European J. Combin. 22(2001), no. 3, 415–429.

EXACT m-COVERS OF GROUPS BY COSETS

Zhi-Wei Sun

Let G be a group covered by its left cosets $a_1G_1, \cdots, a_kG_k$ exactly m times. It is known that $[G : \bigcap_{i=1}^k G_i] \leqslant k!$. When all the G_i are subnormal in G and $\bigcap_{i=1}^k G_i = H$, we are able to determine the least value of k in terms of m, G, H. For any $i = 1, \cdots, k$, providing $G/(G_i)_G$ is solvable we show that $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$, where $f(n) = \sum_{s=1}^r \alpha_s(p_s - 1)$ if $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the standard factorization of n. These extend some previous results on disjoint covers.

1. Introduction

Let G be a (multiplicative) group (whose identity element is denoted by e) and H a subgroup of G. For any $a \in G$ we call $aH = \{ax : x \in H\}$ a left coset of H in G, by G/H we mean *the set of all left cosets of H in G*. As usual, the index of H in G is $[G : H] = |G/H|$, and G/H is called the quotient group of G by H if H is normal in G. We use H_G and H^G to denote the core (i.e. normal interior) and the normal closure of H in G respectively.

Let $a_1G_1, \cdots, a_kG_k$ be finitely many left cosets of a group G. For the system

$$\mathcal{A} = \{a_iG_i\}_{i=1}^k, \tag{1.1}$$

we call

$$w_{\mathcal{A}}(x) = |\{1 \leqslant i \leqslant k : x \in a_iG_i\}| \tag{1.2}$$

the *covering multiplicity* of $x \in G$, and put $m(\mathcal{A}) = \inf_{x \in G} w_{\mathcal{A}}(x)$.

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. (1.1) is said to be an *exact m-cover* of G if $w_{\mathcal{A}}(x) = m$ for all $x \in G$. Exact 1-covers are just partitions into left cosets, they are also called *disjoint covers*. As mentioned in all textbooks on group theory, for any subgroup H of G with finite index, all the $[G : H]$ left cosets of H in G form a disjoint cover of G. Clearly the only exact m-cover of G by subgroups consists of m copies of G. It is known that even for the additive group $\mathbb{Z}$ of integers there exists an exact m-cover no subsystem of which forms an exact n-cover with $0 < n < m$

2000 *Mathematics Subject Classification.* Primary 20D60; Secondary 05A20, 11B75, 20D10, 20E15.

The research is supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, and the National Natural Science Foundation of P. R. China.

(cf. [G,Z]). Exact m-covers are natural extension of disjoint covers, they should have regular properties because any such a cover covers all the elements the same times. Covers of $\mathbb{Z}$ by residue classes were initiated by P. Erdös [E] (see Section 1 of Zhi-Wei Sun [S3] for problems and results in the area), exact m-covers of $\mathbb{Z}$ were first investigated by Š. Porubský [P1], exact m-covers of an arbitrary group have been studied only in the case $m = 1$.

Exact 1-covers are very special in the sense that members of such covers are pairwise disjoint. There has been much research on such covers, see [NZ,P2,Zn3] for results concerning disjoint covers of $\mathbb{Z}$. Often there are no obvious ways to generalize results on disjoint covers to exact m-covers, this is why few properties of exact m-covers are known. By an easy counting argument, if a system

$$A = \{a_i + n_i\mathbb{Z}\}_{i=1}^k \quad (a_1, \cdots, a_k \in \mathbb{Z};\ n_1, \cdots, n_k \in \mathbb{Z}^+) \tag{1.3}$$

of residue classes forms an exact m-cover of $\mathbb{Z}$ then $\sum_{i=1}^k 1/n_i = m$. In [S2,S3,S4] the author revealed some further connections between exact m-covers of $\mathbb{Z}$ and unit fractions. In this paper we shall prove some inequalities for exact m-covers of groups.

The Mycielski function $f : \mathbb{Z}^+ \to \{0, 1, 2, \cdots\}$ is determined as follows:

$$f(p) = p - 1 \text{ for any prime } p \text{ and } f(mn) = f(m) + f(n) \text{ for all } m, n \in \mathbb{Z}^+.$$

Evidently

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{i=1}^r \alpha_i(p_i - 1) \tag{1.4}$$

where $p_1, \cdots, p_r$ are distinct primes and $\alpha_1, \cdots, \alpha_r$ are nonnegative integers. In 1966 J. Mycielski and W. Sierpiński [MS] conjectured that if (1.1) forms a disjoint cover of an abelian group G and all the $[G : G_i]$ are finite then

$$k \geqslant 1 + \max_{1 \leqslant i \leqslant k} f([G : G_i]). \tag{1.5}$$

This was confirmed by Š. Znám [Zn1] for $G = \mathbb{Z}$, and verified by M. Hejný and Znám [HZ] in a special case. In 1968 Znám [Zn2] posed a conjecture that if (1.3) forms a disjoint cover of $\mathbb{Z}$ then $k \geqslant 1 + f(N)$ where N is the least common multiple of $n_1, \cdots, n_k$, later in 1974 his student I. Korec ([K1]) obtained the following stronger result: Let G be a group and (1.1) its disjoint cover with all the G_i normal in G, then each G_i has finite index in G and

$$k \geqslant 1 + f\left(\left[G : \bigcap_{i=1}^k G_i\right]\right). \tag{1.6}$$

In 1988 M.A. Berger, A. Felzenbaum and A.S. Fraenkel [BFF] proved that (1.5) holds if (1.1) forms a disjoint cover of a finite solvable group G.

It seems that Mycielski, Znám and Korec had not known the following basic result established by B.H. Neumann [N1,N2] in 1954: If (1.1) forms a cover of a group G then G is the union of those a_iG_i with $[G : G_i] < \infty$. In 1987 M. J. Tomkinson [T] strengthened the Neumann result by showing that if (1.1) forms a cover of a group G but none of its proper subsystems does then

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \leqslant k! \tag{1.7}$$

where the upper bound $k!$ is best possible. By Corollary 1 of Zhi-Wei Sun [S1], (1.7) holds if $m(\mathcal{A}') < m(\mathcal{A}) = m$ for any proper subsystem $\mathcal{A}'$ of $\mathcal{A}$. In particular, when (1.1) forms an exact m-cover of group G, all the G_i and hence the intersection $\bigcap_{i=1}^{k} G_i$ are of finite index in G.

Let G be a group and H a subnormal subgroup of G with finite index. Let

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G. If the length n is zero (i.e. $H = G$), then we set $d(G, H) = 0$, otherwise we put

$$d(G, H) = \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1). \tag{1.8}$$

By the Jordan–Hölder theorem, $d(G, H)$ does not depend on the choice of the composition series from H to G. Theorem 6 of [S1] indicates that

$$[G : H] - 1 \geqslant d(G, H) \geqslant f([G : H]) \geqslant \log_2[G : H]. \tag{1.9}$$

In contrast with Tomkinson's result, inequality (1.6) implies that

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \leqslant 2^{k-1}. \tag{1.10}$$

In 1990 the author [S1] obtained the following improvement to Korec's result: If (1.1) forms a disjoint cover of a group G with all the G_i subnormal in G, then

$$k \geqslant 1 + d\left(G, \bigcap_{i=1}^{k} G_i\right). \tag{1.11}$$

Let us look at two examples.

