-1)^{|J|} e^{2\pi i \Sigma_{s \in J} \alpha_s / \beta_s} \neq 0.$$

Thus $|V(\theta)| > m$.

(i) Let $v_0 < v_1 < \ldots < v_m$ be the first $m+1$ elements of $V(\theta)$ in ascending order. Clearly

$$1 + |Z(J)| = |Z(J) \cup \{0\}| = \left| \left\{ v - \sum_{s \in J} \frac{1}{\beta_s} : v \in V(\theta) \right\} \right| = |V(\theta)| \geq m + 1$$

and

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m = \sum_{i=0}^{m-1} (v_{i+1} - v_i) + v_0 \geq md(J) + \theta.$$

(ii) If $|V(\theta)| > m+1$ then

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m + 1 \geq 1 + md(J) + \theta.$$

Now suppose that $d(J) \geq [\sum_{s=1}^{k} 1/\beta_s]/m$. Then we must have $|V(\theta)| = m+1$, thus $V(\theta) = \{v_0, v_1, \ldots, v_m\}$ and $|Z(J)| = |V(\theta)| - 1 = m$. As

$$md(J) \geq \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] \geq [v_m] = v_0 - \theta + \sum_{i=0}^{m-1} (v_{i+1} - v_i) \geq [v_0] + md(J),$$

$$md(J) = \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right], \quad [v_0] = 0$$

and

$$[v_n] = v_0 - \theta + \sum_{i=0}^{n-1} (v_{i+1} - v_i) = 0 + \sum_{i=0}^{n-1} d(J) = nd(J)$$

for $n = 1, \ldots, m$.

Choose $0 \leq j \leq m$ such that $v_j = \sum_{s \in J} 1/\beta_s$. Then

$$j = \frac{[v_j]}{d(J)} = m\left[\sum_{s \in J} \frac{1}{\beta_s}\right] \Big/ \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right].$$

Set

$$U'(\theta) = \{v_i : 0 \leq i \leq m,\ i \neq j\}.$$

By Theorem 1, for any $n = 0, 1, \ldots, m$ with $n \neq j$,

$$\begin{aligned} 0 &= \sum_{v \in V(\theta)} \Bigg(\prod_{\substack{x \in U'(\theta) \\ x \neq v_n}} \frac{x - v}{x - v_n} \Bigg) f(v) = \sum_{t=0}^{m} \Bigg(\prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{v_i - v_t}{v_i - v_n} \Bigg) f(v_t) \\ &= \sum_{t=0}^{m} \Bigg(\prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{id(J) + \theta - (td(J) + \theta)}{id(J) + \theta - (nd(J) + \theta)} \Bigg) f(v_t) \\ &= \sum_{t=0}^{m} \Bigg(\prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{i - t}{i - n} \Bigg) f(v_t) \\ &= f(v_n) + \Bigg(\prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{i - j}{i - n} \Bigg) f(v_j). \end{aligned}$$

Since

$$\prod_{\substack{i=0\\ i\neq j,n}}^{m} \frac{i-j}{i-n} = \frac{\prod_{i=0,\, i\neq j}^{m}(i-j)}{n-j} \Big/ \frac{\prod_{i=0,\, i\neq n}^{m}(i-n)}{j-n}$$

$$= -\frac{\prod_{i=0}^{j-1}(i-j)\cdot\prod_{i=j+1}^{m}(i-j)}{\prod_{i=0}^{n-1}(i-n)\cdot\prod_{i=n+1}^{m}(i-n)}$$

$$= -\frac{(-1)^j j!(m-j)!}{(-1)^n n!(m-n)!} = (-1)^{j-n+1}\binom{m}{n}\Big/\binom{m}{j},$$

we have

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} (-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$

$$= f(v_n) = -(-1)^{j-n+1}\binom{m}{n}\binom{m}{j}^{-1} f\Big(\sum_{s\in J}\frac{1}{\beta_s}\Big)$$

