Acta Arith. 97(2001), 87–98.

PRODUCTS OF BINOMIAL COEFFICIENTS MODULO p^2

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1. Introduction

As usual $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the rational field, the real field and the complex field respectively. We also let $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, by (a, n) we mean th greatest common divisor of a and n, if n is odd then the Jacobi symbol $(\frac{a}{n})$ is defined in terms of Legendre symbols (see, e.g. [IR]). For $x \in \mathbb{R}$, $[x]$ and $\{x\}$ stand for the integral and the fractional parts of x respectively. For a prime p and an integer a prime to p, the Fermat quotient $(a^{p-1} - 1)/p$ is denoted by $q_p(a)$. For an odd prime p and $a \in \mathbb{Z}$, we define the Euler quotient

$$\mathrm{eq}_p(a) = \frac{a^{(p-1)/2} - (\frac{a}{p})}{p}. \tag{1.1}$$

The Gauss lemma used to prove the law of quadratic reciprocity is as follows:

Gauss' Lemma. *Let $n > 0$ be an odd integer and a an integer prime to n. Then*

$$\left(\frac{a}{n}\right) = (-1)^{|S_n(a)|} \quad \textit{where } S_n(a) = \left\{k \in \mathbb{Z}^+ : \frac{k}{n} < \frac{1}{2} < \left\{\frac{ka}{n}\right\}\right\}. \tag{1.2}$$

Almost every textbook on number theory only contains Gauss' Lemma with $n = p$ being an odd prime. The general version of Gauss' Lemma was first published by M. Jenkins [J] in 1867 with an elementary proof, in the textbook [R] H. Rademacher supplied a proof using subtle properties of quadratic Gauss sums.

For $x \in \mathbb{R}$ let $\binom{x}{0} = 1$ and $\binom{x}{n} = \frac{1}{n!}\prod_{j=0}^{n-1}(x - j)$ for $n = 1, 2, 3, \cdots$. Recently A. Granville [G] obtained a congruence for $\prod_{0<k<n}\binom{p-1}{[pk/n]} \bmod p^2$ where p is an odd prime not dividing $n \in \mathbb{Z}^+$. With the help of Gauss' Lemma, we are able to get the following more general result.

2000 *Mathematics Subject Classification.* Primary 11B65; Secondary 11A07, 11B68.

The research is supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, and the National Natural Science Foundation of P. R. China.

Theorem 1.1. *Let $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$. Let p be an odd prime not dividing n.*

(i) *If $\delta\in\{0,1\}$ then*

(1.3)
$$\begin{aligned}&(-1)^{\frac{p-1}{2}[\frac{n-\delta}{2}]}\prod_{0<k\leqslant[(n-\delta)/2]}\binom{pm-1}{[pk/n]}\\ \equiv&\begin{cases}(\frac{n}{p})+pmn\mathrm{eq}_p(n)\pmod{p^2}&\text{if } 2\nmid n,\\ (\frac{2n}{p})+pm((-1)^\delta(\frac{n}{p})2\mathrm{eq}_p(2)+(\frac{2}{p})n\mathrm{eq}_p(n))\pmod{p^2}&\text{if } 2\mid n.\end{cases}\end{aligned}$$

(ii) *We have*

(1.4)
$$\sum_{k=0}^{n-1}(-1)^{k+(n-1)[\frac{pk}{n}]}\binom{pm-1}{[pk/n]}\equiv\begin{cases}mn(1-2^{p-1})\pmod{p^2}&\text{if } 2\mid n,\\ 1\pmod{p^2}&\text{if } 2\nmid n.\end{cases}$$

Remark 1.1. In (1.3) we use Euler quotients instead of Fermat quotients, this makes the congruence somewhat symmetric in the case $2\mid n$.

Now we deduce Granville's result from our Theorem 1.1.