Example 1.1. Let G be a group and H a subgroup with $k = [G:H] < \infty$. Suppose that $\{Ha_1, \cdots, Ha_k\}$ is a right coset decomposition of G. Set $G_i = a_i^{-1}Ha_i$ for $i = 1, \cdots, k$. The author [S1] observed that (1.1) forms a disjoint cover of G with $\bigcap_{i=1}^k G_i = H_G$. When $G = S_k$ is the symmetric group on $\{1, \cdots, k\}$ and H is the stabilizer of 1,

$$\{H,\ H(12),\ H(13),\ \cdots,\ H(1k)\} = \{G_1,\ (12)G_2,\ \cdots,\ (1k)G_k\}$$

forms a partition of G where G_i is the stabilizer of i for each $i = 1, \cdots, k$, Tomkinson [T] noticed that in this case $\bigcap_{i=1}^k G_i = H_G = \{e\}$ has index $k!$ in G; we add here that if $k \geqslant 3$ then all the subgroups $G_1, \cdots, G_k$ are distinct, $[G : \cap_{i=1}^k G_i] = k!(> 2^{k-1})$ has some prime divisors (e.g. prime factors of $k-1$) not dividing $k = [G : G_i]$, and $f([G : \bigcap_{i=1}^k G_i]) = \sum_{i=2}^k f(i) \geqslant f(3) + k - 2 = k$.

Example 1.2. Let G be a group and $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ be a chain of subgroups with finite index. Let $a \in G$ and $H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i$ for $i = 0, 1, \cdots, n-1$. In [S1] the author observed that those $ab_j^{(i)}H_i$ ($0 \leqslant i < n$ and $1 \leqslant j < [H_{i+1} : H_i]$) form a partition of $G \setminus aH_0$. Thus, the cosets, together with aH_0 and $m-1$ copies of G, form an exact m-cover of G with the number k of cosets being $m + \sum_{i=0}^{n-1}([H_{i+1} : H_i] - 1)$ and the intersection of the k subgroups being H. If H_i is maximal normal in H_{i+1} for each $i = 0, \cdots, n-1$ then $k = m + d(G, H)$. In general, as $[H_{i+1} : H_i] - 1 \geqslant f([H_{i+1} : H_i])$ we have

$$k - m \geqslant \sum_{i=0}^{n-1} f([H_{i+1} : H_i]) = f\bigg(\prod_{i=0}^{n-1} [H_{i+1} : H_i]\bigg) = f([G : H]).$$

In this paper we study lower bounds for the number k of cosets in an exact m-cover (1.1) of a group G, with the intersection $\bigcap_{i=1}^k G_i$ or a subgroup G_i given. In the next section we prove a key property of exact m-covers. In Section 3 we characterize those subnormal subgroups H of group G for which $[G : H] < \infty$ and $d(G, H) = f([G : H])$. In Section 4 we present our main results with an application in group theory.

Now we state our central results. (Actually we prove more.)

(I) Let G be a group and (1.1) an exact m-cover of G with all the G_i subnormal in G. Then $k \geqslant m + d(G, \bigcap_{i=1}^k G_i)$. Moreover, for any subgroup K of G not contained in all the G_i we have

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geqslant 1 + d\bigg(K, K \cap \bigcap_{i=1}^k G_i\bigg).$$

(II) Let (1.1) be an exact m-cover of a group G. Whenever $G/(G_i)_G$ is solvable, we have $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$.

Concerning result (II) we have a further conjecture.

Conjecture. *Let* (1.1) *be an exact* m*-cover of a group* G *with all the* $G/(G_i)_G$ *solvable. Then* $k \geqslant m + f(N)$ *where* N *is the least common multiple of the indices* $[G : G_1], \cdots, [G : G_k]$.

For a cover (1.1) of group G, if it doesn't form an exact m-cover for any $m = 1, 2, 3, \cdots$, then we don't have a similar inequality in general. When G is cyclic, or $|G|$ is squarefree and all the G_i are subnormal in G, if $m(\mathcal{A}') < m(\mathcal{A})$ for any proper subsystem $\mathcal{A}'$ of $\mathcal{A}$ then we can show that $k \geqslant m(\mathcal{A}) + f([G : \bigcap_{i=1}^k G_i])$. Since this result and its extension are quite involved, we don't present a proof here.

2. A vital property of exact m-covers

Of course, whether two left cosets of subgroups are disjoint, is very essential. Here we give

Lemma 2.1. *Let* H *and* K *be subgroups of a group* G*. Then*

(i) $HK = G$ *if and only if* $xH \cap yK \neq \emptyset$ *for all* $x, y \in G$.

(ii) $HK = KH$ *coincides with* G *or* H *in the following cases:*

(a) H *is maximal and normal in* G*;*

(b) H *is maximal in* G *and* K *is normal in* G*;*

(c) H *is maximal normal and* K *is subnormal in* G.

Proof. i) If $HK = G$, then for any $x, y \in G$ there exist $h \in H$ and $k \in K$ such that $hk = x^{-1}y$ and hence

$$xH \cap yK = xhH \cap xhkK = xh(H \cap K) \neq \emptyset.$$

Conversely, if $xH \cap yK \neq \emptyset$ for all $x, y \in G$, then for any $g \in G$ there is a $h \in H$ for which $h \in gK$ and hence $gK = hK$, therefore $G = HK$.

ii) In each case H or K is normal in G and thus $HK = KH$ is a subgroup of G containing H. Due to the maximality of H, HK coincides with G or H in the cases (a) and (b). In case (c), HK is subnormal in G (by 7.19 of [Ro]), if $HK \neq G$ then $(HK)^G \neq G$ and so $(HK)^G = H$ (i.e. $HK = H$) by the maximal normality of H.

Remark 2.1. For previous combined use of parts (i) and (ii), the reader may consult the proofs of Lemma 6 of [K1], Theorem 1 of [Pa], Lemma 2.II of [BFF] and Theorem 4 of [S1]. For a proper subgroup H of group G, case (a) is equivalent to the following: H is a (maximal) normal subgroup of prime index in G. (Notice that H is maximal in G if it has prime index in G, and that in case (a) G/H is a cyclic group of prime order since G/H and H/H are the only subgroups of G/H.)

Lemma 2.2. *Let* G *be a group and* $G_1, \cdots, G_k$ *be subgroups of* G *with finite index. Then*

$$(2.1) \quad \sum_{i=1}^{k} \frac{1}{[G : G_i]} = \frac{1}{[G : \cap_{j=1}^k G_j]} \sum_{C \in G/\cap_{j=1}^k G_j} |\{1 \leqslant i \leqslant k : C \subseteq a_i G_i\}| \geqslant m(\mathcal{A}).$$

Therefore, (1.1) *forms an exact m-cover of G if and only if*

$$\sum_{i=1}^{k}\frac{1}{[G:G_i]}=m \tag{2.2}$$

and $w_{\mathcal{A}}(x)\leqslant m$ (or $w_{\mathcal{A}}(x)\geqslant m$) for all $x\in G$.

Proof. $H=\bigcap_{j=1}^{k}G_j$ is of finite index in G. Clearly

$$[G:H]\sum_{i=1}^{k}\frac{1}{[G:G_i]}=\sum_{i=1}^{k}[G_i:H]=\sum_{i=1}^{k}\sum_{\substack{C\in G/H\\ C\subseteq a_iG_i}}1=\sum_{C\in G/H}\sum_{\substack{i=1\\ a_iG_i\supseteq C}}^{k}1.$$

So

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{[G:G_i]}&=\frac{1}{[G:H]}\sum_{C\in G/H}|\{1\leqslant i\leqslant k:\ C\subseteq a_iG_i\}|\\ &\geqslant\frac{1}{[G:H]}\sum_{C\in G/H}m(\mathcal{A})=m(\mathcal{A}).\end{aligned}$$

If (1.1) forms an exact m-cover of G, then $|\{1\leqslant i\leqslant k: C\subseteq a_iG_i\}|=m$ for each $C\in G/H$ and hence (2.2) holds. When $w_{\mathcal{A}}(x)\leqslant m$ (or $w_{\mathcal{A}}(x)\geqslant m$) for all $x\in G$, if (2.2) is valid then $|\{1\leqslant i\leqslant k: C\subseteq a_iG_i\}|$ must coincide with m for every $C\in G/H$ and hence $w_{\mathcal{A}}(x)=m$ for any $x\in G$. This completes the proof.