$$= (-1)^{j-n}\binom{m}{n}\binom{m}{j}^{-1}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\alpha_s/\beta_s}$$

and hence

$$\Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s} = \frac{n}{m}\Big[\sum_{s=1}^{k}\frac{1}{\beta_s}\Big] + \Big\{\sum_{s\in J}\frac{1}{\beta_s}\Big\}\Big\}\Big|$$

$$= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} 1$$

$$\geq \Big|\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} (-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Big| = \binom{m}{n}\Big/\binom{m}{j};$$

therefore

$$1+|S(J)| = \Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s}\in V(\theta)\Big\}\Big|$$

$$= \sum_{n=0}^{m}\Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s} = v_n = nd(J)+\theta\Big\}\Big|$$

$$\geq \sum_{n=0}^{m}\binom{m}{n}\Big/\binom{m}{j} = 2^m\Big/\binom{m}{j}.$$

This ends the proof.

Now let us apply Theorem 3 to those m-covers (1) with $\sum_{s=1}^{k} 1/\beta_s = m$.

THEOREM 4. *Let* (1) *be an* m*-cover of* $\mathbb{Z}$ *with* $\sum_{s=1}^{k} 1/\beta_s = m \in \mathbb{Z}^+$, *which happens if* (1) *is an exact* m*-cover of* $\mathbb{Z}$ *by residue classes. Then*

(i) *For every* $l = 1, \ldots, k-1$ *we have*

$$\sum_{s=l+1}^{k} \frac{1}{\beta_s} \geq \frac{1}{\max\{\beta_1, \ldots, \beta_l\}}. \tag{15}$$

(ii) *For any* $\emptyset \neq J \subset \{1, \ldots, k\}$ *there exists an* $I \subseteq \{1, \ldots, k\}$ *with* $I \neq J$ *such that*

$$\sum_{s\in I} \frac{1}{\beta_s} = \sum_{s\in J} \frac{1}{\beta_s}, \tag{16}$$

furthermore when $\sum_{s\in J} 1/\beta_s \in \mathbb{Z}$ *there are at least*

$$\binom{m}{\sum_{s\in J} 1/\beta_s} \geq m > 1$$

subsets I *of* $\{1, \ldots, k\}$ *satisfying* (16).

Proof. (i) For $l = 1, \ldots, k-1$ (15) follows from part (ii) in the case $J = \{l+1, \ldots, k\}$, so we proceed to the proof of part (ii).

(ii) If (11) fails then by part (i) of Theorem 3 and the equality $\sum_{s=1}^{k} 1/\beta_s = m$ we must have

$$\left\{\sum_{s\in J} \frac{1}{\beta_s}\right\} = 0, \quad \text{i.e.} \quad \sum_{s\in J} \frac{1}{\beta_s} \in \mathbb{Z}.$$

Observe that

$$0 < \sum_{s\in J} \frac{1}{\beta_s} < \sum_{s=1}^{k} \frac{1}{\beta_s} = m.$$

If $\sum_{s\in J} 1/\beta_s \in \mathbb{Z}$, then $m > 1$ and $\sum_{s\in J} 1/\beta_s = n$ for some $n = 1, \ldots, m-1$, by part (ii) of Theorem 3 there are at least $\binom{m}{n}/\binom{m}{m} = \binom{m}{n} \geq m$ subsets I of $\{1, \ldots, k\}$ such that

$$\sum_{s\in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \left\{\sum_{1\leq s\leq k} \frac{1}{\beta_s}\right\} = n = \sum_{s\in J} \frac{1}{\beta_s}.$$

We are done.

Remark. In 1992 Z. W. Sun ([17]) proved that if (2) is an exact m-cover of $\mathbb{Z}$ then for each $n = 1, \ldots, m$ there exist at least $\binom{m}{n}$ subsets I of $\{1, \ldots, k\}$ such that $\sum_{s\in I} 1/n_s$ equals n. The lower bounds $\binom{m}{n}$ $(1 \leq n \leq m)$ are best possible, and the Riemann zeta function was used in the proof.

