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EXACT m-COVERS AND THE LINEAR FORM $\sum_{s=1}^{k} x_s/n_s$

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1. INTRODUCTION

For $a, n \in \mathbb{Z}$ with $n > 0$, we let

$$a + n\mathbb{Z} = \{\cdots, a-2n, a-n, a, a+n, a+2n, \cdots\}$$

and call it an *arithmetic sequence*. Given a finite system

$$A = \{a_s + n_s\mathbb{Z}\}_{s=1}^{k} \tag{1}$$

of arithmetic sequences, we associate each $x \in \mathbb{Z}$ with the corresponding covering multiplicity $\sigma(x) = |\{1 \leqslant s \leqslant k : \ x \in a_s + n_s\mathbb{Z}\}|$ ($|S|$ means the cardinality of a set S), and call $m(A) = \inf_{x\in\mathbb{Z}} \sigma(x)$ the *covering multiplicity* of A. Apparently

$$\sum_{s=1}^{k} \frac{1}{n_s} = \frac{1}{N} \sum_{x=0}^{N-1} \sigma(x) \geqslant m(A) \tag{2}$$

where N is the least common multiple of those *common differences* (or *moduli*) $n_1, \cdots, n_k$. For a positive integer m, (1) is said to be an *m-cover of* $\mathbb{Z}$ if its covering multiplicity is not less than m, and an *exact m-cover of* $\mathbb{Z}$ if $\sigma(x) = m$ for all $x \in \mathbb{Z}$. Note that $k \geqslant m$ if (1) forms an m-cover of $\mathbb{Z}$. Clearly the *covering function* $\sigma : \mathbb{Z} \to \mathbb{Z}$ is constant if and only if (1) forms an exact m-cover of $\mathbb{Z}$ for some $m = 1, 2, 3, \cdots$. An exact 1-cover of $\mathbb{Z}$ is a partition of $\mathbb{Z}$ into residue classes.

P. Erdös ([E]) proposed the concept of *cover* (i.e., 1-cover) of $\mathbb{Z}$ in 1930's, Š. Porubský ([P]) introduced the notion of exact m-cover of $\mathbb{Z}$ in 1970's, and the author ([Su3]) studied m-covers of $\mathbb{Z}$ for the first time. The most challenging problem in this field is to describe those $n_1, \cdots, n_k$ in an m-cover (or exact m-cover) (1) of $\mathbb{Z}$ (cf. [Gu]). In [Su2,Su3,Su4] the author revealed some connections between (exact) m-covers of $\mathbb{Z}$ and Egyptian fractions. Here we concentrate on exact m-covers of $\mathbb{Z}$. In [Su3,Su4] results for exact m-covers of $\mathbb{Z}$ were obtained by studying general

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m-covers of $\mathbb{Z}$ and noting that an exact m-cover (1) of $\mathbb{Z}$ is an m-cover of $\mathbb{Z}$ with $\sum_{s=1}^k \frac{1}{n_s} = m$, in section 4 of the present paper we shall directly characterize exact m-covers of $\mathbb{Z}$ in various ways. (Note that in the famous book [Gu] R. K. Guy wrote that how to characterize exact 1-covers of $\mathbb{Z}$ is a main outstanding unsolved problem in the area.) This enables us to make further progress. With the help of the linear form $\sum_{s=1}^k \frac{x_s}{n_s}$ (studied in the next section), we will provide some new properties of exact m-covers of $\mathbb{Z}$ (see section 3). The fifth section is devoted to proofs of the main theorems stated in section 3.

For complex number x and nonnegative integer n, as usual $\binom{x}{n} := \frac{1}{n!}\prod_{j=0}^{n-1}(x-j)$ ($\binom{x}{0}$ refers to 1). For real x we use $[x]$ and $\{x\}$ to represent the integral part and the fractional part of x respectively. For two integers a, b not all zero, (a, b) denotes the greatest common divisor of a and b.

Now we state our central results for an exact m-cover (1) of $\mathbb{Z}$:

(I) For $a = 0, 1, 2, \cdots$ and $t = 1, \cdots, k$ there are at least $\binom{m-1}{[a/n_t]}$ subsets I of $\{1, \cdots, k\}$ for which $t \notin I$ and $\sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t}$, where the lower bounds are best possible.

(II) If $\emptyset \neq I \subseteq \{1, \cdots, k\}$ and $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, then we have $\{\{\sum_{s\in J} \frac{1}{n_s}\} : J \subseteq \{1, \cdots, k\} \setminus I\} \supseteq \{\frac{r}{[n_s]_{s\in I}} : r = 0, 1, \cdots, [n_s]_{s\in I} - 1\}$ where $[n_s]_{s\in I}$ is the least common multiple of those n_s with $s \in I$.

(III) For any rational c, the number of solutions of the equation $\sum_{s=1}^k \frac{x_s}{n_s} = c$ with $x_s \in \{0, 1, \cdots, n_s - 1\}$ for $s = 1, \cdots, k$, is the sum of finitely many (not necessarily distinct) prime factors of $n_1, \cdots, n_k$ if $c \neq 0, 1, 2, \cdots$, and at least $\binom{k-m}{n}$ if c equals a nonnegative integer n.

2. On the Linear Form $\sum_{s=1}^k \frac{x_s}{n_s}$

In this section we shall say something general about the linear form $\sum_{s=1}^k \frac{x_s}{n_s}$ where $n_1, \cdots, n_k$ are positive integers.

Let's first introduce more notations. For x, y in the rational field $\mathbb{Q}$, if $x - y \in \mathbb{Z}$ then we write $x \equiv y \pmod 1$. For $n = 1, 2, 3, \cdots$ we set $R(n) = \{0, \cdots, n-1\}$. When we deal with a finite collection $\{n_s\}_{s\in I}$ of positive integers, the least common multiple $[n_s]_{s\in I}$ and the product $\prod_{s\in I} n_s$ will be regarded as 1 if I is empty.

Definition. Two (finite) sequences $\{n_s\}_{s=1}^k$ and $\{m_t\}_{t=1}^l$ of positive integers are said to be *equivalent* if $k = l$ and $(n_s, n_t) = (m_s, m_t)$ for all $s, t = 1, \cdots, k$ with $s \neq t$. We call $\{n_s\}_{s=1}^k$ a *normal* sequence if n_t divides $[n_s]_{\substack{s=1\\ s\neq t}}^k$ for every $t = 1, \cdots, k$.

Proposition 2.1. *Let $n_1, \cdots, n_k$ be positive integers. Then $\{(n_t, [n_s]_{\substack{s=1\\ s\neq t}}^k)\}_{t=1}^k$ is* *the only normal sequence equivalent to* $\{n_s\}_{s=1}^k$.

Proof. For each $t = 1, \cdots, k$ we let $n'_t = (n_t, [n_s]_{\substack{s=1\\s\neq t}}^k) = [(n_s, n_t)]_{\substack{s=1\\s\neq t}}^k$, clearly n'_t divides $[n'_s]_{\substack{s=1\\s\neq t}}^k$ because $(n_s, n_t) \mid n'_s$ for all $s = 1, \cdots, k$ with $s \neq t$. For $i, j = 1, \cdots, k$ with $i \neq j$, $(n'_i, n'_j) = (n_i, n_j)$ since $n_i \mid [n_s]_{\substack{s=1\\s\neq j}}^k$ and $n_j \mid [n_s]_{\substack{s=1\\s\neq i}}^k$. Hence $\{n'_s\}_{s=1}^k$ is normal and equivalent to $\{n_s\}_{s=1}^k$. If so is $\{m_s\}_{s=1}^k$ where $m_1, \cdots, m_k$ are positive integers, then $m_t = (m_t, [m_s]_{\substack{s=1\\s\neq t}}^k) = [(m_s, m_t)]_{\substack{s=1\\s\neq t}}^k = [(n_s, n_t)]_{\substack{s=1\\s\neq t}}^k = n'_t$ for every $t = 1, \cdots, k$. We are done.

