J. Southeast Univ. (English Edition), 14(1998), no. 2, 106–114.

COVERS WITH LESS THAN 10 MODULI AND THEIR APPLICATIONS

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In this paper we essentially determine all covers $\{a_s(\text{mod } n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with $k < 10$, actually our algorithm is valid for any positive integer k. As an application we provide a somewhat general theorem on (infinite) arithmetic progressions (e.g. $1330319 + 346729110\mathbb{Z}$) consisting of odd integers no term of which can be expressed as a power of two and an odd prime, on the other hand we obtain an interesting result on integers of the form $2^n + cp$ where p is a prime.

Keywords: cover of the integers, arithmetic sequence, integers of the form 2^n+cp.
Chinese Library Subject Classification: O156.1.

1. Introduction

In 1849 A. de Polignac conjectured that any odd integer $m > 1$ is of the form 2^n+p where n is a positive integer and p is either a prime or the number 1. (Actually Euler first mentioned this problem in a letter to Goldbach.) In 1950 P. Erdös [1] proved that there are infinitely many odd positive integers for which the conjecture fails. In fact, he showed that there exists an odd integer a such that any integer

$$x \equiv a \ (\text{mod } 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 31 \cdot 241).$$

cannot be written as the sum of a power of two and a prime. In his ingenious proof he introduced the concept of cover of $\mathbb{Z}$.

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we let

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \ (\text{mod } n)\}.$$

1991 *Mathematics Subject Classification.* Primary 11B25; Secondary 11A07, 11P32.

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The second author is supported by the National Natural Science Foundation of the People's Republic of China and the Return-from-abroad Foundation of the Chinese Educational Committee.

It will be called a residue class (with modulus n) or an arithmetic sequence (with difference n). A finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1}$$

of such sets is said to be a cover (of $\mathbb{Z}$) if each integer lies in at least one of the classes in (1). For a given tuple $\{n_s\}_{s=1}^k$ of positive integers, if there are integer $a_1, \cdots, a_k \in \mathbb{Z}$ such that (1) form a cover of $\mathbb{Z}$ then we call $\{n_s\}_{s=1}^k$ a *covering tuple*. If A is a cover but $\{n_s\}_{s\in I}$ is a covering tuple for no $I \subset \{1, \cdots, k\}$, then we say that cover (1) has order k and call $\{n_s\}_{s=1}^k$ a *minimal covering tuple* with length k.

For covers $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$, we let $C_j = A *_j B$ (where $1 \leqslant j \leqslant k$) be the cover consisting of those $a_s(n_s)$ with $s \neq j$ and those $a_j + b_t n_j(m_t n_j)$ with $1 \leqslant t \leqslant l$, and call C_j a composition of A and B. (Note that $\cup_{t=1}^l a_j + b_t n_j + m_t n_j \mathbb{Z} = a_j + n_j\mathbb{Z}$.) A composition of two covers with more than one moduli is said to be *reducible*. Clearly irreducible covers are of more importance.

Covers of $\mathbb{Z}$ have been investigated for many years, the reader is referred to Erdös [2], R.K. Guy [3] and Š. Porubský [4] for problems and results in this field. Whether a given system (1) forms a cover or not is a co-NP-complete probelm (see [5]), so in general it is extremely hard to make an explicit judgement in polynomial time. It is also not easy to find some new covers with distinct moduli, we recommend the reader to see S.L.G. Choi [6], Jordan [7] and Example 3 of Z.W. Sun [8]. In this paper we aim at determining all covers with order less than 10, and find some applications. In the first section we describe the algorithm and provide its theoretical basis. In section 2 we make use of covers with distinct moduli to provide more arithmetic progressions of odd integers none of which is of the form $2^n + p$ where n is a nonnegative integer and p is a prime, also we will present a result on integers of the form $2^n + cp$. In Appendix we will attach all those irreducible covers $\{a_s(n_s)\}_{s=1}^k$ (up to possible changes of the residues) of order $k < 10$ such that the moduli $n_1, \cdots, n_k$ are distinct and that $\{2, 3, 6\} \not\subseteq \{n_1, \cdots, n_k\}$ if $k = 9$.