Remark 2.2. It has been known that if (1.3) forms an exact m-cover of $\mathbb{Z}$ then $\sum_{i=1}^{k}1/[\mathbb{Z}:n_i\mathbb{Z}]$ equals m (see [P1]). In 1977 Korec and Znám [KZ] proved that $\sum_{i=1}^{k}1/[G:G_i]=1$ for any disjoint cover (1.1) of group G.

Theorem 2.1. *Assume that* (1.1) *forms an exact m-cover of a group G (by left cosets of subgroups $G_1,\cdots,G_k$). For a subgroup H of G we have*

$$\{C\in G/H: C\supseteq a_iG_i \text{ for some } i=1,\cdots,k\}=\emptyset \text{ or } G/H. \tag{2.3}$$

in the following cases:

(a) *H is the group G or a normal subgroup of prime index in G;*

(b) *$G_1,\cdots,G_k$ are normal in G and H is maximal in G;*

(c) *$G_1,\cdots,G_k$ are subnormal and H is maximal normal in G.*

Proof. Suppose on the contrary that (2.3) is false. Then there exist $a,b\in G$ such that $a_jG_j\subseteq aH$ for some $1\leqslant j\leqslant k$ and that $a_iG_i\not\subseteq bH$ for all $i=1,\cdots,k$. Let $I=\{1\leqslant i\leqslant k: G_i\not\subseteq H\}$. Clearly $j\notin I$ since $a_jH=aH\supseteq a_jG_j$. When $i\in I$, by Lemma 2.1 and Remark 2.1 we have $G_iH=G$ and $a_iG_i\cap bH\neq\emptyset$ in all the three cases. If $s\in\{1,\cdots,k\}\setminus I$, then $a_sG_s\subseteq a_sH\neq bH$ and hence $a_sG_s\cap bH=\emptyset$.

As (1.1) forms an exact m-cover of G, I must be nonempty and $\{b^{-1}a_iG_i \cap H\}_{i\in I}$ must be an exact m-cover of H by left cosets of subgroups $G_i \cap H$ $(i \in I)$ in H. In view of Lemma 2.2,

$$\sum_{i\in I} \frac{1}{[H : G_i \cap H]} = m = \sum_{i=1}^{k} \frac{1}{[G : G_i]}.$$

However, for each $i \in I$ we have $[H : G_i \cap H] = [G : G_i]$ because $G = G_iH$ contains exactly $[H : G_i \cap H]$ right cosets of G_i. So I must coincide with $\{1, \cdots, k\}$, which contradicts the fact that $j \notin I$. The proof is ended.
Remark 2.3. When $m = 1$ Theorem 2.1 in case (c) was obtained by the author [S1] in a particular way.

3. When $d(G, H) = f([G : H])$?

For any group G we let $\mathcal{S}(G)$ denote the class of subnormal subgroups H of G for which $[G : H] < \infty$ and $d(G, H) = f([G : H])$. In this section we aim to characterize those $H \in \mathcal{S}(G)$.

Lemma 3.1. *Let G be a group.*
(i) *For subgroups H and K of G with $[G : H]$ finite and H or K subnormal in G, we have $[K : H \cap K] \mid [G : H]$.*
(ii) *If $G_1, \cdots, G_k$ are subnormal subgroups of G with finite index, then*

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \Bigm| \prod_{i=1}^{k} [G : G_i]. \tag{3.1}$$

Proof. i) When H is subnormal in G, there exists a finite chain

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = G$$

of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \cdots, n$. By the second isomorphism theorem (for group H_i) (cf. 3.40 of [Ro])

$$[H_i \cap K : (H_i \cap K) \cap H_{i-1}] = |(H_i \cap K)H_{i-1}/H_{i-1}|$$

and hence $[H_i \cap K : H_{i-1} \cap K] \mid [H_i : H_{i-1}]$. We therefore have

$$\prod_{i=1}^{n} [H_i \cap K : H_{i-1} \cap K] \Bigm| \prod_{i=1}^{n} [H_i : H_{i-1}], \quad \text{i.e.} \quad [K : H \cap K] \mid [G : H].$$

Now we assume that K is subnormal in G. Let $K_0 = K \subseteq K_1 \subseteq \cdots \subseteq K_n = G$ be a chain of subgroups such that K_{i-1} is normal in K_i for any $i = 1, \cdots, n$. In view of Chapter 1 of [Su], the subgroup $K_{i-1}(H \cap K_i)$ of K_i contains exactly

$[K_{i-1} : K_{i-1} \cap (H \cap K_i)] = [K_{i-1} : H \cap K_{i-1}]$ left cosets of $H \cap K_i$ and so $[K_{i-1} : H \cap K_{i-1}] \mid [K_i : H \cap K_i]$. It follows that $[K : H \cap K] = [K_0 : H \cap K_0]$ divides $[K_n : H \cap K_n] = [G : H]$.

This proves the first part.

ii) By part (i), if H and K are subnormal subgroups of G with finite index, then $[G : H \cap K] = [G : K][K : H \cap K]$ divides $[G : H][G : K]$. Thus we can easily show part (ii) by induction. This ends the proof.

Remark 3.1. Let G be a group and H, K be subgroups of G with $[G : H] < \infty$. Although $[K : H \cap K] \leqslant [G : H]$, in general we may have $[K : H \cap K] \nmid [G : H]$ even if G is solvable. For example, the symmetric group $G = S_3$ on $\{1, 2, 3\}$ has subgroups $H = \{e, (12)\}$ and $K = \{e, (13)\}$ with $H \cap K = \{e\}$, apparently $[K : H \cap K] = |K| = 2$, $[G : H] = 3!/2 = 3$ and $[G : H \cap K] \nmid [G : H][G : K]$.

Recall that a group G is called *perfect* if G coincides with its commutator subgroup $G' = [G, G]$, and that every finite group has a unique solvable residual (see 7.50 of [Ro]).

Theorem 3.1. *Let G be a group and H be a subgroup of G.*

(i) *$H \in \mathcal{S}(G)$ if and only if there is a composition series from H to G whose factors are of prime orders.*

(ii) *When $H \in \mathcal{S}(G)$, we have $H \cap K \in \mathcal{S}(K)$ for any subgroup K of G, also $H \cap K, \langle H, K\rangle \in \mathcal{S}(G)$ if $K \in \mathcal{S}(G)$.*

(iii) *If H lies in $\mathcal{S}(G)$ then so do H_G and H^G.*

(iv) *$H \in \mathcal{S}(G)$ if and only if H is subnormal in G and G/H_G is finite and solvable.*

(v) *If $H \in \mathcal{S}(G)$ then H contains all perfect subgroups of G. When G is finite and H is subnormal in G, we have $H \in \mathcal{S}(G)$ if H contains the (perfect) normal subgroup K of G for which G/K is the solvable residual of G.*

(vi) *Assume that H is subnormal in G and $[G : H]$ is finite. Then $H \in \mathcal{S}(G)$ if G is locally solvable, or $[G : H]$ is squarefree or odd or divisible by at most two distinct primes.*

Proof. i) In the case $H = G$, clearly $H \in \mathcal{S}(G)$ since $d(G, H) = 0 = f([G : H])$, and the composition series from H to G has length zero and no factor.