Proposition 2.2. *Let* $n_1, \cdots, n_k$ *be positive integers. For* $\theta \in \mathbb{Q}$ *the equation*

$$\sum_{s=1}^k \frac{x_s}{n_s} \equiv \theta \pmod 1 \qquad \text{with } x_s \in R(n_s) \text{ for } s = 1, \cdots, k \tag{3}$$

is solvable if and only if $[n_1, \cdots, n_k]\theta \in \mathbb{Z}$, *and in the solvable case the number of solutions is* $\frac{n_1 \cdots n_k}{[n_1, \cdots, n_k]}$ *which keeps invariant if we replace* $\{n_s\}_{s=1}^k$ *by an equivalent sequence.*

Proof. Let's prove Proposition 2.2 by induction. The case $k = 1$ is trivial. Let $k > 1$ and assume Proposition 2.2 for smaller values of k. Observe that

$$\frac{1}{[n_1, \cdots, n_k]}\mathbb{Z} = \frac{([n_1, \cdots, n_{k-1}], n_k)}{[n_1, \cdots, n_{k-1}]n_k}\mathbb{Z} = \frac{1}{n_k}\mathbb{Z} + \frac{1}{[n_1, \cdots, n_{k-1}]}\mathbb{Z}.$$

So $[n_1, \cdots, n_k]\theta \in \mathbb{Z}$ if and only if $[n_1, \cdots, n_{k-1}](\theta - x/n_k) \in \mathbb{Z}$ for some $x \in \mathbb{Z}$. For any $a \in \mathbb{Z}$ with $0 \leqslant a < n_k$, the congruence $\sum_{s=1}^{k-1} \frac{x_s}{n_s} \equiv \theta - \frac{a}{n_k} \pmod 1$ is solvable if and only if

$$[n_1, \cdots, n_{k-1}]\left(\theta - \frac{a}{n_k}\right) \in \mathbb{Z}, \quad \text{i.e. } [n_1, \cdots, n_{k-1}]a \equiv [n_1, \cdots, n_{k-1}]n_k\theta \pmod{n_k}.$$

Hence (3) is solvable if and only if $[n_1, \cdots, n_k]\theta \in \mathbb{Z}$. In the solvable case there are exactly $([n_1, \cdots, n_{k-1}], n_k) = [(n_1, n_k), \cdots, (n_{k-1}, n_k)]$ numbers $a \in R(n_k)$ satisfying the last congruence, thus by the induction hypothesis (3) has exactly $\frac{n_1 \cdots n_{k-1}}{[n_1, \cdots, n_{k-1}]}([n_1, \cdots, n_{k-1}], n_k) = \frac{n_1 \cdots n_k}{[n_1, \cdots, n_k]}$ solutoins. As $\frac{n_1 \cdots n_{k-1}}{[n_1, \cdots, n_{k-1}]}$ depends only on those (n_i, n_j) with $1 \leqslant i < j < k$, the number $\frac{n_1 \cdots n_k}{[n_1, \cdots, n_k]}$ depends only on the (n_s, n_t), $1 \leqslant s < t \leqslant k$. This ends the proof.

Corollary 2.1. *Let* a *be an integer and* $n_1, \cdots, n_k$ *positive ones. Then* $\frac{a}{[n_1, \cdots, n_k]}$ *can be written uniquely in the form* $q + \sum_{s=1}^k \frac{x_s}{n_s}$ *with* $q \in \mathbb{Z}$ *and* $x_s \in R(n_s)$ *for* $s = 1, \cdots, k$, *if and only if* $(n_s, n_t) = 1$ *for all* $s, t = 1, \cdots, k$ *with* $s \neq t$.

Proof. By Proposition 2.2, equation (3) with $\theta = \frac{a}{[n_1, \cdots, n_k]}$ has a unique solution if and only if $n_1 \cdots n_k = [n_1, \cdots, n_k]$. So the desired result follows.

Corollary 2.2. *Let $n_1,\cdots,n_k$ be positive integers. Then the number of solutions of the equation*

$$(4)\qquad \sum_{s=1}^k \frac{x_s}{n_s}\equiv 0 \pmod 1 \qquad \text{with } x_s\in\mathbb{Z} \text{ and } 0<x_s<n_s \text{ for } s=1,\cdots,k$$

equals

$$(-1)^k+\sum_{t=1}^k(-1)^{k-t}\sum_{1\leqslant i_1<\cdots<i_t\leqslant k}\frac{n_{i_1}\cdots n_{i_t}}{[n_{i_1},\cdots,n_{i_t}]}$$

which depends only on those (n_s,n_t) with $1\leqslant s<t\leqslant k$.

Proof. For $I\subseteq\{1,\cdots,k\}$ we let $\#I$ denote the number of solutions of the diophantine equation $\sum_{s\in I}\frac{x_s}{n_s}\equiv 0 \pmod 1$ with $x_s\in\{1,\cdots,n_s-1\}$ for $s\in I$, and consider $\#\emptyset$ as 1. By Proposition 2.2 $\sum_{J\subseteq I}\#J=\prod_{s\in I}n_s/[n_s]_{s\in I}$ for all $I\subseteq\{1,\cdots,k\}$, therefore $\#\{1,\cdots,k\}$ coincides with

$$\begin{aligned}&\sum_{J\subseteq\{1,\cdots,k\}}\sum_{s=0}^{k-|J|}(-1)^{k-|J|-s}\binom{k-|J|}{s}\#J=\sum_{J\subseteq\{1,\cdots,k\}}\sum_{J\subseteq I\subseteq\{1,\cdots,k\}}(-1)^{k-|I|}\#J\\ &=\sum_{I\subseteq\{1,\cdots,k\}}(-1)^{k-|I|}\sum_{J\subseteq I}\#J=\sum_{I\subseteq\{1,\cdots,k\}}(-1)^{k-|I|}\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}\\ &=(-1)^k+\sum_{t=1}^k(-1)^{k-t}\sum_{1\leqslant i_1<\cdots<i_t\leqslant k}\frac{n_{i_1}\cdots n_{i_t}}{[n_{i_1},\cdots,n_{i_t}]}.\end{aligned}$$

In view of Proposition 2.2, the number $\#\{1,\cdots,k\}$ remains the same if an equivalent sequence is substituted for $\{n_s\}_{s=1}^k$. The proof is now complete.

Remark 1. Equation (4) is closely related to diagonal hypersurfaces over a finite field. The formula for the number of solutions of (4) was obtained by R. Lill and H. Niederreiter [LN], R. Stanly (cf. C. Small [Sm]), Q. Sun, D.-Q. Wan and D.-G. Ma [SWM] with much more complicated methods. The fact that the number doesn't vary if we replace $\{n_s\}_{s=1}^k$ by the corresponding normal sequence, was recently noted by A. Granville, S.-G. Li and Q. Sun [GLS]. For necessary and sufficient conditions for the solvability of (4), the reader is referred to [SW] where the authors determined when (4) has a unique solution.

Corollary 2.3. *Let* (1) *be a system of arithmetic sequences with $(n_s,n_t)\mid a_s-a_t$ for all $s,t=1,\cdots,k$. Then for any $\theta\in\mathbb{Q}$ with $0\leqslant\theta<1$ we have*

$$(5)\qquad \Bigg|\sum_{\substack{x_s\in R(n_s)\text{ for } s=1,\cdots,k\\ \{\Sigma_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi i\sum_{s=1}^k\frac{a_sx_s}{n_s}}\Bigg|=\begin{cases}\frac{n_1\cdots n_k}{[n_1,\cdots,n_k]} & \text{if } [n_1,\cdots,n_k]\theta\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By the Chinese Remaider Theorem in general form, $\cap_{s=1}^k a_s+n_s\mathbb{Z}\neq\emptyset$ if and only if $a_s+n_s\mathbb{Z}\cap a_t+n_t\mathbb{Z}\neq\emptyset$ for all $s,t=1,\cdots,k$. (For a proof see, e.g. [Su1].) Since $(n_s,n_t)\mid a_s-a_t$ for $s,t=1,\cdots,k$, $\cap_{s=1}^k a_s+n_s\mathbb{Z}$ must contain an integer x. With the help of Proposition 2.2,

$$\sum_{\substack{x_s\in R(n_s)\text{ for }s=1,\cdots,k\\ \{\Sigma_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi i\sum_{s=1}^k\frac{a_sx_s}{n_s}}=\sum_{\substack{x_s\in R(n_s)\text{ for }s=1,\cdots,k\\ \{\Sigma_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi ix\theta}$$

vanishes if $[n_1,\cdots,n_k]\theta\notin\mathbb{Z}$, and otherwise equals $\frac{n_1\cdots n_k}{[n_1,\cdots,n_k]}e^{2\pi ix\theta}$. So (5) holds.