1. Description of the Algorithm

Lemma 1. *Let* (1) *be a cover of order* k. *Then for any* $s = 1, \cdots, k$ *we have*

$$k - 1 \geqslant f(N_s) \quad \text{and hence } N_s \leqslant 2^{k-1}, \tag{2}$$

where $N_s = [n_1, \cdots, n_s]$ *is the least common multiple of* $n_1, \cdots, n_s$, *and the Mycielski function* $f : \mathbb{Z}^+ \to \mathbb{Z}$ *is given by*

$$f\left(\prod_{t=1}^r p_t^{\alpha_t}\right) = \sum_{t=1}^r \alpha_t(p_t - 1) \tag{3}$$

where $p_1, \cdots, p_r$ *are distinct primes and* $\alpha_1, \cdots, \alpha_r$ *are nonnegative integers.*

Proof. That $k \geqslant 1 + f(N_k)$ was conjectured by Š. Znám [9] in 1975 and confirmed by R.J. Simpson [10] in 1985. For $s = 1, \cdots, k$ clearly $N_s \mid N_k$ and therefore $k \geqslant 1 + f(N_s)$. It is known that $n \leqslant 2^{k-1}$ providing $f(n) \leqslant k - 1$ (see [11]). We are done.

It follows from Lemma 1 that there are only finitely many covers of order k. We need more properties of covering tuples.

Lemma 2. *Let* (1) *be a cover of* $\mathbb{Z}$ *with* $n_1 \leqslant \cdots \leqslant n_k$. *Assume that* $\sum_{0<i<s} 1/n_i < 1$ *where* $0 \leqslant s < k$. *Then*

$$n_{s+1} \leqslant \frac{k-s}{1 - \sum_{i=1}^{s} 1/n_i}. \tag{4}$$

Proof. It is well known that $\sum_{i=1}^{k} 1/n_i \geqslant 1$ (see, e.g. [12]). Thus

$$1 - \sum_{0<i<s} \frac{1}{n_i} \leqslant \sum_{s<i\leqslant k} \frac{1}{n_i} \leqslant \sum_{s<i\leqslant k} \frac{1}{n_{s+1}} = \frac{k-s}{n_{s+1}}.$$

This concludes the proof.

Let $1 \leqslant s \leqslant k$. Given positive integers $n_1, \cdots, n_s$ we first check that whether $k - 1 \geqslant f(N_s)$. If not then we cannot extend $n_1, \cdots, n_k$ to a covering tuple with length k. If $s < k$ and $k - 1 \geqslant f(N_s)$, then we select n_{s+1} according to Lemma 2 and then check the inequality $k - 1 \geqslant f(N_{s+1})$.

For a given tuple $\{n_s\}_{s=1}^k$ satisfying $k \geqslant 1 + f(N_k)$ and $\sum_{s=1}^k 1/n_s \geqslant 1$, we use the following recent result to make further judgements.

Lemma 3. *Let* (1) *be a cover of order* k *and* $\varepsilon_1, \cdots, \varepsilon_k \in \{1, -1\}$. *Then*

(i) *For any* $I \subseteq \{1, \cdots, k\}$ *there exists a* $J \subseteq \{1, \cdots, k\}$ *with* $J \neq I$ *such that* $\sum_{s\in J} \varepsilon_s/n_s - \sum_{s\in I} \varepsilon_s/n_s \in \mathbb{Z}$.

(ii) *For each* $t = 1, \cdots, k$ *the set*

$$S_t = \left\{ \left\{ \sum_{s\in I} \frac{\varepsilon_s}{n_s} \right\} : \ I \subseteq \{1, \cdots, k\} \setminus \{t\} \right\} \tag{5}$$

contains an arithmetic progression of length n_t *with common difference* $1/n_t$.

Proof. Part (i) was proved in [8] and part (ii) was obtained by the second author in a recent paper [13].

For a covering tuple $\{n_s\}_{s=1}^k$, there may be many choices of the residues $a_1, \cdots, a_k$ such that $\{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$. We simply want to find a tuple $\{a_s\}_{s=1}^k$ of integers for which (1) is a cover of $\mathbb{Z}$. To make the algorithm more efficient we need the following

Lemma 4. *Let* $n_1, \cdots, n_k$ $(k > 1)$ *be positive integers such that* $\{n_s\}_{s=1}^k$ *is a covering tuple. Then there exist integers* $a_1, \cdots, a_k$ *for which* $0 \leqslant a_s < n_s$ *for* $0 < s < k-1$, $0 \leqslant a_{k-1} \leqslant n_{k-1}/2$, $a_k = 0$ *and* (1) *forms a cover of* $\mathbb{Z}$.