Now let H be proper in G. Apparently H is subnormal and of finite index in G if and only if there is a composition series from H to G whose factors are finite. Suppose that H is such a subgroup of G and that $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ is a composition series from H to G. Then (1.8) holds and

$$f([G : H]) = f\left(\prod_{i=1}^{n} |H_i/H_{i-1}|\right) = \sum_{i=1}^{n} f(|H_i/H_{i-1}|).$$

Since $m - 1 \geqslant f(m)$ for all $m \in \mathbb{Z}^+$, and

$$f(m_1m_2) = f(m_1) + f(m_2) \leqslant m_1 - 1 + m_2 - 1 < m_1m_2 - 1$$

for any integers $m_1, m_2 > 1$, we therefore have

$$\begin{aligned} d(G,H) = f([G:H]) &\iff |H_i/H_{i-1}| - 1 = f(|H_i/H_{i-1}|) \text{ for all } i = 1, \cdots, n \\ &\iff |H_i/H_{i-1}| \text{ is a prime number for any } i = 1, \cdots, n. \end{aligned}$$

This proves part (i).

ii) When $H = G$, part (ii) is obvious. Assume that $H \in \mathcal{S}(G)$ but $H \neq G$. By part (i) there exists a composition series $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ from H to G such that $|H_1/H_0|, \cdots, |H_n/H_{n-1}|$ are primes. Let K be a subgroup of G. For each $i = 1, \cdots, n$, evidently $H_{i-1} \cap K = H_{i-1} \cap (H_i \cap K)$ is normal in $H_i \cap K$ since H_{i-1} is normal in H_i; by Lemma 3.1(i) $[H_i \cap K : H_{i-1} \cap K]$ divides $[H_i : H_{i-1}]$ and hence coincides with one or the prime $|H_i/H_{i-1}|$. Thus, in view of part (i), $H \cap K = H_0 \cap K \in \mathcal{S}(H_n \cap K) = \mathcal{S}(K)$.

Now suppose that $K \in \mathcal{S}(G)$. Then $H \cap K$ is subnormal and of finite index in G. As $H \cap K \in \mathcal{S}(K)$ and $K \in \mathcal{S}(G)$, we have $H \cap K \in \mathcal{S}(G)$ by part (i). Put $L = (H \cap K)_G$. Then H/L, K/L and their join $\langle H/L, K/L\rangle = \langle H, K\rangle/L$ are subnormal in the finite group G/L (cf. 1.8 and 7.22 of [Ro]), so $\langle H, K\rangle$ is subnormal in G. Since $(H \cap K)/L \in \mathcal{S}(K/L)$, by part (i) and 7.25 of [Ro] we have $H/L \in \mathcal{S}(\langle H, K\rangle/L)$ and hence $H \in \mathcal{S}(\langle H, K\rangle)$. As $H \in \mathcal{S}(G)$, $\langle H, K\rangle$ must lie in $\mathcal{S}(G)$.

iii) Let $H \in \mathcal{S}(G)$. By part (i) $g^{-1}Hg \in \mathcal{S}(G)$ for any $g \in G$. As $[G : H] < \infty$ there are only finitely many distinct conjugates of H. Their intersection is H_G and their join is H^G. Applying part (ii) we obtain that $H_G, H^G \in \mathcal{S}(G)$.

iv) Let H be subnormal in G. By part (iii), $H_G \in \mathcal{S}(G)$ providing $H \in \mathcal{S}(G)$. By part (i), if $H_G \in \mathcal{S}(G)$ then $H \in \mathcal{S}(G)$ and $H_G \in \mathcal{S}(H)$. So $H \in \mathcal{S}(G)$ if and only if $H_G \in \mathcal{S}(G)$.

Note that G/H_G is a finite solvable group if and only if there is a composition series from H_G/H_G to G/H_G whose factors are of prime orders (cf. 7.56 of [Ro]). Thus, by part (i), $H_G \in \mathcal{S}(G)$ (i.e. $H \in \mathcal{S}(G)$) if and only if G/H_G is finite and solvable.

v) As usual, for any group F we let $F^{(0)} = F$ and $F^{(n+1)} = (F^{(n)})'$ for $n = 0, 1, 2, \cdots$. Suppose $H \in \mathcal{S}(G)$. Then G/H_G is solvable by part (iv), therefore $(G/H_G)^{(n)} = H_G/H_G$ for some nonnegative integer n (see 7.52 of [Ro]). Notice that $(G/H_G)^{(n)} = (GH_G/H_G)^{(n)} = G^{(n)}H_G/H_G$ (cf. Exercise 164 of [Ro]). So $G^{(n)} \subseteq H_G \subseteq H$. If M is a perfect subgroup of G, then $M = M^{(n)} \subseteq G^{(n)} \subseteq H$.

For the rest of part (v), let G be finite, and H be a subnormal subgroup of G which contains the smallest normal subgroup K of G such that G/K is solvable. As a characteristic subgroup of K, K' is also normal in G (cf. 3.15 and 3.51 of [Ro]). Since K/K' is abelian (cf. 3.52 of [Ro]), G/K' is solvable and so K is perfect by the property of K. Note that $K \subseteq H_G$. Because $G/H_G \cong (G/K)/(H_G/K)$ is solvable, it follows from part (iv) that $H \in \mathcal{S}(G)$.

vi) As $[G : H] < \infty$, G/H_G is finite and so there are finitely many elements $x_1, \cdots, x_n$ of G such that $\langle x_1^*, \cdots, x_n^*\rangle = G/H_G$ where $x_i^* = x_iH_G$ for $i = 1, \cdots, n$.

If G is locally solvable, then finitely generated $K = \langle x_1, \cdots, x_n\rangle$ and the quotient $G/H_G = KH_G/H_G \cong K/(K \cap H_G)$ are solvable, thus $H \in \mathcal{S}(G)$ by part (iv).

Let $[G : H]$ be squarefree. Then the factors of a composition series from H to G are simple groups with squarefree orders. By Corollary 1 to Theorem 2.10 in Ch. 5 of [Su], any group $F \neq \{e\}$ of squarefree order has a normal subgroup with the index being the least prime divisor of $|F|$. So the factors must have prime orders and hence $H \in \mathcal{S}(G)$ by part (i).

Now suppose that $[G : H]$ is odd or divisible by at most two distinct primes. Clearly so is $|G/H_G|$ because we can view H_G as an intersection of finitely many conjugates of H and $|G/H_G|$ must divide a power of $[G : H]$ by Lemma 3.1(ii). Applying the well-known theorems of Feit–Thompson [FT] and W. Burnside (cf. 8.5.3 of [R]), we then obtain the solvability of G/H_G. So $H \in \mathcal{S}(G)$ by part (iv).

The proof of Theorem 3.1 is now complete.

4. The main results

Theorem 4.1. *Let G be a group and* (1.1) *an exact m-cover of G by left cosets of subgroups $G_1, \cdots, G_k$. Then, for any fixed $i \in \{1, \cdots, k\}$,*

$$k \geqslant m + f([G : G_i]) \quad \text{if } G_i \text{ contains some } H \in \mathcal{S}(G). \tag{4.1}$$

Proof. Let $1 \leqslant t \leqslant k$, $H_t \in \mathcal{S}(G)$ and $H_t \subseteq G_t$. We use induction on $[G : H_t]$ to show that $k \geqslant m + f([G : G_t])$.

If $[G : H_t] = 1$, then $G_t = G$ and hence $m + f([G : G_t]) = m = w_{\mathcal{A}}(e) \leqslant k$.