To conclude this section we say a few words. For system (1), $M(A)=\sup_{x\in\mathbb{Z}}\sigma(x)$ keeps invariant if an equivalent sequence takes the place of $\{n_s\}_{s=1}^k$, because for any $I\subseteq\{1,\cdots,k\}$ the set $\cap_{s\in I}a_s+n_s\mathbb{Z}$ is nonempty if and only if $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$. Observe that (1) forms an exact m-cover of $\mathbb{Z}$ if and only if $\sum_{s=1}^k\frac{1}{n_s}=m\geqslant M(A)$. So whether $n_1,\cdots,n_k$ are the moduli of an exact m-cover of $\mathbb{Z}$ only depends on $\sum_{s=1}^k\frac{1}{n_s}$ and the $\frac{k(k-1)}{2}$ numbers (n_s,n_t), $1\leqslant s<t\leqslant k$. For a given exact m-cover (1) of $\mathbb{Z}$, replacing $\{n_s\}_{s=1}^k$ by the unique normal sequence $\{n'_s\}_{s=1}^k$ equivalent to it we find that $\sum_{s=1}^k\frac{1}{n'_s}\leqslant M(A)\leqslant m=\sum_{s=1}^k\frac{1}{n_s}$. As $n'_s\leqslant n_s$ for $s=1,\cdots,k$, the sequence $\{n_s\}_{s=1}^k$ must be identical with $\{n'_s\}_{s=1}^k$ and hence normal. In the light of the above, the reader should not feel strange to connections between exact m-cover (1) of $\mathbb{Z}$ and the linear form $\sum_{s=1}^k\frac{x_s}{n_s}$.

3. Main Theorems and Their Consequences

In this section we let (1) be an exact m-cover of $\mathbb{Z}$ unless it is specified, we also let $I\subseteq\{1,\cdots,k\}$ and $\bar I=\{1,\cdots,k\}\setminus I$. For any rational c, we let $I^*(c)$ be the number of solutions $\langle x_s\rangle_{s\in I}$ to the diophantine equation

$$\sum_{s\in I}\frac{x_s}{n_s}=c\qquad\text{with }x_s\in R(n_s)\text{ for all }s\in I,\tag{6}$$

and $I_*(c)=|\{J\subseteq I:\ \sum_{s\in J}\frac{1}{n_s}=c\}|$ be the number of solutions $\langle\delta_s\rangle_{s\in I}$ to the equation

$$\sum_{s\in I}\frac{\delta_s}{n_s}=c\qquad\text{with }\delta_s\in R(2)=\{0,1\}\text{ for all }s\in I.\tag{7}$$

(When $I=\emptyset$ and $c=0$ we think that each of (6) and (7) only has the zero solution.) We also set

$$I_*^{(0)}(c)=\left|\left\{J\subseteq I:\ 2\mid|J|\text{ and }\sum_{s\in J}\frac{1}{n_s}=c\right\}\right|\tag{8}$$

and

$$I_*^{(1)}(c) = \left|\left\{J \subseteq I:\ 2 \nmid |J| \text{ and } \sum_{s\in J}\frac{1}{n_s} = c\right\}\right|. \tag{9}$$

Let's present our main theorems whose proofs will be given later, and derive a number of interesting corollaries from them.

Theorem 3.1. *Let c be a rational number.*

(i) *When $|I| \leqslant m$, if $I^*(c-n) = 1$ for a nonnegative integer n then*

$$\bar{I}_*(c) + \sum_{\substack{l=0\\ l\neq n}}^{m-|I|}\binom{m-|I|}{l}I^*(c-l) \geqslant \binom{m-|I|}{n}, \tag{10}$$

in particular, if c can be uniquely written in the form $n+\sum_{s\in I}\frac{x_s}{n_s}$ where n and x_s lie in $\{0,1,\cdots,m-|I|\}$ and $\{0,1,\cdots,n_s-1\}$ respectively, then we have the inequality $\bar{I}_*(c) \geqslant \binom{m-|I|}{n}$.

(ii) *When $|I| \geqslant m$, if $\bar{I}_*(c-n) = 1$ for a nonnegative integer n then*

$$I^*(c) + \sum_{\substack{l=0\\ l\neq n}}^{|I|-m}\binom{|I|-m}{l}\bar{I}_*(c-l) \geqslant \binom{|I|-m}{n}, \tag{11}$$

in particular, if c can be uniquely expressed in the form $n+\sum_{s\in J}\frac{1}{n_s}$ where $J \subseteq \bar{I}$ and $n \in \{0,1,\cdots,|I|-m\}$, then we have the inequality $I^*(c) \geqslant \binom{|I|-m}{n}$.

Below are corollaries involving the cases $|I| \leqslant m$, $|I| = m$ and $|I| \geqslant m$.

Corollary 3.1. *Assume that those n_s with $s \in I$ are pairwise relatively prime. Then $|I| \leqslant m$ and*

$$\left|\left\{J \subseteq \bar{I}:\ \sum_{s\in J}\frac{1}{n_s} = n + \sum_{s\in I}\frac{x_s}{n_s}\right\}\right| \geqslant \binom{m-|I|}{n} \tag{12}$$

for all $n = 0,1,2,\cdots$ and $x_s \in R(n_s)$ with $s \in I$, in particular

$$\left\{\sum_{s\in J}\frac{1}{n_s}:\ J \subseteq \bar{I}\right\} \supseteq \left\{\frac{a}{[n_s]_{s\in I}}:\ a \in \mathbb{Z}\ \&\ |I| \leqslant \frac{a}{[n_s]_{s\in I}} \leqslant m-|I|\right\} \tag{13}$$

and

$$\left|\left\{J \subseteq \bar{I}:\ \sum_{s\in J}\frac{1}{n_s} \equiv \frac{a}{\prod_{s\in I}n_s} \pmod 1\right\}\right| \geqslant 2^{m-|I|} \quad \text{for every } a \in \mathbb{Z}. \tag{14}$$

Proof. By the Chinese Remainder Theorem, $\cap_{s\in I} a_s + n_s\mathbb{Z} \neq \emptyset$ if $I \neq \emptyset$. Since any integer lies in exactly m members of (1), $|I|$ doesn't exceed m. Let $N = [n_s]_{s\in I} = \prod_{s\in I} n_s$. By Corollary 2.1, for each $a \in \mathbb{Z}$ the number $\frac{a}{N}$ can be expressed uniquely in the form $q + \sum_{s\in I} \frac{x_s}{n_s}$ with $q \in \mathbb{Z}$ and $x_s \in R(n_s)$ for $s \in I$. Whenever $x_s \in R(n_s)$ for all $s \in I$, by Theorem 3.1 (12) holds for every nonnegative integer n. If $|I|N \leqslant a \leqslant (m-|I|)N$ then the corresponding integer $q = \frac{a}{N} - \sum_{s\in I}\frac{x_s}{n_s}$ lies in the interval $[0,m-|I|]$ and hence

$$\left|\left\{J \subseteq \bar{I} :\ \sum_{s\in J}\frac{1}{n_s} = \frac{a}{N} = q + \sum_{s\in I}\frac{x_s}{n_s}\right\}\right| \geqslant \binom{m-|I|}{q} > 0.$$

From this follows (13). For (14) we observe that

$$\begin{aligned}&\left|\left\{J \subseteq \bar{I} :\ \sum_{s\in J}\frac{1}{n_s} \equiv \frac{a}{N} \pmod 1\right\}\right|\\ \geqslant &\sum_{n=0}^{m-|I|}\left|\left\{J \subseteq \bar{I} :\ \sum_{s\in J}\frac{1}{n_s} = n + \sum_{s\in I}\frac{x_s}{n_s}\right\}\right|\\ \geqslant &\sum_{n=0}^{m-|I|}\binom{m-|I|}{n} = 2^{m-|I|}.\end{aligned}$$

This concludes the proof.

Applying Corollary 3.1 with $I = \emptyset$ we immediately get the theorem of Sun [Su2].

Putting $I = \{t\}$ $(1 \leqslant t \leqslant k)$ in Corollary 3.1 we then obtain result (I) stated in the first section. In the case $m = 1$, result (I) was first observed by the author in [Su4]. When $m > 1$, we noted in [Su4] that, providing $n_1 < \cdots < n_{k-l} < n_{k-l+1} = \cdots = n_k$ for every $r = 0, 1, \cdots, n_k - 1$, there exists a $J \subseteq \{1, \cdots, k-1\}$ with $\sum_{s\in J}\frac{1}{n_s} \equiv \frac{r}{n_k} \pmod 1$. In [Su4] we even conjectured that, if (1) forms an m-cover of $\mathbb{Z}$ with $\sigma(x) = m$ for all $x \equiv a_t \pmod{n_t}$ where $1 \leqslant t \leqslant k$, then

$$\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\} :\ I \subseteq \{1, \cdots, k\}\setminus\{t\}\right\}\bigcap\frac{1}{n_t}\mathbb{Z} = \left\{\frac{r}{n_t} :\ r = 0, \cdots, n_t - 1\right\}. \tag{15}$$

Result (I) confirms the conjecture for exact m-covers of $\mathbb{Z}$. The lower bounds are best possible as is shown by the following example.