From Theorem 3 we can also deduce the following theorem which extends Zhang's result ([19]) and the theorem of Sun [17] even in the case $l = k$.

THEOREM 5. *Let* (1) *be an* m-*cover of* $\mathbb{Z}$ *and* l *a positive integer not exceeding* k *such that*

$$\min\left\{1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_l}\right\} > \sum_{l<t\le k} \frac{1}{\beta_t}, \tag{17}$$

where $\sum_{l<t\le k} 1/\beta_t$ *is considered to be zero for* $l = k$. *Then*

(i) *There are at least* m *positive integers representable by*

$$\sum_{s\in I} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t}, \qquad \textit{where } I \subseteq \{1, \ldots, k\}, \tag{18}$$

thus we have

$$\sum_{s=1}^{l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t} \ge m. \tag{19}$$

(ii) *Provided that any positive integer less than* $[\sum_{s=1}^{k} 1/\beta_s]/m$ *cannot be expressed as the difference of two integers of the form* (18), $[\sum_{s=1}^{k} 1/\beta_s]$ *is divisible by* m *and for each* $n = 0, 1, \ldots, m$ *there are at least* $\binom{m}{n}$ *subsets* I *of* $\{1, \ldots, k\}$ *such that*

$$\sum_{s\in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \sum_{l<t\le k} \frac{1}{\beta_t}, \tag{20}$$

hence there exist at least $2^m - 1$ *subsets* I *of* $\{1, \ldots, k\}$ *with*

$$\sum_{s\in I} \frac{1}{\beta_s} \in \mathbb{Z}^+ + \sum_{l<t\le k} \frac{1}{\beta_t}.$$

Proof. Let $J = \{1 \le t \le k : t > l\}$. By (17),

$$\left[\sum_{t\in J} \frac{1}{\beta_t}\right] = 0 \quad \text{and} \quad \left\{\sum_{t\in J} \frac{1}{\beta_t}\right\} = \sum_{l<t\le k} \frac{1}{\beta_t}.$$

For any $I \subseteq \{1, \ldots, k\}$, if $I \subset J$ then

$$0 < \sum_{t\in J} \frac{1}{\beta_t} - \sum_{s\in I} \frac{1}{\beta_s} < 1,$$

and if $I \not\subseteq J$ then

$$\sum_{s\in I} \frac{1}{\beta_s} - \sum_{t\in J} \frac{1}{\beta_t} \ge \min\left\{\frac{1}{\beta_s} : 1 \le s \le l\right\} - \sum_{l<t\le k} \frac{1}{\beta_t} > 0.$$

So (11) fails, moreover $Z(J)$ given by (12) contains only positive integers. Applying Theorem 3 we obtain the desired results.

Erdős conjectured (before 1950) that if (2) is a cover of $\mathbb{Z}$ with $1 < n_1 < n_2 < \ldots < n_k$ then $\sum_{s=1}^{k} 1/n_s > 1$. H. Davenport, L. Mirsky, D. Newman and R. Radó confirmed this conjecture (independently) by proving that if (2) is an exact cover of $\mathbb{Z}$ with $1 < n_1 \le \ldots \le n_{k-1} \le n_k$ then $n_{k-1} = n_k$. For further improvements see Znám [20], M. Newman [10], Pořubský [11, 12], M. A. Berger, A. Felzenbaum and A. S. Fraenkel [1]. The best record in this direction is the following result due to the author which is partially announced in [15] and completely proved in [16]: Let $\lambda_1, \ldots, \lambda_k$ be complex numbers and $n_0 \in \mathbb{Z}^+$ a period of the function

$$\sigma(x) = \sum_{\substack{s=1 \\ x \equiv a_s \,(\mathrm{mod}\ n_s)}}^{k} \lambda_s.$$