Now assume that $[G : H_t] > 1$. As $H_t \in \mathcal{S}(G)$, by Theorem 3.1(i) there exists a normal subgroup H of G for which H_t is subnormal in H and $p = [G : H]$ is a prime. Write $G/H = \{g_1H, \cdots, g_pH\}$. Put $I_0 = \{1 \leqslant i \leqslant k : G_i \not\subseteq H\}$ and $I_s = \{1 \leqslant i \leqslant k : a_iG_i \subseteq g_sH\}$ for $s = 1, \cdots, p$. Clearly the union of these pairwise disjoint sets coincides with $\{1, \cdots, k\}$. For any $s = 1, \cdots, p$, by Lemma 2.1 we have $a_iG_i \cap g_sH \neq \emptyset$ if and only if $i \in I_0 \cup I_s$, so $I_0 \cup I_s \neq \emptyset$ and $\{g_s^{-1}a_iG_i \cap H\}_{i \in I_0 \cup I_s}$ forms an exact m-cover of H by left cosets of subgroups $G_i \cap H$ $(i \in I_0 \cup I_s)$.

Apparently $H_t \subseteq G_t \cap H$ and $[H : H_t] < [G : H_t]$. Choose $1 \leqslant s \leqslant p$ so that $t \in I_0 \cup I_s$. By the induction hypothesis,

$$|I_0 \cup I_s| \geqslant m + f([H : G_t \cap H]).$$

If $t \in I_0$ then $[G : G_t] = [G_tH : G_t] = [H : G_t \cap H]$ and hence

$$k \geqslant |I_0 \cup I_s| \geqslant m + f([H : G_t \cap H]) = m + f([G : G_t]).$$

In the case $t \in I_s$ (whence $G_t \subseteq H$), by Theorem 2.1 none of $I_1, \cdots, I_p$ is empty, thus

$$\begin{aligned} k = |I_0| + |I_1| + \cdots + |I_p| &\geqslant |I_0 \cup I_s| + p - 1 \\ &\geqslant m + f([H : G_t]) + f([G : H]) = m + f([G : G_t]). \end{aligned}$$

This concludes the induction step.

Remark 4.1. Let (1.1) be an exact m-cover of a group G. Then $[G : G_i]$ and $[G : (G_i)_G]$ are finite. When $G/(G_i)_G$ is solvable, we have $H = (G_i)_G \in \mathcal{S}(G)$ by Theorem 3.1(iv), therefore $k \geqslant m + f([G : G_i])$.

Corollary 4.1. *Let G be a group and* (1.1) *an exact m-cover of G by left cosets. Suppose that*

$$\emptyset \neq I \subseteq \{1 \leqslant i \leqslant k : G_i \text{ contains a subgroup in } \mathcal{S}(G)\}.$$

Then for any subgroup K of G we have

$$(k-m)|I| \geqslant f\bigg(\Big[K : K \cap \bigcap_{i\in I} G_i\Big]\bigg), \tag{4.2}$$

thus

$$\Big[G : \bigcap_{i\in I} G_i\Big] \leqslant 2^{(k-m)|I|}. \tag{4.3}$$

Proof. Let K be an arbitrary subgroup of G and t be any element of I. Put $J = \{1 \leqslant j \leqslant k : a_jG_j \cap a_tK \neq \emptyset\}$. Then $t \in J$ and $\{a_t^{-1}a_jG_j \cap K\}_{j\in J}$ forms an exact m-cover of K by left cosets of $G_j \cap K$ $(j \in J)$. Let H_t be a subgroup in $\mathcal{S}(G)$ contained in G_t. By Theorem 3.1(ii) we have $H_t \cap K \in \mathcal{S}(K)$. In view of Theorem 4.1, $k \geqslant |J| \geqslant m + f([K : G_t \cap K])$.

Write $I = \{i_1, \cdots, i_{|I|}\}$. By the above,

$$\begin{aligned} &k - m \geqslant f([K : K \cap G_{i_1}]),\\ &k - m \geqslant f([K \cap G_{i_1} : K \cap G_{i_1} \cap G_{i_2}]),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,,\\ &k - m \geqslant f([K \cap G_{i_1} \cap \cdots \cap G_{i_{|I|-1}}, K \cap G_{i_1} \cap \cdots \cap G_{i_{|I|-1}} \cap G_{i_{|I|}}]). \end{aligned}$$

Adding these inequalities we then get the desired (4.2).

If we take $K = G$, then (4.2) gives that $(k-m)|I| \geqslant f([G : \bigcap_{i\in I} G_i])$, which implies (4.3) by (1.9). This ends the proof.

Corollary 4.2. *Let $k \geqslant m > 0$ be integers. Then 2^{k-m} is the maximal value that can be the index of a subgroup in a locally solvable group with an exact m-cover by k cosets one of which is a coset of the subgroup.*

Proof. Suppose that a locally solvable group G possesses an exact m-cover consisting of a coset C_1 of subgroup G_1, $\cdots$, a coset C_k of subgroup G_k. For $i = 1, \cdots, k$, we let $G_i^* = G_i$ if C_i is a left coset a_iG_i, and $G_i^* = a_i^{-1}G_ia_i$ if C_i is a right coset G_ia_i of G_i. As $\{a_iG_i^*\}_{i=1}^k$ forms an exact m-cover of G, each G_i^* has finite index in G and hence $(G_i^*)_G \in \mathcal{S}(G)$ by Theorem 3.1(vi). In the light of Theorem 4.1, for each $n_i = [G : G_i] = [G : G_i^*]$, we have $k \geqslant m + f(n_i)$ and hence $n_i \leqslant 2^{k-m}$ by (1.9).

Now it suffices to notice that the following k residue classes

$$\underbrace{\mathbb{Z}, \ \cdots, \ \mathbb{Z}}_{m-1}, \ 2^{k-m}\mathbb{Z}, \ 2^0 + 2\mathbb{Z}, \ \cdots, \ 2^{k-m-1} + 2^{k-m}\mathbb{Z}$$

together form an exact m-cover of the infinite cyclic group $\mathbb{Z}$ with the largest modulus being 2^{k-m}. This follows from Example 1.2 in the case $G = \mathbb{Z}$ and $H = 2^{k-m}\mathbb{Z}$.

Corollary 4.3. *Let* (1.1) *be an exact m-cover of a group G by cosets of subgroups $G_1,\cdots,G_k$. Then $k\geqslant m+f([G:G_i])$ if G_i contains a subnormal subgroup of G with index odd or squarefree or divisible by at most two distinct primes.*

Proof. Let H be any subnormal subgroup of G with $[G:H]$ odd or squarefree or in the form $p^\alpha q^\beta$ where p,q are primes and α,β are nonnegative integers. In view of Theorem 3.1(vi) we have $H\in\mathcal{S}(G)$. If $G_i\supseteq H$, then $k\geqslant m+f([G:G_i])$ by Theorem 4.1. This concludes the proof.

For an exact m-cover of a group G, if all the G_i are subnormal in G then we have a sharp lower bound of k in terms of the intersection $H=\bigcap_{i=1}^k G_i$.

Theorem 4.2. *Let G be a group and $G_1,\cdots,G_k$ be subnormal subgroups of G such that* (1.1) *forms an exact m-cover of G for some $a_1,\cdots,a_k\in G$. Let K be any subgroup of G and set $I(K)=\{1\leqslant i\leqslant k: K\not\subseteq G_i\}$.*

(i) *We have*

$$k\geqslant m+d\Big(K,K\cap\bigcap_{i=1}^k G_i\Big). \tag{4.4}$$

(ii) *If $I(K)\neq\emptyset$, then there exists an $r\in I(K)$ and $x_i\in K\setminus G_i$ for $i\in I(K)\setminus\{r\}$, such that*

$$\left|\left\{x_i\Big(K\cap\bigcap_{s=1}^k G_s\Big): i\in I(K)\setminus\{r\}\right\}\right|\geqslant d\Big(K,K\cap\bigcap_{s=1}^k G_s\Big), \tag{4.5}$$

and hence

$$|\{1\leqslant i\leqslant k: K\not\subseteq G_i\}|\geqslant 1+d\Big(K,K\cap\bigcap_{s=1}^k G_s\Big). \tag{4.6}$$

Proof. We use induction on the finite index $[G:L]$ where $L=\bigcap_{i=1}^k G_i$.