Example. Let $k > m > 0$ be integers. Let $a_s = 0$ and $n_s = 1$ for $s = 1, \cdots, m-1$, $a_s = 2^{s-m}$ and $n_s = 2^{s-m+1}$ for $s = m, \cdots, k-1$, also $a_k = 0$ and $n_k = 2^{k-m}$. It is clear that $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$. As each nonnegative integer can be expressed uniquely in the binary form, the reader can easily check that for $a = 0, 1, 2, \cdots$ and $t = 1, \cdots, k$ we always have

$$\left|\left\{J \subseteq \{1, \cdots, k\}\setminus\{t\} :\ \sum_{s\in J}\frac{1}{n_s} = \frac{a}{n_t}\right\}\right| = \binom{m-1}{[a/n_t]}.$$

Corollary 3.2. *Suppose that $|I| = m$. Then no number occurs exactly once among the $2^{k-m}n_1\cdots n_m$ rationals*

$$\sum_{s\in I}\frac{x_s}{n_s},\ x_s\in R(n_s)\ \textit{for}\ s\in I;\quad \sum_{s\in J}\frac{1}{n_s},\ J\subseteq\bar{I}. \tag{16}$$

Proof. If $I^*(\sum_{s\in I}\frac{x_s}{n_s})=1$ where $x_s\in R(n_s)$ for $s\in I$ then $\bar{I}_*(\sum_{s\in I}\frac{x_s}{n_s})\geqslant\binom{m-|I|}{0}=1$ by part (i) of Theorem 3.1. If $J\subseteq\bar{I}$ and $\bar{I}_*(\sum_{s\in J}\frac{1}{n_s})=1$, then $I^*(\sum_{s\in J}\frac{1}{n_s})\geqslant\binom{|I|-m}{0}=1$ by part (ii) of Theorem 3.1. We are done.

Corollary 3.3. *Assume that $|I|\geqslant m$. For any $J\subseteq\bar{I}$, if*

$$\left|\sum_{s\in J'}\frac{1}{n_s}-\sum_{s\in J}\frac{1}{n_s}\right|\in\{0,1,\cdots,|I|-m\}\quad \textit{for no}\ J'\subseteq\bar{I}\ \textit{with}\ J'\neq J, \tag{17}$$

then

$$I^*\left(n+\sum_{s\in J}\frac{1}{n_s}\right)\geqslant\binom{|I|-m}{n}\qquad \textit{for}\ n=0,1,2,\cdots \tag{18}$$

and hence

$$\prod_{s\in I}n_s\geqslant 2^{|I|-m}[n_s]_{s\in I}. \tag{19}$$

Proof. Let J be a subset of $\bar{I}$ which satisfies (17). Note that $\binom{|I|-m}{n}=0$ for every integer $n>|I|-m$. For $n\in\mathbb{Z}$ with $0\leqslant n\leqslant|I|-m$, if $J'\subseteq\bar{I}$ and $n'\in\{0,1,\cdots,|I|-m\}$ then by (17)

$$n+\sum_{s\in J}\frac{1}{n_s}=n'+\sum_{s\in J'}\frac{1}{n_s}\quad\Longrightarrow\quad J=J'\ \text{and}\ n=n'.$$

So (18) holds in view of the latter part of Theorem 3.1, thus by Proposition 2.2

$$\begin{aligned}\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}\geqslant&\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\ \text{for}\ s\in I\ \&\ \sum_{s\in I}\frac{x_s}{n_s}\equiv\sum_{s\in J}\frac{1}{n_s}\ (\text{mod}\ 1)\right\}\right|\\ \geqslant&\sum_{n=0}^{|I|-m}I^*\left(n+\sum_{s\in J}\frac{1}{n_s}\right)\geqslant\sum_{n=0}^{|I|-m}\binom{|I|-m}{n}=2^{|I|-m}.\end{aligned}$$

Putting $I=\{1,\cdots,k\}$ and $J=\emptyset$ in Corollary 3.3 we obtain the second half of result (III). When $1\leqslant t\leqslant k$ and $n_t>1$, Corollary 3.3 in the case $I=\{1,\cdots,k\}\setminus\{t\}$ and $J=\{t\}$ also yields an interesting result.

Let $p(1)=1$ and $p(n)$ denote the smallest (positive) prime factor of n for $n=2,3,\cdots$. For a positive integer n we also put

$$D(n)=\{\Sigma_{p|n}pm_p:\ m_p\in\mathbb{Z}\ \text{and}\ m_p\geqslant 0\ \text{for any prime divisor}\ p\ \text{of}\ n\}. \tag{20}$$

Theorem 3.2. *Let c be a rational number.*

(i) *When $|I| \leqslant m$, either*

$$\bar{I}_*(c) + \sum_{n=0}^{m-|I|} I^*(c-n) \geqslant p([n_1, \cdots, n_k]) \tag{21}$$

or

$$\bar{I}_*^{(0)}(c) - \bar{I}_*^{(1)}(c) = \sum_{n=0}^{m-|I|} (-1)^n \binom{m-|I|}{n} I^*(c-n), \tag{22}$$

moreover

$$\bar{I}_*(c) + \sum_{n=0}^{m-|I|} \binom{m-|I|}{n} I^*(c-n) \in D([n_1, \cdots, n_k]) \tag{23}$$

if $|S|, |T| \leqslant 1$ and $S \cap T = \emptyset$ where

$$S = \{n \bmod 2 : \ n \in \mathbb{Z}, \ 0 \leqslant n \leqslant m - |I| \text{ and } I^*(c-n) \neq 0\}$$

and

$$T = \left\{ |J| \bmod 2 : \ J \subseteq \bar{I} \text{ and } \sum_{s \in J} \frac{1}{n_s} = c \right\}.$$

(ii) *When $|I| \geqslant m$, either*

$$I^*(c) + \sum_{n=0}^{|I|-m} \bar{I}_*(c-n) \geqslant p([n_1, \cdots, n_k]) \tag{24}$$

or

$$I^*(c) = \sum_{n=0}^{|I|-m} (-1)^n \binom{|I|-m}{n} \left(\bar{I}_*^{(0)}(c-n) - \bar{I}_*^{(1)}(c-n) \right), \tag{25}$$

furthermore

$$I^*(c) + \sum_{n=0}^{|I|-m} \binom{|I|-m}{n} \bar{I}_*(c-n) \in D([n_1, \cdots, n_k]) \tag{26}$$

if $c \neq n + \sum_{s \in J} \frac{1}{n_s}$ for any $n = 0, 1, \cdots, |I| - m$ and $J \subseteq \bar{I}$ with $n \equiv |J| \pmod 2$.

Corollary 3.4. *Let $|I| \leqslant m$ and $J \subseteq \bar{I}$. Suppose that $\sum_{s\in J}\frac{1}{n_s}$ cannot be expressed in the form $n+\sum_{s\in I}\frac{x_s}{n_s}$ where $n\in\{0,1,\cdots,m-|I|\}$ and $x_s\in R(n_s)$ for $s\in I$. Then either $\sum_{s\in J'}\frac{1}{n_s}=\sum_{s\in J}\frac{1}{n_s}$ for at least $p([n_1,\cdots,n_k])$ subsets J' of $\bar{I}$ or the number of such J' is even, either $\sum_{s\in J'}\frac{1}{n_s}=\sum_{s\in J}\frac{1}{n_s}$ for some $J'\subseteq\bar{I}$ with $|J'|\not\equiv|J| \pmod 2$ or $|\{J'\subseteq\bar{I}:\ \sum_{s\in J'}\frac{1}{n_s}=\sum_{s\in J}\frac{1}{n_s}\}|$ can be expressed as the sum of some (not necessarily distinct) prime divisors of $[n_1,\cdots,n_k]$.*

Proof. Let $c=\sum_{s\in J}\frac{1}{n_s}$. As $\bar{I}_*(c)=\bar{I}_*^{(0)}(c)+\bar{I}_*^{(1)}(c)$, and $I^*(c-n)=0$ for every $n=0,1,\cdots,m-|I|$, the desired results follow from the first part of Theorem 3.2.

Remark 2. In the case $I=\emptyset$ Corollary 3.4 was obtained by the author in [Su4].