If $d \in \mathbb{Z}^+$ does not divide n_0 and

$$\sum_{\substack{s=1 \\ d \mid n_s,\ a_s \equiv a \,(\mathrm{mod}\ d)}}^{k} \frac{\lambda_s}{n_s} \ne 0 \quad \text{for some integer } a,$$

then

$$|\{a_s \bmod d : 1 \le s \le k,\ d \mid n_s\}| \ge \min_{\substack{0 \le s \le k \\ d \nmid n_s}} \frac{d}{\gcd(d, n_s)} \ge p(d),$$

where $p(d)$ is the least prime divisor of d. Here we have

THEOREM 6. *Let (1) be an m-cover of $\mathbb{Z}$ with $\beta_1 \le \ldots \le \beta_{k-l} < \beta_{k-l+1} = \ldots = \beta_k$ where $1 \le l < k$. Then either*

(21) $$l \ge \beta_k / \max\{1, \beta_{k-l}\} \quad (> 1 \text{ if } \beta_k > 1),$$

or there are at least m positive integers in the form

(22) $$\sum_{s \in I} \frac{1}{\beta_s} - \frac{l}{\beta_k}, \quad \text{where } I \subseteq \{1, \ldots, k\},$$

and hence

(23) $$\sum_{s=1}^{k} \frac{1}{\beta_s} > \sum_{s=1}^{k-l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \frac{l}{\beta_k} \ge m.$$

(Also, $\sum_{s=1}^{k} 1/\beta_s > \sum_{s=1}^{k} 1/\beta_k \ge k \ge m$ if $\beta_k \le 1$.)

Proof. Clearly $l < \beta_k / \max\{1, \beta_{k-l}\}$ if and only if

$$\min\left\{1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{k-l}}\right\} > \sum_{k-l < t \le k} \frac{1}{\beta_t} \ (= l/\beta_k).$$

Therefore Theorem 6 follows from part (i) of Theorem 5.

Note that when $\beta_{k-l} \ge 1$ and $\beta_k/\beta_{k-l} \in \mathbb{Z}$

$$\beta_k/\max\{1,\beta_{k-l}\} = \beta_k/\beta_{k-l} \ge p(\beta_k/\beta_{k-l}) \quad (\ge p(\beta_k) \text{ if } \beta_{k-l}, \beta_k \in \mathbb{Z}).$$

4. Some local-global results. In 1958 S. K. Stein [14] conjectured that whenever the residue classes in (2) are pairwise disjoint and the moduli $n_1, \ldots, n_k > 1$ are distinct there exists an integer x with $1 \le x \le 2^k$ such that x is not covered by (2). Erdős [6] confirmed this conjecture with $k \cdot 2^k$ instead of 2^k. Since the Davenport–Mirsky–Newman–Radó result indicates that an exact cover of $\mathbb{Z}$ by (finitely many) residue classes cannot have its moduli distinct and greater than one, Erdős proposed the stronger conjecture that any system of k residue classes not covering all the integers omits a positive integer not exceeding 2^k. Both conjectures have some local-global character. In 1969 R. B. Crittenden and C. L. Vanden Eynden [2] claimed their positive answer to the stronger conjecture. Later in [3] a long indirect and awkward proof was given for $k \ge 20$, the authors concluded the paper with the statements: "Of course it remains to show the conjecture is true for $k < 20$. This may be checked by more special arguments."

In 1970 Crittenden and Vanden Eynden [4] conjectured further that if all the moduli n_s in (2) are greater than an integer $0 \le l < k$ then (2) is a cover of $\mathbb{Z}$ if it covers all the integers in the interval $[1, 2^{k-l}(l+1)]$. In contrast with the Crittenden–Vanden Eynden conjecture we give