Proof. Suppose that $\{b_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ where $b_1, \cdots, b_k$ are integers. For any $x \in \mathbb{Z}$ apparently $\{x + b_s(n_s)\}_{s=1}^k$ also forms a cover of $\mathbb{Z}$. So for some integers $c_1, \cdots, c_{k-1}$ the system $\{c_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ where $c_k = b_k - b_k = 0$. If the least nonnegative residue of c_{k-1} modulo n_{k-1} is not more than $n_{k-1}/2$, then we put $\varepsilon = 1$, otherwise we set $\varepsilon = -1$. For $s = 1, \cdots, k$ let a_s be the smallest nonnegative residue of εc_s mod n_s, it is easy to see that $A = \{a_s(n_s)\}_{s=1}^k$ forms a required cover. This ends the proof.

To check whether a given system (1) forms a cover or not, we use the following result of the second author.

Lemma 5. *Let* (1) *be a system of residue classes. Then (1) forms a cover if and only if it covers* $0, 1, \cdots, W-1$ *where*

$$W = \min_{\varepsilon_1, \cdots, \varepsilon_k \in \{1,-1\}} \left| \left\{ \left\{ \sum_{s \in I} \frac{\varepsilon_s}{n_s} \right\} : I \subseteq \{1, \cdots, k\} \right\} \right|. \tag{6}$$

Proof. This follows from the first part of Theorem 1 of [8]. It is stronger than a conjecture of Erdös proved by R.B. Crittenden and C.L. Vanden Eynden [14].

If a cover C is a composition of covers $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$, then by definition we can find the residues in C according to $a_1, \cdots, a_k, b_1, \cdots, b_l$. In our algorithm, we produce reducible covering tuples from those with less length, and corresponding residues are excluded from our consideration.

For brevity we don't mention other techniques in the algorithm. The program was written in C-language and run in a computer of type Pentium 133. To get the datum in Appendix we have run the computer over 250 hours.

2. Applications to Integers Not of the Form $2^n + cp$

For those covers of order less than 10 with distinct moduli, we find that all moduli are contained in the set S consisting of the following numbers:

$$\{2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 15,\ 16,\ 18,\ 20,\ 24,\ 27,\ 30,$$
$$32,\ 36,\ 40,\ 48,\ 54,\ 60,\ 64,\ 72,\ 80,\ 96,\ 108,\ 144,\ 192\}.$$

In the following tables for each value of $n \in S$ we associate it with a prime p dividing $2^n - 1$:

n	2	3, 6	4	8	9	12	16	18	24	27
p	3	7	5	17	73	13	257	19	241	262657
p mod 31	3	7	5	-14	11	13	9	-12	-7	-6

n	32	36	48	54	64	72	96	108	144	192
p	65537	37	97	87211	641	433	193	246241	109	673
p mod 31	3	6	4	8	-10	-1	7	8	-15	-9

n	5	10	15	16	20	30	40	60	80
p	31	11	151	257	41	331	61681	61	4278255361
p mod 43	-12	11	-21	-1	-2	-13	19	18	-12

Theorem 1. *Let* (1) *be any cover of order* $k < 10$ *with* $n_1 < n_2 < \cdots < n_k$ *and* $\{3, 6\} \not\subseteq \{n_1, \cdots, n_k\}$. *For each* n_s *we assign a prime* p_s *according to the above tables. Let* N *be the least common multiple of* $n_1, \cdots, n_k$. *Then we can effectively find an odd integer* $a \in \mathbb{Z}$ *such that any term in* $a + 2p_0p_1 \cdots p_k\mathbb{Z}$ *cannot be written as a sum of a power of two and a prime where*

$$p_0 = \begin{cases} 31 & \text{if } 5 \nmid N, \\ 43 & \text{otherwise.} \end{cases}$$

Proof. Here we make use of the datum given in Appendix. Let's first consider the case $5 \nmid N$. In this case $p_0 = 31 = 2^5 - 1$. If $N \neq 108$, then for each $s = 1, \cdots, k$ we have $n_s \neq 54, 108$, $p_s \not\equiv 0, 8, 12, 14, 15 \pmod{31}$ and hence $p_s + 2^n \not\equiv 16 \pmod{31}$ for any $n = 0, 1, 2, \cdots$. When $N = 108$, for each $s = 1, \cdots, k$ we have $p_s \not\equiv -2, -3, -5, -9, 14 \pmod{31}$ and hence $p_s + 2^n \not\equiv 30 \pmod{31}$ for any $n = 0, 1, 2, \cdots$. Put $r = 16$ if $N \neq 108$, and $r = 30$ if $N = 108$.