If $[G:L]=1$, then $G_1=\cdots=G_k=G$, so $m+d(K,K\cap L)=m=w_{\mathcal{A}}(e)\leqslant k$ and $I(K)=\emptyset$.

Now let's proceed to the induction step and suppose that $[G:L]>1$. Choose $1\leqslant j\leqslant k$ and a maximal normal proper subgroup H of G such that G_j is subnormal in H. Obviously $\{1\leqslant i\leqslant k: G_i\subseteq H\}$ can be partitioned into $h=|G/H|$ sets

$$J(C)=\{1\leqslant i\leqslant k:\ a_iG_i\subseteq C\}\quad (C\in G/H)$$

which are nonempty by Theorem 2.1. Let $I=\{1\leqslant i\leqslant k: G_i\not\subseteq H\}$. If $i\in I$, then by Lemma 2.1 we have $G_iH=G$ and hence $a_iG_i\cap gH\neq\emptyset$ for all $g\in G$.

Let's take the first step. Set $g_1=a_j$ and $I_1=I\cup J(g_1H)$. Then $j\in I_1$ and system $\mathcal{A}_1=\{g_1^{-1}a_iG_i\cap H\}_{i\in I_1}$ forms an exact m-cover of $H_0=H$ by left cosets of subnormal subgroups $G_i\cap H_0$ $(i\in I_1)$ of H_0. Put

$$H_1=H_0\cap\bigcap_{i\in I_1}G_i=\bigcap_{i\in I_1}G_i\quad\text{and}\quad M_1=\{i\in I_1: K\cap H_0\not\subseteq G_i\cap H_0\}.$$

Apparently $M_1 \subseteq I(K)$. Observe that

$$\Big[H_0 : \bigcap_{i\in I_1}(G_i\cap H_0)\Big] = [H_0 : H_1] < [G : L].$$

By the induction hypothesis, we have

$$|I_1| \geqslant m + d\Big(K\cap H_0, (K\cap H_0)\cap \bigcap_{i\in I_1}(G_i\cap H_0)\Big) = m + d(K\cap H_0, K\cap H_1);$$

if $M_1 \neq \emptyset$ (i.e. $K\cap H_0 \neq K\cap H_1$) then there is an $r_1 \in M_1$ and $x_i \in (K\cap H_0)\setminus (G_i\cap H_0) \subseteq K\setminus G_i$ for $i \in M_1\setminus\{r_1\}$ such that

$$|\{x_i(K\cap H_1) : i \in M_1\setminus\{r_1\}\}| \geqslant d(K\cap H_0, K\cap H_1).$$

Suppose that we have found $g_1,\cdots,g_{s-1} \in G$ $(s>1)$ and pairwise disjoint nonempty subsets $I_1,\cdots,I_{s-1}$ of $\{1,\cdots,k\}$ so that for each $1 \leqslant t < s$, either $I_t \subseteq J(g_tH)$ or $t=1$, and $\mathcal{A}_t = \{g_t^{-1}a_iG_i\cap H_{t-1}\}_{i\in I_t}$ forms an exact m_t-cover of H_{t-1} by left cosets of subnormal subgroups $G_i\cap H_{t-1}$ $(i\in I_t)$ of H_{t-1}, where $0 < m_t \leqslant m_1 = m$ and we let

$$H_t = \bigcap_{i\in I_t}(G_i\cap H_{t-1}) = \bigcap_{i\in I_1\cup\cdots\cup I_t} G_i.$$

In the case $I_s^* = \bigcup_{t=1}^{s-1} I_t \subset \{1,\cdots,k\}$, we proceed step s as follows. Select an element g_s in the union of those a_iG_i with $i \notin I_s^*$. Then m is greater than $l_s = |\{i\in I_s^* : g_s \in a_iG_i\}|$, and $|\{i \in I_s^* : g_sx \in a_iG_i\}| = l_s$ for all $x \in H_{s-1} = \cap_{i\in I_s^*}G_i$. Put

$$I_s = \{1\leqslant i\leqslant k :\ i\notin I_s^* \ \&\ a_iG_i\cap g_sH_{s-1}\neq\emptyset\} \neq \emptyset.$$

For $i \in I_s$ we have $a_iG_i\cap g_sH \neq \emptyset$ and hence $i \in J(g_sH)$ since $I_s\cap I = \emptyset$. Let $m_s = m - l_s$. Then $\mathcal{A}_s = \{g_s^{-1}a_iG_i\cap H_{s-1}\}_{i\in I_s}$ forms an exact m_s-cover of H_{s-1} by left cosets of subnormal subgroups $G_i\cap H_{s-1}$ $(i\in I_s)$ of H_{s-1}. Set

$$H_s = H_{s-1}\cap\bigcap_{i\in I_s} G_i = \bigcap_{i\in\bigcup_{t=1}^s I_t} G_i \text{ and } M_s = \{i\in I_s : K\cap H_{s-1}\not\subseteq G_i\cap H_{s-1}\}.$$

Apparently $M_s \subseteq I(K)$, and $M_s = \emptyset$ if and only if $K\cap H_{s-1} = K\cap H_s$. In the light of the induction hypothesis,

$$|I_s| \geqslant m_s + d\Big(K\cap H_{s-1}, K\cap H_{s-1}\cap\bigcap_{i\in I_s}(G_i\cap H_{s-1})\Big) \geqslant 1 + d(K\cap H_{s-1}, K\cap H_s);$$

if $M_s \neq \emptyset$ then there is an $r_s \in M_s$ and $x_i \in (K\cap H_{s-1})\setminus(G_i\cap H_{s-1}) \subseteq K\setminus G_i$ for $i \in M_s\setminus\{r_s\}$ such that

$$|\{x_i(K\cap H_s): i\in M_s\setminus\{r_s\}\}| \geqslant d(K\cap H_{s-1}, K\cap H_s).$$

Since $h = |G/H| > 1$ we have $I_1 \subset \{1,\cdots,k\}$. As $\{1,\cdots,k\}$ is a finite set the above process will terminate after n steps where $1 < n \leqslant k$. Thus $\bigcup_{s=1}^n I_s = \{1,\cdots,k\}$ and $H_n = \bigcap_{i=1}^k G_i = L$. Because

$$\bigcup_{C\in G/H} J(C) = \{1\leqslant i\leqslant k : G_i\subseteq H\} = \bigcup_{s=1}^n \{i\in I_s : G_i \subseteq H\} \subseteq \bigcup_{s=1}^n J(g_sH),$$

we have $n \geqslant h = |G/H| \geqslant l = [K : H\cap K] \geqslant 1 + d(K, H\cap K)$. Therefore

$$\begin{aligned} k =&|I_1| + \sum_{s=2}^n |I_s| \geqslant m + d(K\cap H_0, K\cap H_1) + \sum_{s=2}^n (1 + d(K\cap H_{s-1}, K\cap H_s)) \\ \geqslant& m + d(K, H\cap K) + d(K\cap H_0, K\cap H_n) = m + d(K, K\cap L). \end{aligned}$$

Let

$$M = \bigcup_{\substack{1\leqslant s\leqslant n \\ K\cap H_{s-1}\neq K\cap H_s}} (M_s\setminus\{r_s\}).$$