Corollary 1.1 (Granville [G])**.** *Let n be a positive integer and p an odd prime not dividing n. Then*

$$\prod_{0<k<n}\binom{p-1}{[pk/n]}\equiv(-1)^{\frac{p-1}{2}(n-1)}(n^p-n+1)\pmod{p^2}. \tag{1.5}$$

Proof. Observe that

$$\begin{aligned}&(-1)^{\frac{p-1}{2}(n-1)}\prod_{0<k<n}\binom{p-1}{[pk/n]}\\ =&(-1)^{\frac{p-1}{2}([\frac{n-1}{2}]+[\frac{n}{2}])}\prod_{0<k\leqslant[(n-1)/2]}\binom{p-1}{[pk/n]}\times\prod_{0<k\leqslant[n/2]}\binom{p-1}{[p(n-k)/n]}\\ =&(-1)^{\frac{p-1}{2}[\frac{n-1}{2}]}\prod_{0<k\leqslant[(n-1)/2]}\binom{p-1}{[pk/n]}\times(-1)^{\frac{p-1}{2}[\frac{n}{2}]}\prod_{0<k\leqslant[n/2]}\binom{p-1}{[pk/n]}.\end{aligned}$$

Applying part (i) of Theorem 1.1 with $m=1$ and $\delta=0,1$, we then obtain that

$$(-1)^{\frac{p-1}{2}(n-1)}\prod_{0<k<n}\binom{p-1}{[pk/n]}\equiv1+2pn\Big(\frac{n}{p}\Big)\mathrm{eq}_p(n)\pmod{p^2}.$$

For any integer a prime to p, clearly

$$a^{p-1}-1=\left(a^{\frac{p-1}{2}}+\left(\frac{a}{p}\right)\right)\left(a^{\frac{p-1}{2}}-\left(\frac{a}{p}\right)\right)\equiv 2\left(\frac{a}{p}\right)p\mathrm{eq}_p(a)\pmod{p^2}.$$

So (1.5) follows. □

For $a,n\in\mathbb{Z}$ with $0\leqslant a<n$, we let

$$a(n)=a \bmod n=a+n\mathbb{Z}=\{a+nx : x\in\mathbb{Z}\}.$$

For a finite system $A=\{a_s(n_s)\}_{s=1}^k$ of such residue classes, we define the *covering function* $w_A:\mathbb{Z}\to\{0,1,2,\cdots\}$ by

$$w_A(x)=|\{1\leqslant s\leqslant k : x\in a_s(n_s)\}|. \tag{1.6}$$

When $w_A(x)=m$ for all $x\in\mathbb{Z}$, A is said to be an *exact m-cover* (of $\mathbb{Z}$). We also use the term *disjoint cover* instead of exact 1-cover. (See [S3] and [S4] for problems and results on covers of $\mathbb{Z}$.) For two systems A and B of residue classes, if $w_A=w_B$, then we say that A is *covering equivalent* to B, and denote this by $A\sim B$. For $d,n\in\mathbb{Z}^+$ and $a\in\{0,1,\cdots,d-1\}$, clearly

$$\{a+jd(nd)\}_{j=0}^{n-1}\sim\{(a(d)\}, \tag{1.7}$$

in particular $\{r(n)\}_{r=0}^{n-1}\sim\{0(1)\}$.

In this paper we will also prove the following extension of Corollary 1.1.

Theorem 1.2. *Let p be an odd prime. Let $A=\{a_s(n_s)\}_{s=1}^k$ $(0\leqslant a_s<n_s)$ and $B=\{b_t(m_t)\}_{t=1}^l$ $(0\leqslant b_t<m_t)$ be covering equivalent systems with the moduli n_s and m_t not divisible by p but dividing integer N. Then for any $x\in[0,p)$ we have*

$$\begin{aligned}&\prod_{s=1}^k\binom{p\frac{N}{n_s}-1}{[\frac{x+pa_s}{n_s}]}\Big/\prod_{t=1}^l\binom{p\frac{N}{m_t}-1}{[\frac{x+pb_t}{m_t}]}\\ \equiv&(-1)^{(k-l)\frac{p-1}{2}}\left(1+pN\left(\sum_{s=1}^k\frac{q_p(n_s)}{n_s}-\sum_{t=1}^l\frac{q_p(m_t)}{m_t}\right)\right)\pmod{p^2}.\end{aligned} \tag{1.8}$$

Remark 1.2. Actually we may not require the integer N in Theorem 1.2 to be a common multiple of those moduli n_s and m_t. For example $N=1$ is allowed if we don't mind using $x\notin\mathbb{Z}$ in the notation $\binom{x}{n}$.