Corollary 3.5. *Assume that $|I|=m$. Let l be the total number of ways in which rational c is expressed in the form $\sum_{s\in I}\frac{x_s}{n_s}$ or $\sum_{s\in\bar{I}}\frac{\delta_s}{n_s}$ where $x_s\in R(n_s)$ for $s\in I$ and $\delta_s\in\{0,1\}$ for $s\in\bar{I}$. Then*

$$l\geqslant p([n_1,\cdots,n_k]) \quad or \quad l=2\left|\left\{J\subseteq\bar{I}:\ \sum_{s\in J}\frac{1}{n_s}=c\right\}\right|, \tag{27}$$

and l can be written as the sum of finitely many (not necessarily distinct) prime divisors of $n_1,\cdots,n_k$ providing $\sum_{s\in J}\frac{1}{n_s}=c$ for no $J\subseteq\bar{I}$ with $|J|\equiv 0 \pmod 2$.

Proof. Obviously $l=I^*(c)+\bar{I}_*(c)$, and (22) or (25) says that $\bar{I}_*^{(0)}(c)-\bar{I}_*^{(1)}(c)=I^*(c)$, i.e. $l=2\bar{I}_*^{(0)}(c)$. Therefore Theorem 3.2 yields Corollary 3.5.

Corollary 3.6. *Let $|I|\geqslant m$. Suppose that $\sum_{s\in I}\frac{m_s}{n_s}$ cannot be expressed in the form $n+\sum_{s\in J}\frac{1}{n_s}$ with $n\in\{0,1,\cdots,|I|-m\}$ and $J\subseteq\bar{I}$, where $m_s\in R(n_s)$ for each $s\in I$. Then*

$$\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s) \text{ for } s\in I \text{ and } \sum_{s\in I}\frac{x_s}{n_s}=\sum_{s\in I}\frac{m_s}{n_s}\right\}\right| \tag{28}$$

must be a finite sum of (not necessarily distinct) prime divisors of $[n_1,\cdots,n_k]$.

Proof. Let $c=\sum_{s\in I}\frac{m_s}{n_s}$. Note that $\bar{I}_*(c-n)=0$ for each $n=0,1,\cdots,|I|-m$. By the second part of Theorem 3.2, $I^*(c)$ belongs to $D([n_1,\cdots,n_k])$.

Clearly Corollary 3.6 in the case $I=\{1,\cdots,k\}$ gives the first half of result (III).

Theorem 3.3. (i) *When $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, we have*

$$\sum_{n=0}^{m-1}\bar{I}_*\left(n+\frac{r}{[n_s]_{s\in I}}\right)=\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\geqslant\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}} \tag{29}$$

for each $r=0,1,\cdots,[n_s]_{s\in I}-1$.

(ii) *Assume* $|I|=m$ *and let* $0\leqslant\theta<1$. *Provided that* $[n_s]_{s\in I}\theta\notin\mathbb{Z}$ *or* (n_i,n_j) *doesn't divide* a_i-a_j *for some* $i,j\in I$, *either*

$$\sum_{n=0}^{m-1}\bar{I}_*(n+\theta)=\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}\right|\geqslant p\left([n_s]_{s\in\bar{I}}\right) \tag{30}$$

or

$$\left|\left\{J\subseteq\bar{I}:\ 2\mid|J|\ \&\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}\right|=\left|\left\{J\subseteq\bar{I}:\ 2\nmid|J|\ \&\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}\right|$$

and hence

$$\sum_{n=0}^{m-1}\bar{I}_*(n+\theta)=\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}\right|\equiv 0\pmod 2, \tag{31}$$

moreover

$$\sum_{n=0}^{m-1}\bar{I}_*(n+\theta)=\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}\right|\in D\left([n_s]_{s\in\bar{I}}\right) \tag{32}$$

if all the $|J| \bmod 2$ *with* $J\subseteq\bar{I}$ *and* $\{\sum_{s\in J}\frac{1}{n_s}\}=\theta$ *are the same.*

Remark 3. When those n_s with $s\in I$ are pairwise relatively prime, part (i) of Theorem 3.3 yields the lower bound 1 in (29) while (14) gives the bound $2^{m-|I|}$.

Corollary 3.7. *If* $I\neq\emptyset$ *and* $(n_s,n_t)\mid a_s-a_t$ *for all* $s,t\in I$, *then*

$$\prod_{s\in I}n_s\leqslant 2^{k-|I|},\qquad [n_s]_{s\in I}\mid[n_s]_{s\in\bar{I}}, \tag{33}$$

and

$$\left\{\left\{\sum_{s\in J}\frac{1}{n_s}\right\}:\ J\subseteq\bar{I}\right\}\supseteq\left\{0,\frac{1}{[n_s]_{s\in I}},\dots,\frac{[n_s]_{s\in I}-1}{[n_s]_{s\in I}}\right\}. \tag{34}$$

Proof. (34) follows immediately from the first part of Theorem 3.3. Since $\sum_{s\in J}\frac{1}{n_s}\equiv\frac{1}{[n_s]_{s\in I}}\pmod 1$ for some $J\subseteq\bar{I}$, $[n_s]_{s\in I}$ must divide $[n_s]_{s\in\bar{I}}$. For the inequality in (33) we notice that

$$\begin{aligned}2^{k-|I|}&\geqslant\left|\bigcup_{r=0}^{[n_s]_{s\in I}-1}\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\\&=\sum_{r=0}^{[n_s]_{s\in I}-1}\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\\&\geqslant\sum_{r=0}^{[n_s]_{s\in I}-1}\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}=\prod_{s\in I}n_s.\end{aligned}$$

Remark 4. By checking (33) and (34) with I taken to be $\{1,\cdots,m-1,k\}$ and $\{1,\cdots,m-1,k-1,k\}$ in previous Example, we find that Corollary 3.7 is sharp. When (1) forms an exact 1-cover of $\mathbb{Z}$ and $I\subseteq\{1,\cdots,k\}$ contains at least two elements, we cannot have $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$ with $s\neq t$, and (34) fails to hold because for all $J\subseteq\bar I$ we have

$$\sum_{s\in J}\frac{1}{n_s}\leqslant\sum_{s\in\bar I}\frac{1}{n_s}=1-\sum_{s\in I}\frac{1}{n_s}<1-\frac{1}{[n_s]_{s\in I}}=\frac{[n_s]_{s\in I}-1}{[n_s]_{s\in I}}.$$

For any $a,n\in\mathbb{Z}$ with $n>0$, each integer in $a+n\mathbb{Z}$ belongs to exactly m members of (1) and hence $A_{a(n)}=\{b_s+\frac{n_s}{(n,n_s)}\mathbb{Z}\}_{s\in J}$ also forms an exact m-cover of $\mathbb{Z}$ where $J=\{1\leqslant s\leqslant k:\ (n,n_s)\mid a-a_s\}$, $b_s\in\mathbb{Z}$ and $a+b_sn\equiv a_s\pmod{n_s}$ for $s\in J$. Instead of $A=A_{0(1)}$ we may apply our results to $A_{a(n)}$ so as to give more general ones. See [Su4] for examples of such transformations.

4. Characterizations of Exact m-Covers of $\mathbb{Z}$

Theorem 4.1. *Let* (1) *be a system of arithmetic sequences. Let $I\subseteq\{1,\cdots,k\}$ and $\bar I=\{1,\cdots,k\}\setminus I$. If $|I|\leqslant m$ then* (1) *is an exact m-cover of $\mathbb{Z}$ if and only if*

$$\sum_{\substack{J\subseteq\bar I\\ \Sigma_{s\in J}1/n_s=c}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}a_s/n_s}=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c-n}}e^{2\pi i\Sigma_{s\in I}a_sx_s/n_s}\tag{35}$$

is valid for all rational $c\geqslant 0$. When $|I|\geqslant m$, (1) *forms an exact m-cover of $\mathbb{Z}$ if and only if*

$$\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c}}e^{2\pi i\Sigma_{s\in I}a_sx_s/n_s}=\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar I\\ \sum_{s\in J}\frac{1}{n_s}=c-n}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}a_s/n_s}\tag{36}$$

holds for all rational $c\geqslant 0$.

Proof. Put $N=[n_1,\cdots,n_k]$. We assert that (1) forms an exact m-cover of $\mathbb{Z}$ if and only if we have the identity

$$\prod_{s=1}^{k}\left(1-z^{N/n_s}e^{2\pi ia_s/n_s}\right)=\left(1-z^N\right)^m\tag{37}$$

Apparently any zero of the left hand side of (37) is an Nth root of unity. Observe that for every integer x the number $e^{-2\pi ix/N}$ is a zero of the left hand side of (37) with multiplicity m if and only if x lies in $a_s+n_s\mathbb{Z}$ for exact m of $s=1,\cdots,k$. So the assertion follows from Viete's theorem.