In the case $5 \mid N$, $p_0 = 43$ divides $2^7 + 1$. Since $N \in \{60, 80\}$ and

$$p_s \not\equiv -8, -9, -11, -15, 20, 4, 15, -6, -5, -3, 1, 9, -18, 14 \pmod{43}$$

for $s = 1, \cdots, k$, we have $p_s + 2^n \equiv -7 \equiv 36 \pmod{43}$ for no s and $n \in \mathbb{N}$. We set $r = 36$.

By the Chinese Remainder Theorem, there is a computable integer a such that

$$a \equiv 1 \pmod 2, \ a \equiv r \pmod{p_0}, \ a \equiv 2^{a_s} \pmod{p_s} \ (s = 1, \cdots, k).$$

Let $x \equiv a \pmod{2p_0p_1 \cdots p_k}$ and suppose that $p = x - 2^n$ is a prime for positive integer n. Since (1) forms a cover of $\mathbb{Z}$ there exists an s such that $n \equiv a_s \pmod{n_s}$. Note that $2^{n_s} \equiv 1 \pmod{p_s}$. Thus

$$p = x - 2^n \equiv x - 2^{a_s} \equiv 0 \pmod{p_s},$$

therefore $x = 2^n + p = 2^n + p_s \equiv r \pmod{p_0}$. This contradiction ends our proof.

Example. It can be checked that $A = \{1(2), 2(3), 2(4), 0(8), 4(12), 12(24)\}$ forms a cover of $\mathbb{Z}$. The solution to the system of congruences

$$x \equiv 2^1 \pmod 3,\ x \equiv 2^2 \pmod 7,\ x \equiv 2^2 \pmod 5,\ x \equiv 2^0 \pmod{17},$$
$$x \equiv 2^4 \pmod{13},\ x \equiv 2^{12} \pmod{241},\ x \equiv 1 \pmod 2,\ x \equiv -15 \pmod{31}$$

is $x \equiv 1330319 \pmod{346729110}$. So none of integers in $1330319(346729110)$ can be written in the form $2^n + p$ where n is a nonnegative integer and p is a prime.

With the help of computers, we have searched for positive odd integers not of the form $2^n + p$. Here we list out the first 21 such numbers:

127, 149, 331, 373, 509, 701, 757, 809, 877, 907, 959,
997, 1019, 1087, 1199, 1207, 1211, 1243, 1259, 1271, 1597.

If we take the form $2^n + 3p$ into account, in general we can not get a result similar to Theorem 1 since any cover of order less than 10 with distinct moduli take 2 as its smallest modulus. To make this more explicit we give

Theorem 2. . *Let* (1) *be a minimal cover of* $\mathbb{Z}$ *with distinct moduli. Suppose that for each* $s = 1, \cdots, k$ *prime* p_s *is a primitive factor of* $2^{n_s} - 1$ *(i.e.* $p_s \mid 2^{n_s} - 1$ *but* $p_s \nmid 2^n - 1$ *if* $0 < n < n_s$*). Let the common solution of the congruences* $x \equiv 2^{a_s} \pmod{p_s}$ $(s = 1, \cdots, k)$ *be* $x \equiv a \pmod d$ *where* $a \in \mathbb{Z}$ *and* $d = p_1 \cdots p_k$. *And let* c *be any integer divisible by a unique prime among* $p_1, \cdots, p_k$. *Then exists a positive integer* n *such that* $2^n + cp \in a(d)$ *for infinitely many primes* p.

Remark. A theorem of K. Zsigmondy, rediscovered by G.D. Birkhoff and H.S. Vandiver [15] asserts that for each integer $n > 1$ there exists a primitive prime divisor of $2^n - 1$ unless $n = 6$.

Proof. Assume that $c = p_t^\alpha b$ where $1 \leqslant t \leqslant k$, α is a positive integer and b is an integer divisible by none of $p_1, \cdots, p_k$. Since (1) forms a minimal cover, $a_t(n_t)$ contains an integer n such that $n \not\equiv a_s \pmod{n_s}$ for all $s = 1, \cdots, k$ with $s \neq t$. We may suppose $n > 0$ because each $n' \equiv n \pmod N$ has such a property where $N = [n_1, \cdots, n_k]$.