Corollary 1.2. *Let $A=\{a_s(n_s)\}_{s=1}^k$ $(0\leqslant a_s<n_s)$ be an exact m-cover of $\mathbb{Z}$. Let N be the least common multiple of $n_1,\cdots,n_k$ and p an odd prime not dividing N. Then*

$$\prod_{s=1}^k\binom{p\frac{N}{n_s}-1}{[\frac{pa_s}{n_s}]}\equiv(-1)^{(k-m)\frac{p-1}{2}}\left(1+pN\sum_{s=1}^k\frac{q_p(n_s)}{n_s}\right)\pmod{p^2}. \tag{1.9}$$

Proof. Let B be the system consisting of m copies of $0(1)$. Then $A\sim B$. Since $[\frac{p0}{1}]=\frac{q_p(1)}{1}=0$, Corollary 1.2 follows immediately from Theorem 1.2. $\square$

Remark 1.3. Applying Corollary 1.2 to the trivial disjoint cover $A=\{r(n)\}_{r=0}^{n-1}$ we then get Corollary 1.1 again.

In the next section we will give some examples of uniform maps the concept of which arose from our previous study of covering equivalence (cf. [S1] and [S2]). On the basis of Section 2, we are going to prove Theorems 1.1 and 1.2 in Section 3.

2. Some uniform maps

Definition 2.1. Let m be an integer and M an additive abelian group. Let f be a map from a subset of $\mathbb{C}\times\mathbb{C}$ into M. If for any ordered pair $\langle x,y\rangle$ in the domain $\mathrm{Dom}(f)$ of f and each positive integer n prime to m, we have

$$\left\{\left\langle\frac{x+mr}{n},ny\right\rangle:r=0,1,\cdots,n-1\right\}\subseteq\mathrm{Dom}(f) \tag{2.1}$$

and

$$\sum_{r=0}^{n-1}f\left(\frac{x+mr}{n},ny\right)=f(x,y), \tag{2.2}$$

then we call f an *m-uniform map* (into M).

The functional equation (2.2) with $m=1$ was first introduced by the author in [S1] where he showed the following theorem in the case $m=1$ by a complicated induction method.

Theorem 2.1. *Let m be an integer and M a left R-module where R is a ring with identity. Let f be a map into M with* $\mathrm{Dom}(f)\subseteq\mathbb{C}\times\mathbb{C}$ *such that* (2.1) *holds for any* $\langle x,y\rangle\in\mathrm{Dom}(f)$ *and* $n\in\mathbb{Z}^+$ *with* $(m,n)=1$*. Then the following two statements are equivalent:*

(a) *f is an m-uniform map into M.*

(b) *Whenever*

$$\sum_{\substack{1\leqslant s\leqslant k\\ x\in a_s(n_s)}}\lambda_s=\sum_{\substack{1\leqslant t\leqslant l\\ x\in b_t(m_t)}}\mu_t\qquad\text{for all } x\in\mathbb{Z}\tag{2.3}$$

(*with* $\lambda_s,\mu_t\in R$, $a_s,n_s,b_t,m_t\in\mathbb{Z}$, $0\leqslant a_s<n_s$, $0\leqslant b_t<m_t$ *and* $(n_sm_t,m)=1$), *we have*

$$\sum_{s=1}^{k}\lambda_s f\left(\frac{x+ma_s}{n_s},n_sy\right)=\sum_{t=1}^{l}\mu_t f\left(\frac{x+mb_t}{m_t},m_ty\right)\quad\text{for } \langle x,y\rangle\in\mathrm{Dom}(f).\tag{2.4}$$

Proof. Since $\{r(n)\}_{r=0}^{n-1}\sim\{0(1)\}$ for all $n\in\mathbb{Z}^+$, (b) implies (a).

Now we show (b) under the condition (a). Suppose that (2.3) holds. Let N be the least common multiple of those moduli n_s and m_t. If $\langle x,y\rangle\in\mathrm{Dom}(f)$, then

$$\begin{aligned}\sum_{s=1}^{k}\lambda_s f\left(\frac{x+ma_s}{n_s},n_sy\right)&=\sum_{s=1}^{k}\lambda_s\sum_{j=0}^{N/n_s-1}f\left(\frac{\frac{x+ma_s}{n_s}+jm}{N/n_s},\frac{N}{n_s}(n_sy)\right)\\
&=\sum_{s=1}^{k}\lambda_s\sum_{\substack{r=0\\ r\in a_s(n_s)}}^{N-1}f\left(\frac{x+mr}{N},Ny\right)=\sum_{r=0}^{N-1}\Bigg(\sum_{\substack{1\leqslant s\leqslant k\\ r\in a_s(n_s)}}\lambda_s\Bigg)f\left(\frac{x+mr}{N},Ny\right)\\
&=\sum_{r=0}^{N-1}\Bigg(\sum_{\substack{1\leqslant t\leqslant l\\ r\in b_t(m_t)}}\mu_t\Bigg)f\left(\frac{x+mr}{N},Ny\right)=\sum_{t=1}^{l}\mu_t f\left(\frac{x+mb_t}{m_t},m_ty\right).\end{aligned}$$