Now let us consider the case $|I|\leqslant m$. Clearly the following identities are equivalent:

$$\prod_{s=1}^{k}\left(1-z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)=(1-z^N)^{m-|I|}\prod_{s\in I}\left(1-\left(z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)^{n_s}\right),$$

$$\prod_{s\in\bar{I}}\left(1-z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)=(1-z^N)^{m-|I|}\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{m_s\frac{N}{n_s}}e^{2\pi i m_s\frac{a_s}{n_s}},$$

$$\sum_{J\subseteq\bar{I}}(-1)^{|J|}z^{\Sigma_{s\in J}\frac{N}{n_s}}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}z^{nN}\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{\frac{m_sN}{n_s}}e^{2\pi i\frac{a_sm_s}{n_s}}.$$

By the assertion the first one holds if and only if (1) forms an exact m-cover of $\mathbb{Z}$. Since the third one is valid if and only if (35) is true for every rational $c\geqslant 0$, we get the desired result.

For the case $|I|\geqslant m$, that (1) forms an exact m-cover of $\mathbb{Z}$ is equivalent to any of the identities given below:

$$\prod_{s\in\bar{I}}\left(1-z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)\cdot\prod_{s\in I}\left(1-z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)=(1-z^N)^m,$$

$$\prod_{s\in\bar{I}}\left(1-z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)\cdot\prod_{s\in I}\left(1-\left(z^{\frac{N}{n_s}}e^{2\pi i\frac{a_s}{n_s}}\right)^{n_s}\right)=(1-z^N)^m\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{\frac{m_sN}{n_s}}e^{2\pi i\frac{a_sm_s}{n_s}},$$

$$\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}z^{nN}\sum_{J\subseteq\bar{I}}(-1)^{|J|}z^{\Sigma_{s\in J}\frac{N}{n_s}}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}=\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{\frac{m_sN}{n_s}}e^{2\pi i\frac{a_sm_s}{n_s}}.$$

As the last one holds if and only if (36) does for all rational $c\geqslant 0$, we are done.

Remark 5. In the case $I=\emptyset$ and $c\in\{1,\cdots,m\}$, that (35) holds for any exact m-cover (1) of $\mathbb{Z}$ was first observed by the author in [Su2] with the help of the Riemann zeta function.

The characterization of exact m-cover (1) of $\mathbb{Z}$ given in Theorem 4.1 involves a fixed subset I of $\{1,\cdots,k\}$. Now we present a new one which depends on all the $I\subseteq\{1,\cdots,k\}$ with $|I|=m$.

Theorem 4.2. *Let* (1) *be a system of arithmetic sequences. Then* (1) *forms an exact m-cover of $\mathbb{Z}$ if and only if the relation*

$$(38)\qquad \sum_{\substack{J\in\{1,\cdots,k\}\setminus I\\ \{\Sigma_{s\in J}1/n_s\}=\theta}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}a_s/n_s}=\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \{\Sigma_{s\in I}x_s/n_s\}=\theta}}e^{2\pi i\Sigma_{s\in I}a_sx_s/n_s}$$

holds for all $\theta\in[0,1)$ and $I\subseteq\{1,\cdots,k\}$ with $|I|=m$.

Proof. Let $N=[n_1,\cdots,n_k]$ and $\bar I=\{1,\cdots,k\}\setminus I$ for all $I\subseteq\{1,\cdots,k\}$. First suppose that (1) forms an m-cover of $\mathbb{Z}$. Let x be any integer and I a subset of $\{1,\cdots,k\}$ with $|I|=m$. By taking $z=r^{1/N}e^{2\pi ix/N}$ in (37), we get the equality

$$\prod_{s=1}^{k}\left(1-r^{\frac{1}{n_s}}e^{2\pi i\frac{x+a_s}{n_s}}\right)=(1-r)^m$$

for all $r\geqslant 0$. If $I=\{1\le s\le k:\ n_s\mid x+a_s\}$, then

$$\begin{aligned}&\prod_{s\in\bar I}\left(1-e^{2\pi i\frac{x+a_s}{n_s}}\right)\Big/\prod_{s\in I}\sum_{x_s=0}^{n_s-1}e^{2\pi i\frac{x+a_s}{n_s}x_s}\\ &=\lim_{r\to1}\prod_{s\in\bar I}\left(1-r^{\frac{1}{n_s}}e^{2\pi i\frac{x+a_s}{n_s}}\right)\Big/\prod_{s\in I}\lim_{\bar r\to e^{2\pi i\frac{x+a_s}{n_s}}}\frac{1-\bar r^{n_s}}{1-(\bar r^{n_s})^{1/n_s}}\\ &=\lim_{r\to1}\prod_{s\in\bar I}\left(1-r^{\frac{1}{n_s}}e^{2\pi i\frac{x+a_s}{n_s}}\right)\cdot\prod_{s\in I}\frac{1-r^{1/n_s}}{1-r}\\ &=\lim_{r\to1}(1-r)^{-|I|}\prod_{s=1}^{k}\left(1-r^{\frac{1}{n_s}}e^{2\pi i\frac{x+a_s}{n_s}}\right)=\lim_{r\to1}(1-r)^{-|I|}(1-r)^m=1.\end{aligned}$$

If $I\neq\{1\le s\le k:\ n_s\mid x+a_s\}$, then $n_s\mid x+a_s$ for some $s\in\bar I$ and $n_t\nmid x+a_t$ for some $t\in I$, therefore $\prod_{s\in\bar I}(1-e^{2\pi i\frac{x+a_s}{n_s}})=0=\prod_{t\in I}\sum_{x_t=0}^{n_t-1}e^{2\pi i\frac{x+a_t}{n_t}x_t}$. So we always have the identity

$$(39)\qquad \prod_{s\in\bar I}\left(1-e^{2\pi i\frac{x+a_s}{n_s}}\right)=\prod_{s\in I}\sum_{x_s=0}^{n_s-1}e^{2\pi i\frac{x+a_s}{n_s}x_s}.$$

Next assume (39) for all $x\in\mathbb{Z}$ and $I\subseteq\{1,\cdots,k\}$ with $|I|=m$. For each integer x, if $|\{1\le s\le k:\ n_s\mid x+a_s\}|>m$, then we can choose a proper subset I of $\{1\le s\le k:\ n_s\mid x+a_s\}$ with cardinality m, for which the left hand side of (39) is zero but the right hand side of (39) is nonzero; if $|\{1\le s\le k:\ n_s\mid x+a_s\}|<m$, then we can select an $I\supset\{1\le s\le k:\ n_s\mid x+a_s\}$ with $|I|=m$, for this I the left

hand side of (39) is nonzero while the right hand side of (39) is zero. So (1) forms an exact m-cover of $\mathbb{Z}$.

Now let I be any subset of $\{1,\cdots,k\}$ with $|I|=m$. For every $x\in\mathbb{Z}$,

$$\begin{aligned}\prod_{s\in\bar I}\left(1-e^{2\pi i\frac{x+a_s}{n_s}}\right)&=\sum_{J\subseteq\bar I}(-1)^{|J|}e^{2\pi i\left(\sum_{s\in J}\frac{a_s}{n_s}+x\sum_{s\in J}\frac{1}{n_s}\right)}\\&=\sum_{r=0}^{N-1}e^{2\pi i\frac{rx}{N}}\sum_{\substack{J\subseteq\bar I\\ \{\Sigma_{s\in J}1/n_s\}=r/N}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}\end{aligned}$$

while $\prod_{s\in I}\sum_{x_s=0}^{n_s-1}e^{2\pi i\frac{x+a_s}{n_s}x_s}$ coincides with

$$\sum_{x_s\in R(n_s)\ \text{for}\ s\in I}e^{2\pi i\left(\Sigma_{s\in I}\frac{a_sx_s}{n_s}+x\Sigma_{s\in I}\frac{x_s}{n_s}\right)}=\sum_{r=0}^{N-1}e^{2\pi i\frac{rx}{N}}\sum_{\substack{x_s\in R(n_s)\ \text{for}\ s\in I\\ \{\Sigma_{s\in I}x_s/n_s\}=r/N}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}.$$

