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# ON COVERING MULTIPLICITY

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ABSTRACT. Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be a system of arithmetic sequences which forms an $m$-cover of $\mathbb{Z}$ (i.e. every integer belongs at least to $m$ members of $A$). In this paper we show the following surprising properties of $A$: (a) For each $J\subseteq\{1,\cdots,k\}$ there exist at least $m$ subsets $I$ of $\{1,\cdots,k\}$ with $I\neq J$ such that $\sum_{s\in I}1/n_s-\sum_{s\in J}1/n_s\in\mathbb{Z}$. (b) If $A$ forms a minimal $m$-cover of $\mathbb{Z}$, then for any $t=1,\cdots,k$ there is an $\alpha_t\in[0,1)$ such that for every $r=0,1,\cdots,n_t-1$ there exists an $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ for which $[\sum_{s\in I}1/n_s]\geqslant m-1$ and $\{\sum_{s\in I}1/n_s\}=(\alpha_t+r)/n_t$.

## 1. INTRODUCTION

For integer $a$ and positive integer $n$ we call

$$a(n)=\{x\in\mathbb{Z}:\ x\equiv a\ (\mathrm{mod}\ n)\}=a+n\mathbb{Z}$$

an *arithmetic sequence* with *common difference* $n$ or a *residue class* with *modulus* $n$. For a finite system

$$A=\{a_s(n_s)\}_{s=1}^k \tag{1}$$

of such sets, we define its *covering multiplicity* by

$$m(A)=\inf_{x\in\mathbb{Z}}|S(x)| \tag{2}$$

where $S(x)=\{1\leqslant s\leqslant k:\ x\equiv a_s\ (\mathrm{mod}\ n_s)\}$. It is easy to show that

$$\sum_{s=1}^k\frac{1}{n_s}\geqslant m(A), \tag{3}$$

and the equality holds if and only if (1) covers each integer exactly $m$ times for some $m=1,2,3,\cdots$. (Cf. [S2,S4].)

Let $m$ be a nonnegative integer. If system (1) has covering multiplicity at least $m$ then we call (1) an *$m$-cover* (of $\mathbb{Z}$). A *minimal $m$-cover* (of $\mathbb{Z}$) is an $m$-cover

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whose proper subsystems are not. If $|S(x)| = m$ for all $x \in \mathbb{Z}$ then we say that $A$ forms an *exact $m$-cover* (of $\mathbb{Z}$). Notice that an exact 1-cover is a partition of $\mathbb{Z}$ into (finitely many) periodic sets. The Chinese Remainder Theorem tells that the intersection of residue classes $a_1(n_1), \cdots, a_k(n_k)$ is empty if and only if two of them are disjoint. So, as a dual question, when (1) forms a 1-cover is fundamental and important. In fact, 1-covers and exact $m$-covers (especially exact 1-covers) have been investigated for many years, also some famous conjectures remain open. (See R. K. Guy [G].)

Now we introduce some notations. As usual, if $m$ and $n$ are integers then $(m, n)$ represents the greatest common divisor of $m$ and $n$. For a real number $x$, we set $\binom{x}{0} = 1$ and let $\binom{x}{n} = \prod_{j=0}^{n-1} \frac{x-j}{n-j}$ for $n = 1, 2, 3, \cdots$, also $[x]$ and $\{x\}$ denote the integral and the fractional parts of $x$ respectively.

In this paper we study the covering multiplicity of a general system of residue classes. Our main result is as follows.