This ends the proof. □

Proposition 2.1. (i) *Let* $m\in\mathbb{Z}$. *Then the function* $[\ ]_m:\mathbb{R}\times\mathbb{R}\to\mathbb{Q}$ *given by*

$$[\ ]_m(x,y)=[x]+\frac{1-m}{2}\tag{2.5}$$

is an m*-uniform map into the rational field* $\mathbb{Q}$.

(ii) *For each* $m=0,1,2,\cdots$ *the functions* $b_m:\mathbb{C}\times\mathbb{C}^*\to\mathbb{C}$ *and* $e_m:\mathbb{C}\times\mathbb{Z}\to\mathbb{C}$ *given by*

$$b_m(x,y)=y^{m-1}B_m(x)\tag{2.6}$$

and

$$e_m(x,y)=\begin{cases}e^{\pi ixy}y^mE_m(x) & \text{if } y \text{ is odd},\\ -\frac{2}{m+1}e^{\pi ixy}y^mB_{m+1}(x) & \text{if } y \text{ is even},\end{cases}\tag{2.7}$$

are 1*-uniform maps into the complex field* $\mathbb{C}$*, where* $B_m(x)$ *and* $E_m(x)$ *are the* m*th Bernoulli polynomial and the* m*th Euler polynomial respectively.*

Proof. Let n be any positive integer.

i) If $(m,n)=1$ then

$$\begin{aligned}&\sum_{r=0}^{n-1}\left(\left[\frac{x+mr}{n}\right]+\frac{1-m}{2}\right)=\sum_{r=0}^{n-1}\left(\frac{x+mr}{n}+\frac{1-m}{2}-\left\{\frac{x+mr}{n}\right\}\right)\\ =&x+m\sum_{r=0}^{n-1}\left(\frac{r}{n}-\frac{1}{2}\right)-\sum_{r=0}^{n-1}\left(\left\{\frac{\{x\}+[x]+mr}{n}\right\}-\frac{1}{2}\right)\\ =&x-\frac{m}{2}-\sum_{s=0}^{n-1}\left(\frac{\{x\}+s}{n}-\frac{1}{2}\right)=x-\frac{m}{2}-\left(\{x\}-\frac{1}{2}\right)=[x]+\frac{1-m}{2}.\end{aligned}$$

This proves part (i).

ii) Let m be an arbitrary nonnegative integer. Raabe's identity states that

$$\sum_{r=0}^{n-1}B_m\left(z+\frac{r}{n}\right)=n^{1-m}B_m(nz). \tag{2.8}$$

Another known identity (cf. [B]) asserts that

$$E_m(nz)=\begin{cases}n^m\sum_{r=0}^{n-1}(-1)^rE_m(z+\frac{r}{n}) & \text{if } 2\nmid n,\\ -\frac{2n^m}{m+1}\sum_{r=0}^{n-1}(-1)^rB_{m+1}(z+\frac{r}{n}) & \text{if } 2\mid n.\end{cases} \tag{2.9}$$

By these two identities we can easily check that

$$\sum_{r=0}^{n-1}b_m\left(\frac{x+r}{n},ny\right)=b_m(x,y)\quad\text{for } x\in\mathbb{C} \text{ and } y\in\mathbb{C}^*$$

and

$$\sum_{r=0}^{n-1}e_m\left(\frac{x+r}{n},ny\right)=e_m(x,y)\quad\text{for } x\in\mathbb{C} \text{ and } y\in\mathbb{Z}.$$

The proof is now complete. □

Remark 2.1. In 1989 the author [S1] briefly mentioned the basic things for Proposition 2.1. For more examples of 1-uniform maps, the reader is referred to [S5].