If (38) holds for all $\theta\in[0,1)$ then (39) follows from the above for each $x\in\mathbb{Z}$. Conversely, providing (39) for all $x\in\mathbb{Z}$, for any $a=0,1,\cdots,N-1$ we have

$$\begin{aligned}&N\sum_{\substack{J\subseteq\bar I\\ \{\Sigma_{s\in J}1/n_s\}=a/N}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}\\&=\sum_{r=0}^{N-1}\sum_{\substack{J\subseteq\bar I\\ \{\Sigma_{s\in J}1/n_s\}=r/N}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}\sum_{x=0}^{N-1}e^{2\pi i\frac{r-a}{N}x}\\&=\sum_{x=0}^{N-1}e^{-2\pi i\frac{ax}{N}}\left(\sum_{r=0}^{N-1}e^{2\pi i\frac{rx}{N}}\sum_{\substack{J\subseteq\bar I\\ \{\Sigma_{s\in J}1/n_s\}=r/N}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}\right)\\&=\sum_{x=0}^{N-1}e^{-2\pi i\frac{ax}{N}}\left(\sum_{r=0}^{N-1}e^{2\pi i\frac{rx}{N}}\sum_{\substack{x_s\in R(n_s)\ \text{for}\ s\in I\\ \{\Sigma_{s\in I}x_s/n_s\}=r/N}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}\right)\\&=\sum_{r=0}^{N-1}\sum_{\substack{x_s\in R(n_s)\ \text{for}\ s\in I\\ \{\Sigma_{s\in I}x_s/n_s\}=r/N}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}\sum_{x=0}^{N-1}e^{2\pi i\frac{r-a}{N}x}\\&=N\sum_{\substack{x_s\in R(n_s)\ \text{for}\ s\in I\\ \{\Sigma_{s\in I}x_s/n_s\}=a/N}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}},\end{aligned}$$

therefore (38) is valid for every $\theta\in[0,1)$.

Combining the above we obtain the desired result.

5. Proofs of Theorems 3.1–3.3

Proof of Theorem 3.1. (i) Assume $|I| \leqslant m$ and $I^*(c-n)=1$ where n is a nonnegative integer. Let $\langle m_s\rangle_{s\in I}$ be the unique tuple for which $\sum_{s\in I}\frac{m_s}{n_s}=c-n$ and $m_s\in R(n_s)$ for all $s\in I$. Since $\binom{m-|I|}{n}=0$ if $n>m-|I|$, by (35) we have

$$\sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J}\frac{1}{n_s}=c}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}-(-1)^n\binom{m-|I|}{n}e^{2\pi i\Sigma_{s\in I}\frac{a_sm_s}{n_s}}$$
$$=\sum_{\substack{l=0\\ l\neq n}}^{m-|I|}(-1)^l\binom{m-|I|}{l}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c-l}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}.$$

Therefore $\bar I_*(c)+\sum_{\substack{l=0\\ l\neq n}}^{m-|I|}\binom{m-|I|}{l}I^*(c-l)$ is greater than or equal to

$$\Bigg|\sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J}\frac{1}{n_s}=c}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}-\sum_{\substack{l=0\\ l\neq n}}^{m-|I|}(-1)^l\binom{m-|I|}{l}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c-l}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}\Bigg|$$
$$=\left|(-1)^n\binom{m-|I|}{n}e^{2\pi i\Sigma_{s\in I}\frac{a_sm_s}{n_s}}\right|=\binom{m-|I|}{n}.$$

(ii) Now we suppose $|I|\geqslant m$ and $\bar I_*(c-n)=1$ where n is a nonnegative integer. Let I' be the unique subset of $\bar I$ such that $\sum_{s\in I'}\frac{1}{n_s}=c-n$. By (36) we have

$$\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c}}e^{2\pi i\Sigma_{s\in I}\frac{a_sx_s}{n_s}}-(-1)^n\binom{|I|-m}{n}(-1)^{|I'|}e^{2\pi i\Sigma_{s\in I'}\frac{a_s}{n_s}}$$
$$=\sum_{\substack{l=0\\ l\neq n}}^{|I|-m}(-1)^l\binom{|I|-m}{l}\sum_{\substack{J\subseteq\bar I\\ \Sigma_{s\in J}1/n_s=c-l}}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\frac{a_s}{n_s}}.$$

Thus (11) follows.

Lemma. *Let $c_1,\cdots,c_k$ be nonnegative integers and $d_1,\cdots,d_l$ positive ones. Assume that there exist nonzero numbers $z_1,\cdots,z_k$ for which $\sum_{s=1}^k c_sz_s^t=0$ for those positive integers t not divisible by any of $d_1,\cdots,d_l$. Then $c_1+\cdots+c_k$ is the sum of some (not necessarily distinct) numbers among $d_1,\cdots,d_l$.*

This is Lemma 9 of [Su4] and the initial idea is due to Y.-G. Chen.

Proof of Theorem 3.2. Let d be an integer prime to $N=[n_1,\cdots,n_k]$. Since any integer can be written in the form $dx+Ny$ where $x,y\in\mathbb{Z}$, and $dx+Ny\equiv da_s \pmod{n_s}$ if and only if $x\equiv a_s \pmod{n_s}$, $A_d=\{da_s+n_s\mathbb{Z}\}_{s=1}^k$ also forms an exact m-cover of $\mathbb{Z}$. When $|I|\leqslant m$, by applying Theorem 4.1 to A_d we get

$$\sum_{\substack{J\subseteq\bar I\\ \sum_{s\in J}\frac{1}{n_s}=c}}(-1)^{|J|}e^{2\pi i d\Sigma_{s\in J}\frac{a_s}{n_s}}=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c-n}}e^{2\pi i d\Sigma_{s\in I}\frac{a_sx_s}{n_s}},$$

that is, $\sum_{w\in W_1}B_1(c,w)e^{2\pi i dw}$ coincides with zero, where W_1 is the union of sets

$$\left\{\left\{\sum_{s\in J}\frac{a_s}{n_s}\right\}:\ J\subseteq\bar I\ \&\ \sum_{s\in J}\frac{1}{n_s}=c\right\}$$

and

$$\left\{\left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}:\ x_s\in R(n_s)\text{ for }s\in I,\ c-\sum_{s\in I}\frac{x_s}{n_s}\in\{0,1,\cdots,m-|I|\}\right\},$$

and

$$B_1(c,w)=\sum_{\substack{J\subseteq\bar I\\ \Sigma_{s\in J}1/n_s=c\\ \{\Sigma_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}-\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}$$

$$\cdot\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for }s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c-n\ \&\ \left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}=w\right\}\right|$$

for $w\in W_1$. If $|I|\geqslant m$, then by applying Theorem 4.1 to A_d we obtain the equality

$$\begin{aligned}&\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \Sigma_{s\in I}x_s/n_s=c}}e^{2\pi i d\Sigma_{s\in I}\frac{a_sx_s}{n_s}}\\ &=\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar I\\ \sum_{s\in J}\frac{1}{n_s}=c-n}}(-1)^{|J|}e^{2\pi i d\Sigma_{s\in J}\frac{a_s}{n_s}},\end{aligned}$$

i.e., $\sum_{w\in W_2}B_2(c,w)e^{2\pi i dw}=0$, where W_2 is the union of

$$\left\{\left\{\sum_{s\in J}\frac{a_s}{n_s}\right\}:\ J\subseteq\bar I\text{ and }\sum_{s\in J}\frac{1}{n_s}=c-n\text{ for some }n=0,1,\cdots,|I|-m\right\}$$

and

$$\left\{\left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}:\ x_s\in R(n_s)\text{ for }s\in I\text{ and }\sum_{s\in I}\frac{x_s}{n_s}=c\right\},$$

and

$$B_2(c,w)=\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for }s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c\ \&\ \left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}=w\right\}\right|$$
$$-\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar I\\ \Sigma_{s\in J}1/n_s=c-n\\ \{\Sigma_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}$$

for $w\in W_2$.

Case 1. $|I|\leqslant m$. In this case (22) and (23) are obvious if $W_1=\emptyset$. Suppose that W_1 is nonempty. If the inequality

$$\bar I_*(c)+\sum_{n=0}^{m-|I|}I^*(c-n)\geqslant|W_1|\geqslant p(N)$$

fails or $N=1$, then $\sum_{w\in W_1}B_1(c,w)e^{2\pi idw}=0$ for every $d=1,\cdots,|W_1|$, since

$$\left|\left(e^{2\pi idw}\right)_{\substack{1\leqslant d\leqslant|W_1|\\ w\in W_1}}\right|\Big/\prod_{w\in W_1}e^{2\pi iw}$$

is a determinant of Vandermonde's type $B_1(c,w)=0$ for all $w\in W_1$ and hence (22) follows. When $|S|,|T|\leqslant 1$ and $S\cap T=\emptyset$ where S and T are as in Theorem 3.2, there is an $\varepsilon\in\{1,-1\}$ such that

$$\varepsilon B_1(c,w)=|B_1(c,w)|$$
$$=\left|\left\{J\subseteq\bar I:\ \sum_{s\in J}\frac{1}{n_s}=c\ \&\ \left\{\sum_{s\in J}\frac{a_s}{n_s}\right\}=w\right\}\right|+\sum_{n=0}^{m-|I|}\binom{m-|I|}{n}$$
$$\cdot\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for }s\in I,\sum_{s\in I}\frac{x_s}{n_s}=c-n\ \&\ \left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}=w\right\}\right|$$

for every $w\in W_1$, if $N\neq 1$ then $\sum_{w\in W_1}|B_1(c,w)|(e^{2\pi iw})^d=0$ for all positive integers d divisible by none of prime divisors of N and therefore by Lemma

$$\bar I_*(c)+\sum_{n=0}^{m-|I|}\binom{m-|I|}{n}I^*(c-n)=\sum_{w\in W_1}|B_1(c,w)|\in D(N),$$

if $N=1$ then the last equality also holds because $B_1(c,w)=0$ for every $w\in W_1$.