By the above $M \subseteq I(K)$. For $x, y \in K$, if $1 \leqslant s \leqslant n$ and $x(K\cap H_s) \neq y(K\cap H_s)$, then $x(K\cap L) \neq y(K\cap L)$ (otherwise $x^{-1}y \in K\cap L \subseteq K\cap H_s$). If $1 \leqslant t < s \leqslant n$, $x \in K\cap H_{s-1}$ and $y \in (K\cap H_{t-1})\setminus(G_i\cap H_{t-1})$ for some $i \in I_t$, then $x(K\cap L) \subseteq H_{s-1} \subseteq H_t \subseteq G_i\cap H_{t-1}$ and $y(K\cap L)\cap(G_i\cap H_{t-1}) = \emptyset$. Therefore

$$\begin{aligned} |\{x_i(K\cap L): i\in M\}| =& \sum_{\substack{1\leqslant s\leqslant n \\ K\cap H_{s-1}\neq K\cap H_s}} |\{x_i(K\cap L): i\in M_s\setminus\{r_s\}\}| \\ \geqslant& \sum_{\substack{1\leqslant s\leqslant n \\ K\cap H_{s-1}\neq K\cap H_s}} |\{x_i(K\cap H_s): i\in M_s\setminus\{r_s\}\}| \\ \geqslant& \sum_{\substack{1\leqslant s\leqslant n \\ K\cap H_{s-1}\neq K\cap H_s}} d(K\cap H_{s-1}, K\cap H_s) = \sum_{s=1}^n d(K\cap H_{s-1}, K\cap H_s) \\ =& d(K\cap H_0, K\cap H_n) = d(H\cap K, K\cap L). \end{aligned}$$

If $1 \leqslant s \leqslant n$ and $M_s = \emptyset$, then we let r_s be an element of I_s. Clearly $M' = \{r_s : 1 < s \leqslant l\}$ has cardinality $l-1$, $M'\cap M = \emptyset$ and $M\cup M' \subseteq I(K)\setminus\{r_1\}$. If $1 < s \leqslant l$, then $G_{r_s} \subseteq H$ since $r_s \in I_s \subseteq J(g_sH)$. Write $K/(H\cap K) = \{H\cap K, b_1(H\cap K), \cdots, b_l(H\cap K)\}$. When $1 < s \leqslant l$, we have $b_s \notin H$ (otherwise

$b_s \in H \cap K$, which is impossible) and hence $x_{r_s} = b_s \in K \setminus G_{r_s}$. Recall that $x_i \in K \cap H_{s-1} \subseteq H \cap K$ for $i \in M_s \setminus \{r_s\}$. For $x, y \in K$ with $x(H\cap K) \neq y(H \cap K)$, obviously $x(K \cap L) \neq y(K \cap L)$. For $i \in (I(K) \setminus \{r_1\}) \setminus (M \cup M')$ let x_i be any element of $K \setminus G_i$. Then

$$\begin{aligned}&|\{x_i(K\cap L) : i \in I(K) \setminus \{r_1\}\}|\\ \geqslant&|\{x_i(K\cap L) : i \in M \cup M')\}| = |\{x_i(K\cap L) : i \in M'\}| + |\{x_i(K\cap L) : i \in M\}|\\ \geqslant&l - 1 + d(H\cap K, K\cap L) \geqslant d(K, H\cap K) + d(H \cap K, K\cap L) = d(K, K\cap L)\end{aligned}$$

and therefore $|I(K)| \geqslant 1 + d(K, K \cap L)$.

The proof by induction is now complete.

Remark 4.2. (a) Clearly $I(K) \subseteq \{1 \leqslant i \leqslant k : G_i \neq G\}$, so we cannot substitute m ($\geqslant |\{1 \leqslant i \leqslant k : G_i = G\}|$) for the first term 1 on the right hand side of (4.6). (b) Let H be a subnormal subgroup of finite index in group G. If (1.1) forms an exact m-cover of G with $G_1, \cdots, G_k$ subnormal in G and $\bigcap_{i=1}^k G_i = H$, then by taking $K = G$ in Theorem 4.2(i) we get the inequality $k \geqslant m + d(G, H)$. On the other hand, by Example 1.2 there indeed exists an exact m-cover (1.1) of G with all the G_i subnormal in G, $\bigcap_{i=1}^k G_i = H$ and $k = m + d(G, H)$.

Corollary 4.4. *Let G be a group and H, K be subnormal subgroups of G with finite index. Let $G_1, \cdots, G_k$ be subnormal subgroups of G for which all the G_i contain H but $I(K) = \{1 \leqslant i \leqslant k : K \not\subseteq G_i\} \neq \emptyset$. Suppose that $X = \bigcup_{i=1}^k a_iG_i$ is a union of left cosets of K and $w_{\mathcal{A}}(x) = m$ for all $x \in X$, where $a_1, \cdots, a_k \in G$. Then there are $C_i \in K/(H\cap K)$ for those $i \in I(K)$ such that $C_i \cap G_i \neq \emptyset$ for a unique $i \in I(K)$ and that*

$$|\{C_i : i \in I(K)\}| \geqslant 1 + d\Big(K, K \cap \bigcap_{i=1}^k G_i\Big). \tag{4.7}$$

Proof. The set $G \setminus X$ can be written as a union of finitely many distinct left cosets $b_1K, \cdots, b_lK$ of K. Let $G_{k+1} = \cdots = G_{k+lm} = K$ and $a_{k+(j-1)m+1} = \cdots = a_{k+jm} = b_j$ for $j = 1, \cdots, l$. Clearly $\mathcal{A}' = \{a_iG_i\}_{i=1}^{k+lm}$ forms an exact m-cover of G by left cosets of subnormal subgroups of G. Observe that $\bigcap_{i=1}^{k+l} G_i = L$ where $L = \bigcap_{i=1}^k G_i$. Also, $\{1 \leqslant i \leqslant k + l : K \not\subseteq G_i\} = I(K) \neq \emptyset$. In the light of Theorem 4.2(ii), there is an $r \in I(K)$ and $x_i \in K \setminus G_i$ for $i \in I(K) \setminus \{r\}$ such that $|\{x_i(K\cap L) : i \in I(K)\setminus\{r\}\}| \geqslant d(K, K\cap L)$. Put $C_r = H\cap K$ and $C_i = x_i(H\cap K)$ for $i \in I(K) \setminus \{r\}$. Note that $e \in C_r \cap G_r$. For $i \in I(K) \setminus \{r\}$ we have $C_i \cap G_i = \emptyset$ because $C_i \subseteq x_iG_i$ and $x_i \notin G_i$. If s and t are distinct elements of $I(K)\setminus\{r\}$, then

$$C_s = C_t \Rightarrow x_s^{-1}x_t \in H \cap K \subseteq K \cap L \Rightarrow x_s(K\cap L) = x_t(K \cap L).$$

So

$$\begin{aligned}&|\{C_i :\ i \in I(K)\}| = 1 + |\{C_i :\ i \in I(K) \setminus \{r\}\}|\\ \geqslant&1 + |\{x_i(K\cap L) :\ i \in I(K)\setminus\{r\}\}| \geqslant 1 + d(K, K\cap L).\end{aligned}$$

This ends the proof.

Remark 4.3. When $m = 1$ and $G_1, \cdots, G_k, K$ are normal in G, the inequality $k \geqslant 1 + d(K, K \cap \bigcap_{i=1}^k G_i)$ was obtained by Korec [K2] in another way. Theorem 9 of [S1] is essentially Corollary 4.4 in the case $m = 1$ and $X = G$.