Corollary 2.1. *Let p be an odd prime and $n > 0$ an even integer prime to p. Then*

$$\sum_{r=0}^{n-1}(-1)^r B_{p-1}\left(\frac{r}{n}\right) \equiv -nq_p(2) \pmod{p}. \tag{2.10}$$

Proof. By Proposition 2.1,

$$\frac{2n^{p-2}}{1-p}\sum_{r=0}^{n-1}(-1)^r B_{p-1}\left(\frac{r}{n}\right) = \sum_{r=0}^{n-1} e_{p-2}\left(\frac{r}{n}, n\right) = e_{p-2}(0,1)$$

doesn't depend on the value of positive even integer n. So

$$\begin{aligned} n^{p-2}\sum_{r=0}^{n-1}(-1)^r B_{p-1}\left(\frac{r}{n}\right) =& 2^{p-2}\left(2B_{p-1} - \sum_{r=0}^{2-1} B_{p-1}\left(\frac{r}{2}\right)\right) \\ =& 2^{p-1}B_{p-1} - B_{p-1}. \end{aligned}$$

Since $pB_{p-1} \equiv \sum_{r=1}^{p-1} r^{p-1} \equiv -1 \pmod{p}$ (see, e.g. [IR]), (2.10) follows at once. □

Proposition 2.2. *Let p be an odd prime. For $x \geqslant 0$ and $m \in \mathbb{Z} \setminus p\mathbb{Z}$ let*

$$q(x,m) = \frac{q_p(m)}{m} + \sum_{\substack{0<j\leqslant[x] \\ p\nmid j}} \frac{1}{jm} \tag{2.11}$$

Then the function $\bar{q}(x,m) = q(x,m) \bmod p$ is a p-uniform map into the finite field $\mathbb{Z}/p\mathbb{Z}$.

Proof. Let $m \in \mathbb{Z} \setminus p\mathbb{Z}$ and $n \in \mathbb{Z}^+ \setminus p\mathbb{Z}$. Since

$$q_p(mn) = \frac{m^{p-1}-1}{p} + m^{p-1}\frac{n^{p-1}-1}{p} \equiv q_p(m) + q_p(n) \pmod{p},$$

for $x \geqslant 0$ the congruence

$$\sum_{k=0}^{n-1} q\left(\frac{x+pk}{n}, nm\right) \equiv q(x,m) \pmod{p}$$

is equivalent to the following

$$q_p(n) \equiv \sum_{\substack{0<j\leqslant[x] \\ p\nmid j}} \frac{1}{j} - \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\substack{0<j\leqslant[\frac{x+pk}{n}] \\ p\nmid j}} \frac{1}{j} \pmod{p}. \tag{2.12}$$

Now it suffices to show (2.12) for all $x=0,1,2,\cdots$.

By pp. 125–126 of [GS] we have

$$B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\equiv-\sum_{0<j\leqslant[pk/n]}\frac{1}{j}\pmod p \tag{2.13}$$

for any $k=0,1,\cdots,n-1$. Observe that

$$\begin{aligned}&\sum_{k=0}^{n-1}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right)=\sum_{r=0}^{n-1}B_{p-1}\left(\frac{r}{n}\right)-nB_{p-1}\\ =&n^{2-p}B_{p-1}-nB_{p-1}=\frac{n}{n^{p-1}}\cdot\frac{1-n^{p-1}}{p}(pB_{p-1})\equiv nq_p(n)\pmod p.\end{aligned}$$

Thus (2.12) holds for $x=0$.

Let $r\in\mathbb{Z}^+$. Assume (2.12) for $x=r-1$. Denote by k_0 the unique integer $k\in[0,n)$ such that $r+pk\equiv 0\pmod n$. Clearly $p\mid r$ if and only if p divides $j_0=(r+pk_0)/n$. For $k\in\{0,1,\cdots,n-1\}$, we have

$$\left[\frac{r+pk}{n}\right]=\left[\frac{r-1+pk}{n}\right]+\begin{cases}1 & \text{if } k=k_0,\\ 0 & \text{otherwise.}\end{cases}$$

If $p\nmid r$, then

$$\frac{1}{r}-\frac{1}{n}\cdot\frac{1}{j_0}=\frac{1}{r}-\frac{1}{r+pk_0}\equiv 0\pmod p.$$

Thus

$$\begin{aligned}&\sum_{\substack{0<j\leqslant r\\ p\nmid j}}\frac{1}{j}-\frac{1}{n}\sum_{k=0}^{n-1}\sum_{\substack{0<j\leqslant[\frac{r+pk}{n}]\\ p\nmid j}}\frac{1}{j}\\ \equiv&\sum_{\substack{0<j\leqslant r-1\\ p\nmid j}}\frac{1}{j}-\frac{1}{n}\sum_{k=0}^{n-1}\sum_{\substack{0<j\leqslant[\frac{r-1+pk}{n}]\\ p\nmid j}}\frac{1}{j}\equiv q_p(n)\pmod p.\end{aligned}$$