**Theorem 1.** *Let* (1) *be a system of arithmetic sequences, and $J$ a subset of $\{1, \cdots, k\}$. Put $J^- = \{1, \cdots, k\} \setminus J$.*

(i) *For any $m_1, \cdots, m_k \in \mathbb{Z}$ we have*

$$(4) \qquad \left|\left\{I \subseteq \{1, \cdots, k\} : \ I \neq J \ \& \ \left\{\sum_{s \in I} \frac{m_s}{n_s}\right\} = \left\{\sum_{s \in J} \frac{m_s}{n_s}\right\}\right\}\right| \geqslant m(A).$$

(ii) *Suppose $\emptyset \neq J \subseteq S(x)$ for some $x \in \mathbb{Z}$ with $|S(x)| = m(A)$. For each $s \in J^-$ let $m_s$ be a positive integer prime to $n_s$. Then there exists an $\alpha \in [0, 1)$ such that*

$$(5) \qquad \begin{aligned} &\left\{\left\{\sum_{s \in I} \frac{m_s}{n_s}\right\} : I \subseteq J^-, \ \left[\sum_{s \in I} \frac{m_s}{n_s}\right] \geqslant m(A) - |J|\right\} \\ &\supseteq \left\{\frac{a}{N(J)} : 0 \leqslant a < N(J), \ \{a\} = \alpha\right\}, \end{aligned}$$

*where $N(J)$ denotes the least common multiple of those $n_s$ with $s \in J$.*

In view of Theorem 1, an $m$-cover $A = \{a_s(n_s)\}_{s=1}^k$ possesses the following properties:

(a) For each $J \subseteq \{1, \cdots, k\}$, there exist at least $m$ subsets $I$ of $\{1, \cdots, k\}$ with $I \neq J$ such that $\sum_{s \in I} 1/n_s - \sum_{s \in J} 1/n_s \in \mathbb{Z}$.

(b) If $A$ forms a minimal $m$-cover of $\mathbb{Z}$, then for any $t = 1, \cdots, k$ there is an $\alpha_t \in [0, 1)$ such that, for every $r = 0, 1, \cdots, n_t - 1$, there exists an $I \subseteq \{1, \cdots, k\} \setminus \{t\}$ for which $[\sum_{s \in I} 1/n_s] \geqslant m - 1$ and $\{\sum_{s \in I} 1/n_s\} = (\alpha_t + r)/n_t$.

Part (i) of Theorem 1 can be strengthened in the case $J = \emptyset$. By Theorem 1, if (1) forms a 1-cover then $\sum_{s \in I} 1/n_s \in \mathbb{Z}$ for some nonempty subset $I$ of $\{1, \cdots, k\}$, which is the main result of M. Z. Zhang [Z] obtained by means of the Riemann zeta function. For an exact $m$-cover (1), the author proved in [S1] that for each $n = 0, 1, \cdots, m$ there exist at least $\binom{m}{n}$ subsets $I$ of $\{1, \cdots, k\}$ with $\sum_{s \in I} 1/n_s = n$.

When (1) is an $m$-cover and $m_1,\cdots,m_k$ are positive integers, it was shown in [S3] that there are at least $m$ positive integers in the form $\sum_{s\in I} m_s/n_s$ where $I\subseteq\{1,\cdots,k\}$, we even conjecture that there exist nonempty subsets $I_1,\cdots,I_m$ of $\{1,\cdots,k\}$ for which $I_1\subset\cdots\subset I_m$ and $\sum_{s\in I_t} m_s/n_s\in\mathbb{Z}$ for all $t=1,\cdots,m$.

The first part of Theorem 1 yields

**Corollary 1.** *Let* (1) *be an $m$-cover of $\mathbb{Z}$ and $m_1,\cdots,m_k$ any integers. Then*

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\leqslant\frac{2^k}{m+1}. \tag{6}$$

Proof. By part (i) of Theorem 1, for any $J\subseteq\{1,\cdots,k\}$ there are at least $m+1$ subsets $I$ of $\{1,\cdots,k\}$ with $\{\sum_{s\in I} m_s/n_s\}=\{\sum_{s\in J} m_s/n_s\}$. Since $\{1,\cdots,k\}$ has exactly $2^k$ subsets, Corollary 1 follows immediately.