Corollary 4.5. *Let m be a positive integer and H a subgroup of a group G with $[G : H]$ finite.*

(i) *If G is locally nilpotent, then $m + f([G : H])$ is the least positive integer k such that there exists an exact m-cover of G by k left cosets of subgroups whose intersection is H.*

(ii) *Providing H is subnormal in G, if G is locally solvable, or $[G : H]$ odd or squarefree or in the form $p^\alpha q^\beta$ where p, q are distinct primes and α, β are nonnegative integers, then $m + f([G : H])$ is the smallest $k \in \mathbb{Z}^+$ such that there exists an exact m-cover (1.1) of G with all the G_i subnormal in G and $\bigcap_{i=1}^k G_i$ equal to H.*

Proof. In view of Remark 4.2(b), it suffices to make the following observations:

i) When G is locally nilpotent, by Theorem 7 of [S1] H is in $\mathcal{S}(G)$ and any subgroup containing H is subnormal in G.

ii) Suppose that H is subnormal in G, and that G is locally solvable or $[G : H]$ is odd or squarefree or divisible by at most two distinct primes. Then $d(G, H) = f([G : H])$ by Theorem 3.1(vi).

Corollary 4.6. *Let H be a subnormal subgroup of a group G with $[G : H] < \infty$. Then H is normal in G if and only if*

$$|N_G(H)/H| + d(H, H_G) \geqslant [G : H] \tag{4.8}$$

where $N_G(H)$ denotes the normalizer of H in G.

Proof. If H is normal in G then (4.8) holds because $N_G(H) = G$ and $H_G = H$.

Below we assume that H is not normal in G. Let k be the index $[G : H]$ and $\{Ha_1, \cdots, Ha_k\}$ a partition of G into right cosets of H. Clearly $G_i = a_i^{-1} H a_i$ is subnormal in G for each $i = 1, \cdots, k$. As mentioned in Example 1.1 system (1.1) forms a disjoint cover of G with $\bigcap_{i=1}^k G_i = H_G$. Since $H \supset H_G = \bigcap_{i=1}^k G_i$, it follows from Theorem 4.2(ii) that

$$|\{1 \leqslant i \leqslant k : \ H \not\subseteq G_i\}| \geqslant 1 + d\biggl(H, H \cap \bigcap_{i=1}^k G_i\biggr) = 1 + d(H, H_G).$$

For $i = 1, \cdots, k$, as $[G : G_i] = [G : H] < \infty$, $H \subseteq G_i$ if and only if $H = G_i$. We have $[N_G(H) : H] = |\{1 \leqslant i \leqslant k : G_i = H\}|$, because

$$\begin{aligned} N_G(H) =& \{g \in G : g^{-1} H g = H\} = \bigcup_{i=1}^k \{ha_i : \ h \in H \ \& \ (ha_i)^{-1} H h a_i = H\} \\ =& \bigcup_{i=1}^k \{ha_i : \ h \in H \ \& \ a_i^{-1} H a_i = H\} = \bigcup_{\substack{i=1 \\ a_i^{-1} H a_i = H}}^k H a_i. \end{aligned}$$

Therefore

$$\begin{aligned} d(H,H_G) <&|\{1 \leqslant i \leqslant k : H \not\subseteq G_i\}| = |\{1 \leqslant i \leqslant k : G_i \neq H\}| \\ =&k - |\{1 \leqslant i \leqslant k : G_i = H\}| = [G:H] - |N_G(H)/H|. \end{aligned}$$

So (4.8) fails to hold. We are done.

Remark 4.4. Let G be a group and H be a subnormal subgroup of G with finite index. By Corollary 3 of [S1], $[G:H] \geqslant 1 + d(G,H_G)$ and consequently $|G/H_G| \leqslant 2^{[G:H]-1}$. In view of Corollary 4.6 and (1.9), if H is not normal in G then $|H/H_G| \leqslant 2^{d(H,H_G)} \leqslant 2^{[G:H]-1-|N_G(H)/H|}$. It seems that $|N_G(H)/H| \geqslant d(G,H)$.

References

[BFF] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, *Mycielski–Sierpiński conjecture and Korec–Znám result*, Colloq. Math. **56** (1988), 241–249. MR 90d:20044.

[E] P. Erdös, *On integers of the form* $2^k + p$ *and some related problems*, Summa Brasil. Math. **2** (1950), 113–123. MR 13, 437.

[FT] W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029. MR 29#3538.

[G] R. K. Guy, *Unsolved Problems in Number Theory* (2nd, ed.), Springer-Verlag, New York, 1994, Sections A19, B21, E23, F13, F14.

[HZ] M. Hejný and Š. Znám, *Coset decomposition of Abelian groups*, Acta Fac. Rerum Natur. Univ. Comenian. Math. Publ. **25** (1971), 15–19. MR 46#3644.

[K1] I. Korec, *On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups*, Fund. Math. **85** (1974), 41–47. MR 50#10025.

[K2] I. Korec, *On number of cosets in nonnatural disjoint covering systems*, in: Number Theory, Vol. I, Colloq. Math. Soc. János Bolyai, 51, North-Holland, Amsterdam, 1990, pp. 265–278. MR 91k:11016.

[KZ] I. Korec and Š. Znám, *On disjoint covering of groups by their cosets*, Math. Slovaca **27** (1977), 3–7. MR 58#5260.

[MS] J. Mycielski and W. Sierpiński, *Sur une propriété des ensembles linéaires*, Fund. Math. **58** (1966), 143–147. MR 34#4140.

[N1] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248. MR 15, 931.

[N2] B. H. Neumann, *Groups covered by finitely many cosets*, Publ. Math. Debrecen **3** (1954), 227–242. MR 17, 234.

[NZ] B. Novák, and Š. Znám, *Disjoint covering systems*, Amer. Math. Monthly **81** (1974), 42–45. MR 48#10958.

[Pa] M. M. Parmenter, *Exact covering systems for groups*, Fund. Math. **123** (1984), 133–136. MR 86h:20029.

[P1] Š. Porubský, *On m times covering systems of congruences*, Acta Arith. **29** (1976), 159–169. MR 53#2884.

[P2] Š. Porubský, *Results and Problems on Covering Systems of Residue Classes*, Mitt. Math. Sem. Giessen, Heft 150, Giessen Univ., 1981, pp. 1–85. MR 83b:10068.

[R] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[Ro] J. S. Rose, *A Course on Group Theory*, Cambridge Univ. Press, 1978.

[S1] Z. W. Sun, *Finite coverings of groups*, Fund. Math. **134** (1990), 37–53. MR 91g:20031.

[S2] Z. W. Sun, *On exactly m times covers*, Israel J. Math. **77** (1992), 345–348. MR 93k:11007.

[S3] Z. W. Sun, *Covering the integers by arithmetic sequences* II, Trans. Amer. Math. Soc. **348** (1996), 4279–4320. MR 97c:11011.

[S4] Z. W. Sun, *Exact m-covers and the linear form* $\sum_{s=1}^k x_s/n_s$, Acta Arith. **81** (1997), 175–198. MR 98h:11019.

[Su] M. Suzki, *Group Theory* I–II (Grundlehren der mathematischen Wissenschaften; 247–248), Springer-Verlag, New York, 1982, 1986.

[T] M. J. Tomkinson, *Groups covered by finitely many cosets or subgroups*, Comm. Algebra **15** (1987), 845–859. MR 88c:20049.

[Z] M. Z. Zhang, *On irreducible exactly m times covering system of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **28** (1991), 403–408. MR 92j:11001.

[Zn1] Š Znám, *On Mycielski's problem on systems of arithmetic progressions*, Colloq. Math. **15** (1966), 201–204. MR 34#134.

[Zn2] Š. Znám,, *On exactly covering systems of arithmetic sequences,* in: Number Theory (Colloq. János Bolyai Math. Soc., Debrecen 1968), North-Holland, Amsterdam, 1970, 221–225. MR 42#7586.

[Zn3] Š. Znám, *A survey of covering systems of congruences*, Acta Math. Univ. Comenian. **40/41** (1982), 59–79. MR 84e:10004.

Department of Mathematics, Nanjing University, Nanjing 210093, the People's Republic of China

E-mail address: zwsun@nju.edu.cn