Let $1 \leqslant s \leqslant k$ and $s \neq t$. For some integer u, v we have $(a_s - n, n_s) = (a_s - n)u + n_s v$. (As usual (m_1, m_2) denotes the greatest common divisor of integers m_1 and m_2.) If $2^{a_s} \equiv 2^n \pmod{p_s}$, then by the above $2^{(a_s - n, n_s)} \equiv 1 \pmod{p_s}$. but $(a_s - n, n_s) < n_s$ since $n \not\equiv a_s \pmod{n_s}$, so this contradicts the choice of p_s. Thus $2^{a_s} - 2^n$ is prime to p_s. By a well-known fact in number theory, the congruence $p_t^\alpha bx \equiv 2^{a_s} - 2^n \pmod{p_s}$ has a unique solution $x \equiv r_s \pmod{p_s}$ where $r_s \in \mathbb{Z}$ and $(r_s, p_s) = 1$.

Put $r_t = 1$. In view of the Chinese Remainder Theorem, the system of congruences $x \equiv r_s \pmod{p_s}$ $(1 \leqslant s \leqslant k)$ has a unique solution $x \equiv r \pmod d$ where $r \in \mathbb{Z}$. Clearly $(r, d) = 1$ since $(r, p_s) = (r_s, p_s) = 1$ for each $s \neq t$. By Dirichlet's theorem there are infinitely many primes p for which $p \equiv r \pmod d$, hence

$$2^n + cp \equiv 2^n + p_t^\alpha b r_s \equiv 2^{a_s} \pmod{p_s}$$

for all $s = 1, \cdots, k$ and so $2^n + cp \in a(d)$. We are done.

Appendix. Datum for covers with less than 10 moduli

In the following k and N stand for the number of moduli and the least common multiple of moduli in a cover of $\mathbb{Z}$.

Case $k=5$: $\{1(2),1(3),2(4),2(6),0(12)\}$.

Case $k=6$:

$$\{1(2),1(3),2(4),2(6),4(8),0(24)\},\{1(2),2(3),2(4),4(8),4(12),0(24)\},$$
$$\{1(2),1(3),2(6),4(8),6(12),0(24)\}.$$

Case $k=7$:

$N=36$:

$$\{1(2),1(3),2(4),2(6),3(9),6(18),0(36)\},\{1(2),2(3),2(4),3(9),4(12),6(18),0(36)\},$$
$$\{1(2),1(3),2(6),3(9),6(12),6(18),0(36)\},\{1(2),2(4),2(6),3(9),4(12),6(18),0(36)\}.$$

$N=48$:

$$\{1(2),1(3),2(4),2(6),4(8),8(16),0(48)\},\{1(2),1(3),2(4),2(6),8(16),12(24),0(48)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),8(16),0(48)\},\{1(2),1(3),2(4),4(8),8(16),8(24),0(48)\},$$
$$\{1(2),2(3),2(4),4(12),8(16),12(24),0(48)\},\{1(2),1(3),2(6),4(8),6(12),8(16),0(48)\},$$
$$\{1(2),1(3),2(6),6(12),8(16),12(24),0(48)\}.$$

Case $k=8$:

$N=96$:

$$\{1(2),1(3),2(4),2(6),4(8),8(16),16(32),0(96)\},$$
$$\{1(2),1(3),2(4),2(6),4(8),16(32),24(48),0(96)\},$$
$$\{1(2),1(3),2(4),2(6),8(16),12(24),16(32),0(96)\},$$
$$\{1(2),1(3),2(4),2(6),12(24),16(32),24(48),0(96)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),8(16),16(32),0(96)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),16(32),24(48),0(96)\},$$
$$\{1(2),1(3),2(4),4(8),8(16),8(24),16(32),0(96)\},$$
$$\{1(2),2(3),2(4),4(8),8(16),16(32),16(48),0(96)\},$$
$$\{1(2),1(3),2(4),4(8),8(24),16(32),24(48),0(96)\},$$
$$\{1(2),2(3),2(4),4(12),8(16),12(24),16(32),0(96)\},$$
$$\{1(2),2(3),2(4),4(12),12(24),16(32),24(48),0(96)\},$$
$$\{1(2),1(3),2(6),4(8),6(12),8(16),16(32),0(96)\},$$
$$\{1(2),1(3),2(6),4(8),6(12),16(32),24(48),0(96)\},$$
$$\{1(2),1(3),2(6),6(12),8(16),12(24),16(32),0(96)\},$$
$$\{1(2),1(3),2(6),6(12),12(24),16(32),24(48),0(96)\}.$$