Case 2. $|I|\geqslant m$. Apparently (25) and (26) are valid if $W_2=\emptyset$. Now assume $|W_2|\geqslant 1$. If the equality

$$I^*(c)+\sum_{n=0}^{|I|-m}\bar{I}_*(c-n)\geqslant |W_2|\geqslant p(N)$$

fails or N equals one, then $\sum_{w\in W_2}B_2(c,w)e^{2\pi idw}=0$ for every $d=1,\cdots,|W_2|$, hence $B_2(c,w)=0$ for all $w\in W_2$ and so we have (25). When $c\neq n+\sum_{s\in J}\frac{1}{n_s}$ for any $n=0,1,\cdots,|I|-m$ and $J\subseteq\bar{I}$ with $n\equiv|J|\pmod 2$,

$$\begin{aligned}B_2(c,w)=&\left|\left\{\langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for }s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c\ \&\ \left\{\sum_{s\in I}\frac{a_sx_s}{n_s}\right\}=w\right\}\right|\\&+\sum_{n=0}^{|I|-m}\binom{|I|-m}{n}\left|\left\{J\subseteq\bar{I}:\ \sum_{s\in J}\frac{1}{n_s}=c-n\ \&\ \left\{\sum_{s\in J}\frac{a_s}{n_s}\right\}=w\right\}\right|\end{aligned}$$

for all $w\in W_2$, so with the help of Lemma whether $N=1$ or not (26) always holds.

Proof of Theorem 3.3. Apparently $0\leqslant\sum_{s\in\bar{I}}\frac{1}{n_s}<\sum_{s=1}^k\frac{1}{n_s}=m$.

(i) For part (i) of Theorem 3.3, we first suppose $|I|=m$. Let $r\in R\left([n_s]_{s\in I}\right)$. In the light of Theorem 4.2,

$$\sum_{\substack{J\subseteq\bar{I}\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}\frac{a_s}{n_s}}=\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \left\{\sum_{s\in I}\frac{x_s}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}}}e^{2\pi i\sum_{s\in I}\frac{a_sx_s}{n_s}}.$$

As $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, by Corollary 2.3 the absolute value of the right hand side is equal to $\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}$. So

$$\begin{aligned}&\sum_{n=0}^{m-1}\bar{I}_*\left(n+\frac{r}{[n_s]_{s\in I}}\right)=\left|\left\{J\subseteq\bar{I}:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\\ \geqslant&\left|\sum_{\substack{J\subseteq\bar{I}\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}\frac{a_s}{n_s}}\right|=\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}.\end{aligned}$$

Next we handle the case $|I|\neq m$. Notice that there exists an $x\in\cap_{s\in I}a_s+n_s\mathbb{Z}$. Let $I'=\{1\leqslant s\leqslant k:\ x\equiv a_s\pmod{n_s}\}$. Then $|I'|=m$ and $I'\supset I$. By the previous argument,

$$\left|\left\{J\subseteq\{1,\cdots,k\}\setminus I':\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{a}{[n_s]_{s\in I'}}\right\}\right|\geqslant\frac{\prod_{s\in I'}n_s}{[n_s]_{s\in I'}}$$

for every $a \in R([n_s]_{s\in I'})$. So, for any $r \in R([n_s]_{s\in I})$, we have

$$\begin{aligned}&\left|\left\{J\subseteq \bar I:\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\\ \geqslant&\left|\left\{J\subseteq \{1,\cdots,k\}\setminus I':\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r\frac{[n_s]_{s\in I'}}{[n_s]_{s\in I}}}{[n_s]_{s\in I'}}\right\}\right|\\ \geqslant&\frac{\prod_{s\in I'}n_s}{[n_s]_{s\in I'}}=\frac{\prod_{s\in I}n_s\cdot\prod_{s\in I'\setminus I}n_s}{\left[[n_s]_{s\in I},[n_s]_{s\in I'\setminus I}\right]}\geqslant\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}.\end{aligned}$$

(ii) Now we turn to show the second part. If $[n_s]_{s\in I}\theta \notin \mathbb{Z}$, then $\{\sum_{s\in I}\frac{x_s}{n_s}\}\neq\theta$ whenever $x_s \in R(n_s)$ for all $s \in I$, and thus by Theorem 4.2

$$(*)\qquad \sum_{w\in W}e^{2\pi i w}\sum_{\substack{J\subseteq \bar I\\ \{\Sigma_{s\in J}1/n_s\}=\theta\\ \{\Sigma_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=\sum_{\substack{J\subseteq\bar I\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}\frac{a_s}{n_s}}=0$$

where

$$W=\left\{\left\{\sum_{s\in J}\frac{a_s}{n_s}\right\}:\ J\subseteq\bar I \text{ and } \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta\right\}.$$

If $(n_{s_1},n_{s_2})\nmid a_{s_1}-a_{s_2}$ for some $s_1,s_2\in I$, then $\{a_s+n_s\mathbb{Z}\}_{s\in I}$ covers each integer at most $m-1$ times because $a_{s_1}+n_{s_1}\mathbb{Z}\cap a_{s_2}+n_{s_2}\mathbb{Z}=\emptyset$, therefore system $\{a_s+n_s\mathbb{Z}\}_{s\in\bar I}$ forms a cover of $\mathbb{Z}$ and $(*)$ holds by Theorem 1 of [Su3]. For each integer a prime to $[n_s]_{s\in\bar I}$, by applying the automorphism σ_a of the cyclotomic field $\mathbb{Q}\left(e^{2\pi i/[n_s]_{s\in\bar I}}\right)$ with $\sigma_a\left(e^{2\pi i/[n_s]_{s\in\bar I}}\right)=e^{2\pi i a/[n_s]_{s\in\bar I}}$ we obtain from $(*)$ the equality

$$(*_a)\qquad \sum_{w\in W}\left(e^{2\pi i w}\right)^a\sum_{\substack{J\subseteq \bar I\\ \{\Sigma_{s\in J}1/n_s\}=\theta\\ \{\Sigma_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=0.$$

Observe that $|W|\leqslant|\{J\subseteq\bar I:\ \{\sum_{s\in J}\frac{1}{n_s}\}=\theta\}|=\sum_{n=0}^{m-1}\bar I_*(n+\theta)$. If $0<|W|<p([n_s]_{s\in\bar I})$, then $(*_a)$ holds for every $a=1,\cdots,|W|$, hence

$$\sum_{\substack{J\subseteq \bar I\\ \{\Sigma_{s\in J}1/n_s\}=\theta\\ \{\Sigma_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=0\qquad\text{for all } w\in W$$

and in particular

$$\sum_{\substack{J\subseteq\bar I,\ 2\mid |J|\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta}}1-\sum_{\substack{J\subseteq\bar I,\ 2\nmid |J|\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta}}1=\sum_{\substack{J\subseteq\bar I\\ \left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\theta}}(-1)^{|J|}=0,$$

for, the determinant of matrix $\left((e^{2\pi i w})^a\right)_{\substack{1\leqslant a\leqslant |W|\\ w\in W}}$ is nonzero. In the case $W=\emptyset$ we obviously have the last equality and (32). Assume $W\neq\emptyset$ below. Provided that all the $|J|$ mod 2 with $J\subseteq \bar{I}$ and $\{\sum_{s\in J}\frac{1}{n_s}\}=\theta$ are the same, if $[n_s]_{s\in\bar{I}}=1$ then $\theta=0$ and we must have $\bar{I}=\emptyset$, i.e. $k=|I|=m$, which contradicts that $(n_i,n_j)\nmid a_i-a_j$ for some $i,j\in I$; when $[n_s]_{s\in\bar{I}}>1$, (32) follows from Lemma and the validity of $(*_a)$ for all integers a prime to $[n_s]_{s\in\bar{I}}$. This ends the proof.

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