*Remark* 1. A conjecture of P. Erdös proved by R. B. Crittenden and C. L. Vanden Eynden [CV] states that (1) forms a 1-cover of $\mathbb{Z}$ if it covers $1,\cdots,2^k$. In [S2,S3] the author showed that (1) forms an $m$-cover of $\mathbb{Z}$ if there exist $W$ consecutive integers each of which lies in at least $m$ members of (1), where $W$ is the least integer equal to the left hand side of (6) for some integers $m_1,\cdots,m_k$ prime to $n_1,\cdots,n_k$ respectively.

As for part (ii) of Theorem 1 we should mention the following result obtained by the author ([S4]) recently: Let (1) be an exact $m$-cover of $\mathbb{Z}$ , and $J$ a nonempty subset of $\{1,\cdots,k\}$ with $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in J$ (i.e. $\emptyset\neq J\subseteq S(x)$ for some $x\in\mathbb{Z}$). Then

$$\left|\left\{I\subseteq J^-:\left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{a}{N(J)}\right\}\right|\geqslant\frac{\prod_{s\in J}n_s}{N(J)}$$

for every $a=0,1,\cdots,N(J)-1$, and

$$\left|\left\{I\subseteq J^-:\sum_{s\in I}\frac{1}{n_s}=\frac{a}{N(J)}\right\}\right|\geqslant\binom{m-1}{[a/N(J)]}$$

for all $a=0,1,2,\cdots$ if $|J|=1$.

**Corollary 2.** *Let* (1) *be an $m$-cover of $\mathbb{Z}$ with $n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$. Suppose that $B=\{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an $m$-cover of $\mathbb{Z}$. If $\sum_{s=1}^{k-1}1/n_s=m$, then $n_{k-1}=n_k>1$ and*

$$\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k-1\}\right\}\supseteq\left\{\frac{r}{n_k}:\ r=0,1,\cdots,n_k-1\right\}. \tag{7}$$

Proof. Assume that $\sum_{s=1}^{k-1}1/n_s=m$. By part (ii) of Theorem 1 there exists an $\alpha\in[0,1)$ such that

$$\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:I\subseteq\{1,\cdots,k-1\},\ \left[\sum_{s\in I}\frac{1}{n_s}\right]\geqslant m-1\right\}\supseteq\left\{\frac{a}{n_k}:0\leqslant a<n_k,\ \{a\}=\alpha\right\}.$$

Thus

$$\begin{aligned}&\bigg\{\sum_{s\in J}\frac{1}{n_s}:\ J\subseteq\{1,\cdots,k-1\},\ \sum_{s\in J}\frac{1}{n_s}\notin\mathbb{Z}\bigg\}\\ \supseteq&\bigg\{\sum_{s=1}^{k-1}\frac{1}{n_s}-\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k-1\},\ m-1<\sum_{s\in I}\frac{1}{n_s}<m=\sum_{s=1}^{k-1}\frac{1}{n_s}\bigg\}\\ =&\bigg\{1-\bigg\{\sum_{s\in I}\frac{1}{n_s}\bigg\}:\ I\subseteq\{1,\cdots,k-1\},\ \bigg[\sum_{s\in I}\frac{1}{n_s}\bigg]\geqslant m-1\bigg\}\setminus\{1\}\\ \supseteq&\bigg\{1-\frac{a}{n_k}:\ 0\leqslant a<n_k,\ \{a\}=\alpha\bigg\}\setminus\{1\}=\bigg\{\frac{b}{n_k}:\ 0<b<n_k,\ \{b\}=\{-\alpha\}\bigg\}.\end{aligned}$$

Observe that (7) follows if $\alpha=0$. Since $B$ doesn't form an $m$-cover of $\mathbb{Z}$, we cannot have $n_1=\cdots=n_{k-1}=1$ (otherwise $k-1=\sum_{s=1}^{k-1}1/n_s=m$). So $n_k\geqslant n_{k-1}>1$, hence by the above for some nonempty $J\subseteq\{1,\cdots,k-1\}$ we have

$$\frac{1}{n_{k-1}}\leqslant\min_{s\in J}\frac{1}{n_s}\leqslant\sum_{s\in J}\frac{1}{n_s}=\frac{1-\alpha}{n_k}\leqslant\frac{1}{n_k}\leqslant\frac{1}{n_{k-1}}.$$

Therefore $n_k=n_{k-1}$ and $\alpha=0$. We are done.