$N = 72$:

$$\{1(2),1(3),2(4),2(6),4(8),3(9),6(18),0(72)\},$$
$$\{1(2),1(3),2(4),2(6),4(8),6(9),12(36),0(72)\},$$
$$\{1(2),1(3),2(4),2(6),4(8),6(18),12(36),0(72)\},$$
$$\{1(2),1(3),2(4),2(6),3(9),6(18),12(24),0(72)\},$$
$$\{1(2),1(3),2(4),2(6),6(9),12(24),12(36),0(72)\},$$
$$\{1(2),1(3),2(4),2(6),6(18),12(24),12(36),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),4(12),6(18),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),6(9),4(12),12(36),0(72)\},$$
$$\{1(2),2(3),2(4),0(8),3(9),6(18),4(24),0(36)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),6(18),8(24),0(72)\},$$
$$\{1(2),1(3),2(4),4(8),6(9),8(24),12(36),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),6(18),12(36),0(72)\},$$
$$\{1(2),1(3),2(4),4(8),6(18),8(24),12(36),0(72)\},$$
$$\{1(2),2(3),2(4),3(9),4(12),6(18),12(24),0(72)\},$$
$$\{1(2),2(3),2(4),6(9),4(12),12(24),12(36),0(72)\},$$
$$\{1(2),2(3),2(4),4(12),6(18),12(24),12(36),0(72)\},$$
$$\{1(2),1(3),2(6),4(8),3(9),6(12),6(18),0(72)\},$$
$$\{1(2),1(3),2(6),4(8),6(9),6(12),12(36),0(72)\},$$
$$\{1(2),1(3),2(6),2(8),3(9),6(18),6(24),0(36)\},$$
$$\{1(2),1(3),2(6),4(8),3(9),6(18),18(36),0(72)\},$$
$$\{1(2),1(3),2(6),4(8),6(12),6(18),12(36),0(72)\},$$
$$\{1(2),1(3),2(6),3(9),6(12),6(18),12(24),0(72)\},$$
$$\{1(2),1(3),2(6),6(9),6(12),12(24),12(36),0(72)\},$$
$$\{1(2),1(3),2(6),3(9),6(18),12(24),18(36),0(72)\},$$
$$\{1(2),1(3),4(8),8(9),5(9),6(12),2(18),0(24)\},$$
$$\{1(2),2(4),2(6),4(8),3(9),4(12),6(18),0(72)\},$$
$$\{1(2),2(4),2(6),4(8),6(9),4(12),12(36),0(72)\},$$
$$\{1(2),2(4),2(6),0(8),3(9),6(18),4(24),0(36)\},$$
$$\{1(2),2(4),4(6),4(8),3(9),6(18),8(24),0(72)\},$$
$$\{1(2),2(4),4(6),4(8),6(9),8(24),12(36),0(72)\},$$
$$\{1(2),2(4),2(6),3(9),4(12),6(18),12(24),0(72)\},$$
$$\{1(2),2(4),2(6),6(9),4(12),12(24),12(36),0(72)\},$$
$$\{1(2),2(4),0(8),3(9),8(12),6(18),4(24),0(36)\},$$
$$\{1(2),2(4),4(8),3(9),4(12),6(18),8(24),0(72)\},$$
$$\{1(2),2(4),4(8),6(9),4(12),8(24),12(36),0(72)\}.$$

Case $k = 9$:

$N = 60$:

$$\{1(2), 1(3), 0(4), 1(5), 4(10), 2(12), 3(15), 2(20), 0(30)\},$$
$$\{1(2), 2(3), 2(4), 1(5), 2(10), 4(12), 3(15), 4(20), 0(60)\},$$
$$\{1(2), 2(3), 2(4), 4(5), 2(10), 4(12), 3(15), 6(30), 0(60)\},$$
$$\{1(2), 2(3), 2(4), 3(5), 2(10), 4(12), 4(20), 6(30), 0(60)\},$$
$$\{1(2), 2(3), 2(4), 2(5), 4(12), 3(15), 4(20), 6(30), 0(60)\},$$
$$\{1(2), 2(3), 2(4), 2(10), 4(12), 3(15), 4(20), 6(30), 0(60)\},$$
$$\{1(2), 0(4), 1(5), 4(6), 4(10), 2(12), 3(15), 2(20), 0(30)\},$$
$$\{1(2), 2(4), 1(5), 2(6), 2(10), 4(12), 3(15), 4(20), 0(60)\},$$
$$\{1(2), 2(4), 4(5), 2(6), 2(10), 4(12), 3(15), 6(30), 0(60)\},$$
$$\{1(2), 2(4), 3(5), 2(6), 2(10), 4(12), 4(20), 6(30), 0(60)\},$$
$$\{1(2), 2(4), 2(5), 2(6), 4(12), 3(15), 4(20), 6(30), 0(60)\}.$$

$N = 80$: $\{1(2), 2(4), 1(5), 4(8), 2(10), 8(16), 4(20), 8(40), 0(80)\}$.