THEOREM 7. *For any $m \in \mathbb{Z}^+$, (1) is an m-cover of $\mathbb{Z}$ if it covers $2^{k-M}(M+1)$ consecutive integers at least m times, where*

$$M = \max_{1 \le t \le k} |\{1 \le s \le k : \beta_s = \beta_t\}|. \tag{24}$$

Proof. Let $\beta > 0$ be a number such that $J = \{1 \le s \le k : \beta_s = \beta\}$ has cardinality M. As

$$\begin{aligned}
&\left|\left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} : I \subseteq \{1,\ldots,k\}\right\}\right| \\
&\quad \le \left|\left\{\sum_{s\in I\cap J}\frac{1}{\beta_s} + \sum_{s\in I\setminus J}\frac{1}{\beta_s} : I \subseteq \{1,\ldots,k\}\right\}\right| \\
&\quad \le \left|\left\{\sum_{s\in I}\frac{1}{\beta} : I \subseteq J\right\}\right| \cdot \left|\left\{\sum_{s\in I}\frac{1}{\beta_s} : I \subseteq \{1,\ldots,k\}\setminus J\right\}\right| \\
&\quad \le \left|\left\{\frac{|I|}{\beta} : I \subseteq J\right\}\right| \cdot |\{I : I \subseteq \{1,\ldots,k\}\setminus J\}| \\
&\quad = (|J|+1)\cdot 2^{k-|J|} = 2^{k-M}(M+1),
\end{aligned}$$

Theorem 1 implies Theorem 7.

The following example noted by Crittenden and Vanden Eynden [4] shows that the number $g(k, M) = 2^{k-M}(M+1)$ in Theorem 7 is best possible.

EXAMPLE. Let $M = n - 1 \in \mathbb{Z}^+$. Consider the system A consisting of the following $k \geq M$ residue classes:

$$\begin{gathered} 1 + n\mathbb{Z}, \quad 2 + n\mathbb{Z}, \quad \ldots, \quad M + n\mathbb{Z}, \\ n + 2n\mathbb{Z}, \quad 2n + 2^2 n\mathbb{Z}, \quad \ldots, \quad 2^{k-M-1}n + 2^{k-M}n\mathbb{Z}. \end{gathered}$$

Observe that A together with $2^{k-M}n\mathbb{Z}$ forms an exact cover of $\mathbb{Z}$. So A covers positive integers from 1 to $2^{k-M}(M+1) - 1$, but it does not cover all the integers.

Result (II) stated in Section 1 follows from Theorem 7 and Example, since (1) covers $\alpha + \beta x$ (where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $x \in \mathbb{Z}$) at least m times if and only if $\{\frac{\alpha_s - \alpha}{\beta} + \frac{\beta_s}{\beta}\mathbb{Z}\}_{s=1}^k$ covers x at least m times, and $2^{k-l}(l+1) \geq 2^{k-M}(M+1)$ if $k \geq M \geq l > 0$. (The case $l = 0$ can be reduced to the case $l = 1$.)

5. Several open problems. Theorem 1 tells us that (2) is a cover of $\mathbb{Z}$ if it covers integers from 1 to

$$\left| \left\{ \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq \{1, \ldots, k\} \right\} \right| \leq 2^k \leq 2^{n_1 + \ldots + n_k}.$$

This suggests

PROBLEM 1. Can we find a polynomial P with integer coefficients such that a finite system (2) of residue classes forms a cover of $\mathbb{Z}$ whenever it covers all positive integers not exceeding $P(n_1 + \ldots + n_k)$?

In 1973 L. J. Stockmeyer and A. R. Meyer proved that the problem whether there exists an integer not covered by a given (2) is NP-complete. In 1991 S. P. Tung [18] extended this result to algebraic integer rings. If the required P in Problem 1 exists, then there is a polynomial time algorithm to decide whether (2) covers all the integers or not. So a positive answer to Problem 1 would imply that NP = P.

By appearances Theorems 2–7 involve no roots of unity. Perhaps vast generalizations of them could be made.