*Remark* 2. Let (1) be an $m$-cover of $\mathbb{Z}$ with $n_1\leqslant\cdots\leqslant n_{k-1}<n_k$. By part (iv) of Theorem I of [S3], $\sum_{s=1}^{k-1}1/n_s\geqslant m$. In view of Corollary 2, if $\{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an $m$-cover of $\mathbb{Z}$, then $\sum_{s=1}^{k-1}1/n_s$ must be greater than $m$. This extends and improves a confirmed conjecture of Erdös which states that $\sum_{s=1}^{k}1/n_s>1$ for any 1-cover (1) with $1<n_1<\cdots<n_{k-1}<n_k$ (see [E] and [G]).

**Corollary 3.** *Let* (1) *be an $m$-cover of $\mathbb{Z}$, and $J$ a nonempty subset of $\{1,\cdots,k\}$ with $|\{s\in J^-:\ x\in a_s(n_s)\}|=m-|J|$ for some $x\in\mathbb{Z}$. Let $\varepsilon_s\in\{1,-1\}$ for those $s\in J^-$. Then*

$$\bigg|\bigg\{\bigg\{\sum_{s\in I}\frac{\varepsilon_s}{n_s}\bigg\}:\ I\subseteq J^-\bigg\}\bigg|\geqslant N(J).\tag{8}$$

Proof. This follows immediately from the second part of Theorem 1.

*Remark* 3. With the help of a local-global result proved in [S2], in 1994 the author found Corollary 3 in the case $|J|=1$ (see Section 3 of [S3]).

**Corollary 4.** *Let* (1) *be a minimal $m$-cover of $\mathbb{Z}$, and $m_1,\cdots,m_k$ any positive integers prime to $n_1,\cdots,n_k$ respectively. Then for every $t=1,\cdots,k$ all the numbers $0,1/n_t,\cdots,(n_t-1)/n_t$ lie in the set*

$$\bigg\{\bigg\{\sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\bigg\}:\ I,J\subseteq\{1,\cdots,k\}\setminus\{t\}\ \&\ \sum_{s\in I}\frac{m_s}{n_s},\sum_{s\in J}\frac{m_s}{n_s}\geqslant m-1\bigg\}.\tag{9}$$

Proof. By part (ii) of Theorem 1 there is an $\alpha_t \in [0,1)$ such that

$$\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\setminus\{t\},\ \left[\sum_{s\in I}\frac{m_s}{n_s}\right]\geqslant m-1\right\}$$

contains $S_t=\{a/n_t:0\leqslant a<n_t,\ \{a\}=\alpha_t\}$. As $r/n_t=(\alpha_t+r)/n_t-\alpha_t/n_t$ for each $r=0,1,\cdots,n_t-1$, the desired result follows.

*Remark* 4. In [S3] the author was able to prove Corollary 4 with $\sum_{s\in J}m_s/n_s\geqslant m-1$ in (9) replaced by $\sum_{s\in J}m_s/n_s\geqslant m-2$.

## 2. Proof of Theorem 1

Let's recall a key result given by the author in [S2].