$N = 108$:

$$\{1(2), 2(3), 2(4), 6(9), 4(12), 9(27), 12(36), 18(54), 0(108)\},$$
$$\{1(2), 2(3), 2(4), 4(12), 6(18), 9(27), 12(36), 18(54), 0(108)\},$$
$$\{1(2), 2(4), 2(6), 6(9), 4(12), 9(27), 12(36), 18(54), 0(108)\}.$$

$N = 192$:

$$\{1(2), 2(3), 2(4), 4(8), 4(12), 16(32), 24(48), 32(64), 0(192)\},$$
$$\{1(2), 1(3), 2(4), 4(8), 8(16), 8(24), 16(32), 32(64), 0(192)\},$$
$$\{1(2), 1(3), 2(4), 4(8), 8(16), 8(24), 32(64), 48(96), 0(192)\},$$
$$\{1(2), 1(3), 2(4), 4(8), 8(24), 16(32), 24(48), 32(64), 0(192)\},$$
$$\{1(2), 2(3), 2(4), 4(12), 8(16), 12(24), 16(32), 32(64), 0(192)\},$$
$$\{1(2), 2(3), 2(4), 4(12), 8(16), 12(24), 32(64), 48(96), 0(192)\},$$
$$\{1(2), 2(3), 2(4), 4(12), 12(24), 16(32), 24(48), 32(64), 0(192)\}.$$

$N = 144$:

$$\{1(2),2(3),2(4),4(8),3(9),4(12),8(16),6(18),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),6(9),4(12),8(16),12(36),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),4(12),8(16),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),4(12),6(18),24(48),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),6(9),4(12),12(36),24(48),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),4(12),24(48),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),8(16),6(18),8(24),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),7(9),8(16),10(18),4(36),0(48)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),0(16),6(18),8(48),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),8(16),6(18),16(48),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),6(9),8(16),8(24),12(36),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),8(16),8(24),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),6(9),0(16),12(36),8(48),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),6(9),8(16),12(36),16(48),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),3(9),8(16),16(48),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),6(18),8(24),24(48),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),6(9),8(24),12(36),24(48),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),3(9),8(24),24(48),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),8(16),6(18),12(36),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),8(16),12(18),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),8(16),12(36),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),4(12),6(18),12(36),24(48),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),8(16),6(18),8(24),12(36),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),8(16),12(18),8(24),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),0(16),6(18),12(36),8(48),0(72)\},$$
$$\{1(2),2(3),2(4),4(8),8(16),6(18),12(36),16(48),0(144)\},$$
$$\{1(2),2(3),2(4),4(8),8(16),12(18),16(48),24(72),0(144)\}.$$
$$\{1(2),2(3),2(4),4(8),8(16),12(36),16(48),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),6(18),8(24),12(36),24(48),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),12(18),8(24),24(48),24(72),0(144)\},$$
$$\{1(2),1(3),2(4),4(8),8(24),12(36),24(48),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),3(9),4(12),8(16),6(18),12(24),0(144)\},$$
$$\{1(2),2(3),2(4),3(9),4(12),4(16),6(18),12(48),0(72)\},$$
$$\{1(2),2(3),2(4),3(9),4(12),8(16),6(18),36(72),0(144)\},$$
$$\{1(2),2(3),2(4),6(9),4(12),8(16),12(24),12(36),0(144)\},$$
$$\{1(2),2(3),2(4),3(9),4(12),8(16),12(24),24(72),0(144)\},$$
$$\{1(2),2(3),2(4),6(9),4(12),4(16),12(36),12(48),0(72)\},$$
$$\{1(2),2(3),2(4),6(9),4(12),8(16),12(36),36(72),0(144)\}.$$