This concludes the induction step. We are done. $\square$

3. Proofs of Theorems 1.1 and 1.2

Lemma 3.1. (i) *Let $a\in\mathbb{Z}$, $n\in\mathbb{Z}^+$ and $(2a,n)=1$. Then*

$$|S_n(a)|\equiv\sum_{0<k<n/2}\left[\frac{ka}{n}\right]+\frac{n^2-1}{8}(a-1)\pmod 2. \tag{3.1}$$

(ii) *Let $m,n\in\mathbb{Z}^+$ and $(m,n)=1$. Then for $\delta\in\{0,1\}$ we have*

$$\sum_{0<k\leqslant(n-\delta)/2}\left[\frac{km}{n}\right]+\sum_{0<k\leqslant(m-\delta)/2}\left[\frac{kn}{m}\right]=\left[\frac{m-\delta}{2}\right]\left[\frac{n-\delta}{2}\right]. \tag{3.2}$$

The above lemma is well-known and usually stated in textbooks with a,m,n being odd primes.

Lemma 3.2. *Let $k, m, n \in \mathbb{Z}$ and $0 \leqslant k < n$. Let p be an odd prime not dividing n. Then*

$$(3.3)\quad (-1)^{[pk/n]}\binom{pm-1}{[pk/n]} \equiv 1+pm\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right) \pmod{p^2}.$$

Proof. For any $l \in \{0,1,\cdots,p-1\}$,

$$(3.4)\qquad (-1)^l\binom{pm-1}{l} = \prod_{0<j\leqslant l}\left(1-p\frac{m}{j}\right) \equiv 1-pm\sum_{0<j\leqslant l}\frac{1}{j} \pmod{p^2}.$$

Combining this with (2.13) we then obtain (3.3). □

Proof of Theorem 1.1. As $p-1$ is even, we have $B_{p-1}(1-x) = B_{p-1}(x)$.

i) Let $l = [(n-\delta)/2]$ and $\varepsilon_n = (1+(-1)^n)/2$. By Lemma 3.2,

$$\begin{aligned}&\prod_{0<k\leqslant l}(-1)^{[\frac{pk}{n}]}\binom{pm-1}{[pk/n]}\\ \equiv&1+pm\sum_{0<k\leqslant l}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right) \pmod{p^2}.\end{aligned}$$

Observe that

$$\begin{aligned}&2\sum_{0<k\leqslant l}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right)-\varepsilon_n(-1)^\delta\left(B_{p-1}\left(\frac{1}{2}\right)-B_{p-1}\right)\\ =&\sum_{0<k\leqslant l}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)+B_{p-1}\left(\left\{\frac{p(n-k)}{n}\right\}\right)-2B_{p-1}\right)\\ &-\varepsilon_n(-1)^\delta\left(B_{p-1}\left(\left\{\frac{p}{2}\right\}\right)-B_{p-1}\right)\\ =&\sum_{k=0}^{n-1}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right)\equiv nq_p(n) \pmod{p}\end{aligned}$$

where the last step is taken as in the proof of Proposition 2.2. By Corollary 2.1, $B_{p-1}(\frac{1}{2})-B_{p-1} \equiv 2q_p(2) \pmod{p}$. Recall that $q_p(a) \equiv 2(\frac{a}{p})\mathrm{eq}_p(a) \pmod{p}$ for any $a \in \mathbb{Z}$ with $(a,p)=1$. So

$$\begin{aligned}&\sum_{0<k\leqslant l}\left(B_{p-1}\left(\left\{\frac{pk}{n}\right\}\right)-B_{p-1}\right)\\ \equiv&n\left(\frac{n}{p}\right)\mathrm{eq}_p(n)+\varepsilon_n(-1)^\delta 2\left(\frac{2}{p}\right)\mathrm{eq}_p(2) \pmod{p}.\end{aligned}$$

By Lemma 3.1 and Gauss' Lemma,

$$(-1)^{\sum_{0<k\leqslant l}[\frac{pk}{n}]}=(-1)^{\frac{p-1}{2}l-\sum_{0<k<p/2}[\frac{nk}{p}]}=(-1)^{\frac{p-1}{2}l}\left(\frac{n}{p}\right)\left(\frac{2}{p}\right)^{n-1}.$$