**Proposition 1.** *Let $\mathcal{A}=\{\alpha_s+\beta_s\mathbb{Z}\}_{s=1}^k$ where $\alpha_1,\cdots,\alpha_k$ are real numbers and $\beta_1,\cdots,\beta_k$ are positive reals. Let $m$ be a positive integer. Then $\mathcal{A}$ forms an $m$-cover of $\mathbb{Z}$ (i.e. $|\{1\leqslant s\leqslant k:\ (x-\alpha_s)/\beta_s\in\mathbb{Z}\}|\geqslant m$ for all $x\in\mathbb{Z}$) if and only if*

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \{\sum_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\sum_{s\in I}\alpha_s/\beta_s}=0 \tag{10}$$

*holds for all $\theta\in[0,1)$ and $n=0,1,\cdots,m-1$.*

**Lemma 1.** *Let $k,m,n$ be positive integers with $k>m-n\geqslant 0$. Then* (1) *forms an $m$-cover of $\mathbb{Z}$ if and only if for each $I\subseteq\{1,\cdots,k\}$ with $|I|=m-n$ system $A_I=\{a_s(n_s)\}_{s\in I^-}$ forms an $n$-cover of $\mathbb{Z}$.*

Proof. If (1) is an $m$-cover of $\mathbb{Z}$ and $I$ is a subset of $\{1,\cdots,k\}$ with $|I|=m-n$, then for any integer $x$ we have

$$|\{s\in I^-:\ x\equiv a_s\ (\mathrm{mod}\ n_s)\}|\geqslant m-|I|=n,$$

therefore $A_I$ is an $n$-cover of $\mathbb{Z}$.

Now suppose that $A_I$ forms an $n$-cover of $\mathbb{Z}$ for all $I\subseteq\{1,\cdots,k\}$ with $|I|=m-n$. Let's show that $A=A_\emptyset$ forms an $m$-cover of $\mathbb{Z}$. Assume on the contrary that for some integer $x$ set $J=\{1\leqslant s\leqslant k:\ x\equiv a_s\ (\mathrm{mod}\ n_s)\}$ has cardinality $l<m$. Choose a subset $I$ of $\{1,\cdots,k\}$ with cardinality $m-n$ such that either $I\subseteq J$ or $I\supseteq J$. Observe that $x$ belongs to less than $n$ members of $A_I$. This contradiction ends our proof.

*Remark* 5. Apparently for (1) to be an $m$-cover of $\mathbb{Z}$ it is necessary that $k\geqslant m$.

**Proof of part (i) of Theorem 1**. It suffices to handle the case $m=m(A)>0$.

At first we let that $n_1,\cdots,n_k$ are all greater than one. Since $m\leqslant\sum_{s=1}^k 1/n_s\leqslant k/2$, either $J$ or $J^-$ has cardinality not less than $m$.

*Case* 1. $|J^-| \geqslant m$. Among $I \subseteq \{1, \cdots, k\}$ with $\{\sum_{s\in I} m_s/n_s\} = \{\sum_{s\in J} m_s/n_s\}$, we select a $J_0$ with the least cardinality. Apparently $|J_0^-| \geqslant |J^-| \geqslant m$. Let $I_0 = \{s_1, \cdots, s_{m-1}\}$ be a subset of $\{1, \cdots, k\}$ with $|I_0| = m-1$ and $I_0 \cap J_0 = \emptyset$. By Lemma 1 and Remark 5, system $\{a_s(n_s)\}_{s\in I_0^-}$ forms a 1-cover of $\mathbb{Z}$ and hence so does $\mathcal{A}_0 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s\in I_0^-}$. As $J_0 \subseteq I_0^-$, by Proposition 1 or Theorem 2 of [S2] there is a $J_1 \subseteq I_0^-$ for which $J_1 \neq J_0$ and $\{\sum_{s\in J_1} m_s/n_s\} = \{\sum_{s\in J_0} m_s/n_s\}$. According to the choice of $J_0$ we must have $J_1 \not\subseteq J_0$. Choose $t_1 \in J_1 \setminus J_0$ and put $I_1 = \{t_1, s_2, \cdots, s_{m-1}\}$. Observe that $I_1 \cap J_0 = \emptyset$. Since $\mathcal{A}_1 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s\in I_1^-}$ forms a 1-cover of $\mathbb{Z}$, there exists a $J_2 \subseteq I_1^-$ with $J_2 \neq J_0$ such that $\{\sum_{s\in J_2} m_s/n_s\} = \{\sum_{s\in J_0} m_s/n_s\}$. Choose $t_2 \in J_2 \setminus J_0$ and put $I_2 = \{t_1, t_2, s_3, \cdots, s_{m-1}\}$. Then continue this procedure to find