PROBLEM 2. Let $A_1, \ldots, A_k$ be sets of natural numbers having positive densities $d(A_1), \ldots, d(A_k)$ respectively. If no A_s contains $m_s \in \mathbb{Z}^+$ consecutive integers, does $\bigcup_{s=1}^k A_s$ have density 1 when it covers $m_1 \ldots m_k$ arbitrarily large consecutive integers? Suppose that $\{A_s\}_{s=1}^k$ covers all the natural numbers; does there exist, for any $J \subseteq \{1, \ldots, k\}$, an $I \subseteq \{1, \ldots, k\}$

with $I \neq J$ such that

$$\sum_{s\in I} d(A_s) - \sum_{s\in J} d(A_s) \in \mathbb{Z}?$$

PROBLEM 3. Let K be an algebraic number field and O_K the ring of algebraic integers in K. Let $a_1, \ldots, a_k \in O_K$ and $A_1, \ldots, A_k$ be ideals of O_K with norms $N(A_1), \ldots, N(A_k)$ respectively. If $\{a_s + A_s\}_{s=1}^k$ forms an exact m-cover of O_K for some $m \in \mathbb{Z}^+$, is it true that for any $\emptyset \neq J \subseteq \{1, \ldots, k\}$ there exists a subset I of $\{1, \ldots, k\}$ with $I \neq J$ such that

$$\sum_{s\in I} \frac{1}{N(A_s)} = \sum_{s\in J} \frac{1}{N(A_s)}?$$

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References

[1] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, *Improvements to the Newman–Znám result for disjoint covering systems*, Acta Arith. 50 (1988), 1–13.

[2] R. B. Crittenden and C. L. Vanden Eynden, *A proof of a conjecture of Erdős*, Bull. Amer. Math. Soc. 75 (1969), 1326–1329.

[3] —, —, *Any n arithmetic progressions covering the first 2^n integers cover all integers*, Proc. Amer. Math. Soc. 24 (1970), 475–481.

[4] —, —, *The union of arithmetic progressions with differences not less than k*, Amer. Math. Monthly 79 (1972), 630.

[5] P. Erdős, *On integers of the form $2^k + p$ and some related problems*, Summa Brasil. Math. 2 (1950), 113–123.

[6] —, *Remarks on number theory IV: Extremal problems in number theory I*, Mat. Lapok 13 (1962), 228–255.

[7] —, *Problems and results on combinatorial number theory III*, in: Number Theory Day, M. B. Nathanson (ed.), Lecture Notes in Math. 626, Springer, New York, 1977, 43–72.

[8] —, *Problems and results in number theory*, in: Recent Progress in Analytic Number Theory, H. Halberstam and C. Hooley (eds.), Vol. 1, Academic Press, London, 1981, 1–14.

[9] R. K. Guy, *Unsolved Problems in Number Theory*, Springer, New York, 1981.

[10] M. Newman, *Roots of unity and covering sets*, Math. Ann. 191 (1971), 279–282.

[11] Š. Porubský, *Covering systems and generating functions*, Acta Arith. 26 (1974/75), 223–231.

[12] —, *On m times covering systems of congruences*, ibid. 29 (1976), 159–169.

[13] —, *Results and problems on covering systems of residue classes*, Mitt. Math. Sem. Giessen 1981, no. 150, 1–85.

[14] S. K. Stein, *Unions of arithmetic sequences*, Math. Ann. 134 (1958), 289–294.

[15] Z. W. Sun, *Several results on systems of residue classes*, Adv. in Math. (Beijing) 18 (1989), 251–252.

[16] —, *An improvement to the Znám–Newman result*, Chinese Quart. J. Math. 6 (3) (1991), 90–96.

[17] Z. W. Sun, *On exactly m times covers*, Israel. J. Math. 77 (1992), 345–348.
[18] S. P. Tung, *Complexity of sentences over number rings*, SIAM J. Comp. 20 (1991), 126–143.
[19] M. Z. Zhang, *A note on covering systems of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) 26 (1989), Special Issue, 185–188.
[20] Š. Znám, *On exactly covering systems of arithmetic sequences*, Math. Ann. 180 (1969), 227–232.
[21] —, *A survey of covering systems of congruences*, Acta Math. Univ. Comenian. 40/41 (1982), 59–79.

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