Therefore

$$\begin{aligned}(-1)^{\frac{p-1}{2}l}\left(\frac{n}{p}\right)\left(\frac{2}{p}\right)^{n-1}\prod_{0<k\leqslant l}\binom{pm-1}{[pk/n]}&=\prod_{0<k\leqslant l}(-1)^{[\frac{pk}{n}]}\binom{pm-1}{[pk/n]}\\ \equiv&1+pm\bigg(n\left(\frac{n}{p}\right)\mathrm{eq}_p(n)+\varepsilon_n(-1)^{\delta}2\left(\frac{2}{p}\right)\mathrm{eq}_p(2)\bigg)\pmod{p^2}\end{aligned}$$

and hence (1.3) follows.

ii) Write S for the left hand side of (1.4) and $S'=\sum_{r=0}^{n-1}(-1)^rB_{p-1}(\frac{r}{n})$. By Lemma 3.2,

$$\begin{aligned}S\equiv&\sum_{k=0}^{n-1}(-1)^{\{pk\}_n}\bigg(1+pm\bigg(B_{p-1}\bigg(\frac{\{pk\}_n}{n}\bigg)-B_{p-1}\bigg)\bigg)\\ \equiv&(1-pmB_{p-1})\Delta+pmS'\pmod{p^2}\end{aligned}$$

where

$$\{pk\}_n=n\left\{\frac{pk}{n}\right\}=pk-n\left[\frac{pk}{n}\right]\quad\text{and}\quad\Delta=\sum_{r=0}^{n-1}(-1)^r=\frac{1-(-1)^n}{2}.$$

If $2\nmid n$, then $S'=B_{p-1}$ since $(-1)^{n-r}B_{p-1}(\frac{n-r}{n})=-(-1)^rB_{p-1}(\frac{r}{n})$, therefore $S\equiv1\pmod{p^2}$. When $2\mid n$ we may apply Corollary 2.1. This concludes the proof. □

Proof of Theorem 1.2. Since $A\sim B$, by Theorem 2.1 and Proposition 2.1 we have

$$\sum_{s=1}^{k}\left(\left[\frac{x+pa_s}{n_s}\right]+\frac{1-p}{2}\right)=\sum_{t=1}^{l}\left(\left[\frac{x+pb_t}{m_t}\right]+\frac{1-p}{2}\right).$$

So (1.8) is equivalent to the following

$$\begin{aligned}P_A=&\prod_{s=1}^{k}(-1)^{[\frac{x+pa_s}{n_s}]}\binom{p\frac{N}{n_s}-1}{[\frac{x+pa_s}{n_s}]}\times\left(1-pN\sum_{s=1}^{k}\frac{q_p(n_s)}{n_s}\right)\\ \equiv P_B=&\prod_{t=1}^{l}(-1)^{[\frac{x+pb_t}{m_t}]}\binom{p\frac{N}{m_t}-1}{[\frac{x+pb_t}{m_t}]}\times\left(1-pN\sum_{t=1}^{l}\frac{q_p(m_t)}{m_t}\right)\pmod{p^2}.\end{aligned}$$

By (3.4) we have

$$\begin{aligned} P_A \equiv& \prod_{s=1}^{k}\left(1-p\frac{N}{n_s}\sum_{0<j\leqslant[\frac{x+pa_s}{n_s}]}\frac{1}{j}\right)\left(1-pN\frac{q_p(n_s)}{n_s}\right) \\ \equiv& \prod_{s=1}^{k}\left(1-p\frac{N}{n_s}\left(q_p(n_s)+\sum_{0<j\leqslant[\frac{x+pa_s}{n_s}]}\frac{1}{j}\right)\right) \\ \equiv& \prod_{s=1}^{k}\left(1-pNq\left(\frac{x+pa_s}{n_s},n_s\right)\right) \\ \equiv& 1-pN\sum_{s=1}^{k}q\left(\frac{x+pa_s}{n_s},n_s\right) \pmod{p^2}; \end{aligned}$$

similarly

$$P_B \equiv 1-pN\sum_{t=1}^{l}q\left(\frac{x+pb_t}{m_t},m_t\right) \pmod{p^2}.$$

In view of Theorem 2.1 and Proposition 2.2, $P_A \equiv P_B \pmod{p^2}$. We are done. □

Acknowledgment. The author is indebted to professor Andrew Granville for his comments, and the referee for his suggestions.

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