$$J_3, t_3, I_3;\ \cdots;\ J_{m-1}, t_{m-1}, I_{m-1};\ J_m, t_m$$

in the same way. Apparently $J_1, J_2, \cdots, J_m$ are all different from $J_0$. If $1 \leqslant i < j \leqslant m$ then $t_i \in J_i \setminus J_j$ because $t_i \in I_{j-1}$ and $J_j \cap I_{j-1} = \emptyset$. So the $m+1$ subsets $J_0, J_1, J_2, \cdots, J_m$ of $\{1, \cdots, k\}$ are distinct, therefore

$$\begin{aligned}&\left|\left\{I \subseteq \{1, \cdots, k\}:\ I \neq J\ \&\ \sum_{s\in I}\frac{m_s}{n_s} - \sum_{s\in J}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \\ &\qquad \geqslant |\{J_i:\ 0 \leqslant i \leqslant m\ \&\ J_i \neq J\}| \geqslant m.\end{aligned}$$

*Case* 2. $|J| \geqslant m$, i.e. $|(J^-)^-| \geqslant m$. It follows from the above that

$$\left|\left\{I \subseteq \{1, \cdots, k\}:\ I \neq J^-\ \&\ \sum_{s\in I}\frac{m_s}{n_s} - \sum_{s\in J^-}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \geqslant m.$$

Thus

$$\begin{aligned}&\left|\left\{I' \subseteq \{1, \cdots, k\}:\ I' \neq J\ \&\ \sum_{s\in I'}\frac{m_s}{n_s} - \sum_{s\in J}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \\ =&\left|\left\{I^-:\ I \subseteq \{1, \cdots, k\},\ I^- \neq J\ \&\ \sum_{s\in I^-}\frac{m_s}{n_s} - \sum_{s\in J}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \\ =&\left|\left\{I \subseteq \{1, \cdots, k\}:\ I \neq J^-\ \&\ \sum_{s\in I}\frac{m_s}{n_s} - \sum_{s\in J^-}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \geqslant m.\end{aligned}$$

So far we have proven (4) in both cases.

Next let's consider the situation in which $K = \{1 \leqslant s \leqslant k:\ n_s = 1\}$ is nonempty. If $|K| < m$, then $\{a_s(n_s)\}_{s\in K^-}$ forms an $m - |K|$-cover of $\mathbb{Z}$ with all the moduli greater than one, hence by the above

$$\left|\left\{I \subseteq K^-:\ \sum_{s\in I}\frac{m_s}{n_s} - \sum_{s\in J\setminus K}\frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \geqslant m - |K| + 1.$$

Therefore

$$\begin{aligned}&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\=&\left|\left\{I\cup I':\ I\subseteq K,\ I'\subseteq K^-\ \&\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&\left|\left\{I\cup I':\ I\subseteq K,\ |I|\leqslant 1,\ I'\subseteq K^-\ \&\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&|K|+\left|\left\{I'\subseteq K^-:\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&|K|+\max\{m-|K|+1,1\}\geqslant m+1.\end{aligned}$$

This completes the proof.

**Lemma 2.** *Let* (1) *be a system of arithmetic sequences, and $J$ a nonempty subset of $\{1,\cdots,k\}$ with $|J|\leqslant m(A)$ and $\cap_{s\in J}a_s(n_s)\neq\emptyset$. For each $s\in J^-$ let $m_s$ be a positive integer. Let $0\leqslant a<N(J)$ and*

$$C(a)=\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a}{N(J)}}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m(A)-|J|}e^{2\pi i\sum_{s\in I}\frac{m_s}{n_s}(a_s-a_J)}\tag{11}$$

*where $a_J$ is the unique integer in $\cap_{s\in J}a_s(n_s)$ with $0\leqslant a_J<N(J)$. Then $C(a)=C(\{a\})$.*

Proof. Apparently it suffices to show $C(a)=C(a-1)$ providing $a\geqslant 1$.