$\{1(2), 2(3), 2(4), 3(9), 4(12), 6(18), 12(24), 24(48), 0(144)\}$,
$\{1(2), 2(3), 2(4), 3(9), 4(12), 6(18), 24(48), 36(72), 0(144)\}$,
$\{1(2), 2(3), 2(4), 7(9), 8(16), 10(18), 12(24), 4(36), 0(48)\}$,
$\{1(2), 2(3), 2(4), 4(12), 8(16), 6(18), 12(24), 12(36), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 8(16), 12(18), 12(24), 24(72), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 4(16), 6(18), 12(36), 12(48), 0(72)\}$,
$\{1(2), 2(3), 2(4), 4(12), 8(16), 6(18), 12(36), 36(72), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 8(16), 12(24), 12(36), 24(72), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 6(18), 12(24), 12(36), 24(48), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 12(18), 12(24), 24(48), 24(72), 0(144)\}$,
$\{1(2), 2(3), 2(4), 4(12), 6(18), 12(36), 24(48), 36(72), 0(144)\}$,
$\{1(2), 2(4), 2(6), 4(8), 3(9), 4(12), 8(16), 6(18), 0(144)\}$,
$\{1(2), 2(4), 2(6), 4(8), 6(9), 4(12), 8(16), 12(36), 0(144)\}$,
$\{1(2), 2(4), 2(6), 4(8), 3(9), 4(12), 8(16), 24(72), 0(144)\}$,
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$\{1(2), 2(4), 2(6), 4(8), 3(9), 4(12), 24(48), 24(72), 0(144)\}$,
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$\{1(2), 2(4), 2(6), 4(8), 7(9), 8(16), 10(18), 4(36), 0(48)\}$,
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$\{1(2), 2(4), 2(6), 4(8), 3(9), 8(16), 6(18), 16(48), 0(144)\}$,
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$\{1(2), 2(4), 2(6), 7(9), 8(16), 10(18), 12(24), 4(36), 0(48)\}$.

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$$\{1(2), 2(4), 4(8), 7(9), 8(12), 8(16), 10(18), 4(36), 0(48)\},$$
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$$\{1(2), 2(4), 4(8), 3(9), 8(12), 8(16), 6(18), 16(48), 0(144)\},$$
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$$\{1(2), 2(4), 7(9), 8(12), 8(16), 10(18), 12(24), 4(36), 0(48)\}.$$

References

1. P.Erdös, *On integers of the form $2^k + p$ and some related problems*, Summa Brasil. Math., 1950, vol. 2, 113–123.
2. P.Erdös, *Problems and results in number theory*, in: H. Halberstam and C. Holley, eds., Recent Progress in Analytic Number Theory, vol. 1, Academic Press, New York, 1981, pp. 1–13.
3. R. K. Guy, *Unsolved Problems in Number Theorey* (2nd, ed.), Springer-Verlag, New York, 1994, pp. 251–256.
4. Š.Porubský, *Results and Problems on Covering Systems of Residue Classes*, Mitt. Math. Sem. Giessen, Heft 150, Giessen Univ., 1981, pp.1–85.
5. S.P. Tung, *Complexity of sentences over number rings*, SIAM J. Comp., 1991, vol. 20, 126–143.
6. S.L.G. Choi, *Covering the set of integers by congruence classes of distinct moduli*, Math. Comput., 1971, vol.25, 885–895.
7. J.H. Jordan, *Covering classes of residues*, Canad. J. Math., 1967, vol.19, 514–519.
8. Zhi-Wei Sun, *Covering the integers by arithmetic sequences* II, Trans. Amer. Math. Soc., 1996, 348(11), 4279–4320.
9. Š.Znám On properties of systems of arithmetic sequences, Acta Arith., 1975, vol. 26, 279–283.
10. R.J. Simpson, *Regular coverings of the integers by arithmetic progressions*, Acta Arith., 1985, vol.45, 145–152.
11. Zhi-Wei Sun, *Finite coverings of groups*, Fund. Math., 1990, 134(1): 37–53.
12. Zhi-Wei Sun, *Covering the integers by arithmetic sequences*, Acta Arith., 1995, 72(2), 109–129.

13. Zhi-Wei Sun, *On covering multiplicity*, Proc. Amer. Math. Soc., 127(1999), in press.
14. R. B. Crittenden and C. L. Vanden Eynden, *Any n arithmetic progressions covering the first 2^n integers cover all integers*, Proc. Amer. Math. Soc., 1970, vol.24, 475–481.
15. G.D. Birkhoff and H.S. Vandiver, *On the integral divisors of $a^n - b^n$*, Ann. Math., 1904, vol.5, 173–180.