Let $m=m(A)$. Observe that the sequences $a_s+(n_s/m_s)\mathbb{Z}$ $(s\in J^-)$ together with $a_J+N(J)\mathbb{Z}$ form an $m-|J|+1$-cover of $\mathbb{Z}$. In view of Proposition 1,

$$\begin{aligned}&\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a}{N(J)}}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m-|J|}e^{2\pi i\sum_{s\in I}\frac{a_sm_s}{n_s}}\\&+\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}+\frac{1}{N(J)}\}=\frac{a}{N(J)}}}(-1)^{|I|+1}\binom{[\sum_{s\in I}\frac{m_s}{n_s}+\frac{1}{N(J)}]}{m-|J|}e^{2\pi i(\sum_{s\in I}\frac{a_sm_s}{n_s}+\frac{a_J}{N(J)})}\end{aligned}$$

vanishes. So

$$\begin{aligned}
&e^{2\pi i a a_J/N(J)}C(a)\\
=&\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a}{N(J)}}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m-|J|}e^{2\pi i\sum_{s\in I}\frac{a_s m_s}{n_s}}\\
=&\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a-1}{N(J)}}}(-1)^{|I|}\binom{[[\sum_{s\in I}\frac{m_s}{n_s}]+\frac{a}{N(J)}]}{m-|J|}e^{2\pi i(\sum_{s\in I}\frac{a_s m_s}{n_s}+\frac{a_J}{N(J)})}\\
=&e^{2\pi i a_J/N(J)}\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a-1}{N(J)}}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m-|J|}e^{2\pi i\sum_{s\in I}\frac{a_s m_s}{n_s}}\\
=&e^{2\pi i a_J/N(J)}e^{2\pi i(a-1)a_J/N(J)}C(a-1)=e^{2\pi i a a_J/N(J)}C(a-1).
\end{aligned}$$

Therefore $C(a)=C(a-1)$. We are done.

*Remark* 6. If we replace $m(A)-|J|$ in (11) by a smaller nonnegative integer $n$ then the new $C(a)$ will equal zero by Proposition 1, because system $\{a_s+(n_s/m_s)\mathbb{Z}\}_{s\in J^-}$ forms an $m(A)-|J|$-cover of $\mathbb{Z}$.

**Proof of part (ii) of Theorem 1**. Since $|\{s\in J^-: x\in a_s(n_s)\}|=m(A)-|J|$ for some integer $x$ and $(m_s,n_s)=1$ for all $s\in J^-$, system $\{a_s+(n_s/m_s)\mathbb{Z}\}_{s\in J^-}$ fails to form an $m(A)-|J|+1$-cover of $\mathbb{Z}$ as well as $\{a_s(n_s)\}_{s\in J^-}$. As $x\in\cap_{s\in J}a_s(n_s)$, there is a unique integer $a_J$ with $0\leqslant a_J<N(J)$ such that $a_J\equiv a_s \pmod{n_s}$ for all $s\in J$. By Proposition 1 and Remark 6 there exists a $\theta\in[0,1)$ such that

$$C(N(J)\theta)e^{2\pi i a_J\theta}=\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m(A)-|J|}e^{2\pi i\sum_{s\in I}\frac{a_s m_s}{n_s}}\neq 0.$$

Put $\alpha=\{N(J)\theta\}$. If $0\leqslant a<N(J)$ and $\{a\}=\alpha$, then $a-N(J)\theta\in\mathbb{Z}$ and hence $C(a)=C(N(J)\theta)\neq 0$ by Lemma 2, therefore $\{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a}{N(J)}$ for some $I\subseteq J^-$ with $[\sum_{s\in I}\frac{m_s}{n_s}]\geqslant m(A)-|J|$. This concludes the proof.

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