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ALGEBRAIC APPROACHES TO PERIODIC ARITHMETICAL MAPS

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Abstract. A residue class $a+n\mathbb{Z}$ with weight λ is denoted by $\langle\lambda,a,n\rangle$. For a finite system $\mathcal{A}=\{\langle\lambda_s,a_s,n_s\rangle\}_{s=1}^k$ of such triples, the periodic map $w_{\mathcal{A}}(x)=\sum_{n_s\mid x-a_s}\lambda_s$ is called the covering map of $\mathcal{A}$. Some interesting identities for those $\mathcal{A}$ with a fixed covering map have been known, in this paper we mainly determine out all those functions $f:\Omega\to\mathbb{C}$ such that $\sum_{s=1}^k\lambda_s f(a_s+n_s\mathbb{Z})$ depends only on $w_{\mathcal{A}}$ where Ω denotes the family of all residue classes. We also study algebraic structures related to such maps f, and periods of arithmetical functions $\psi(x)=\sum_{s=1}^k\lambda_s e^{2\pi i a_s x/n_s}$ and $\omega(x)=|\{1\leqslant s\leqslant k:(x+a_s,n_s)=1\}|$.

1. Introduction and Preliminaries

Let S be a set. For an arithmetical map $\psi:\mathbb{Z}\to S$, if for some $n\in\mathbb{Z}^+=\{1,2,3,\cdots\}$ we have $\psi(x+n)=\psi(x)$ for all $x\in\mathbb{Z}$, then we say that ψ is *periodic modulo* n and n is a *period* of ψ. Let $\mathrm{P}(S)$ denote the set of all periodic maps $\psi:\mathbb{Z}\to S$. If $m,n\in\mathbb{Z}^+$ are periods of a map $\psi\in\mathrm{P}(S)$, then the greatest common divisor (m,n) is also a period of ψ, for we can write (m,n) in the form $am+bn$ with $a,b\in\mathbb{Z}$. Thus, any period of $\psi\in\mathrm{P}(S)$ is a multiple of the smallest (positive) period $n(\psi)$ of ψ.

A *monoid* is a semigroup with identity. Let M be a commutative monoid (considered as an additive one). If $\psi_1,\psi_2\in\mathrm{P}(M)$, then the map $\psi_1+\psi_2:x\mapsto\psi_1(x)+\psi_2(x)$ also lies in $\mathrm{P}(M)$ because $\psi_1+\psi_2$ is periodic modulo the least common multiple $[n(\psi_1),n(\psi_2)]$. In 1989 the author [S1] introduced triples of the form $\langle\lambda,a,n\rangle$ where $\lambda\in M$, $n\in\mathbb{Z}^+$ and $a\in R(n)=\{0,1,\cdots,n-1\}$. We can view $\langle\lambda,a,n\rangle$ as the *residue class* (or *arithmetic sequence*)

$$a(n)=a+n\mathbb{Z}=\{a+nx:x\in\mathbb{Z}\}\tag{1}$$

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associated with *weight* (or *multiplier*) λ. Let $\mathrm{S}(M)$ denote the family of all finite systems of such triples. For

$$\mathcal{A}=\{\langle\lambda_s,a_s,n_s\rangle\}_{s=1}^k\in\mathrm{S}(M), \tag{2}$$

the *covering map* $w_{\mathcal{A}}:\mathbb{Z}\to M$ is given by

$$w_{\mathcal{A}}(x)=\sum_{\substack{s=1\\ x\in a_s(n_s)}}^k\lambda_s\quad\text{for }x\in\mathbb{Z} \tag{3}$$

($w_{\emptyset}$ refers to the zero map), clearly $w_{\mathcal{A}}$ is periodic modulo the least common multiple $[n_1,\cdots,n_k]$ of all the moduli $n_1,\cdots,n_k$. Observe that any $\psi\in\mathrm{P}(M)$ periodic mod n is the covering map of the system $\{\langle\psi(r),r,n\rangle\}_{r=0}^{n-1}\in\mathrm{S}(M)$. For $\mathcal{A},\mathcal{B}\in\mathrm{S}(M)$ we define their formal union $\mathcal{A}\sqcup\mathcal{B}$ by putting triples in $\mathcal{A}$ and $\mathcal{B}$ altogether. Then $w:\mathcal{A}\mapsto w_{\mathcal{A}}$ gives an epimorphism of the commutative monoid $\mathrm{S}(M)$ (with respect to the formal union) onto the commutative monoid $\mathrm{P}(M)$. Two systems $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{S}(M)$ are said to be *equivalent* if they have the same covering map. We use $\sim$ to denote this congruence relation on $\mathrm{S}(M)$. Note that the quotient monoid $\mathrm{S}(M)/\sim$ is isomorphic to $\mathrm{P}(M)$.

When the additive monoid M is an abelian group, for system (2) we let $-\mathcal{A}$ be the system $\{\langle-\lambda_s,a_s,n_s\rangle\}_{s=1}^k$ and for $\psi\in\mathrm{P}(M)$ we let $-\psi$ be given by $-\psi(x)=-(\psi(x))$. Notice that $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{S}(M)$ are equivalent if and only if $\mathcal{A}\sqcup-\mathcal{B}\sim\emptyset$. By the fundamental theorem of homomorphisms the group $\mathrm{S}(M)/\mathrm{K}(M)$ is isomorphic to the abelian group $\mathrm{P}(M)$ where

$$\mathrm{K}(M)=\{\mathcal{A}\in\mathrm{S}(M):\ \mathcal{A}\sim\emptyset\ \ (\text{i.e. }w_{\mathcal{A}}=0)\}. \tag{4}$$

If M is an R-module where R is a ring with identity, then we can make $\mathrm{P}(M)$ and $\mathrm{S}(M)$ be R-modules by letting $a\psi(x)=a(\psi(x))$ and $a\mathcal{A}=\{\langle a\lambda_s,a_s,n_s\rangle\}_{s=1}^k$ for $a\in R$, $\psi\in\mathrm{P}(M)$ and system (2). Observe that the R-module $\mathrm{S}(M)/\mathrm{K}(M)$ is isomorphic to the R-module $\mathrm{P}(M)$. The so-called vector-covers of $\mathbb{Z}$ studied by Š. Znám [Z1, Z2] are those $\mathcal{A}\in\mathrm{S}(\mathbb{R})$ with $\mathcal{A}\sim\{\langle1,0,1\rangle\}$ where $\mathbb{R}$ is the field of real numbers.

For any $m,n\in\mathbb{Z}^+$ and $a\in R(n)$, apparently $\{\langle0,a,n\rangle\}\sim\emptyset$, also

$$\{\langle m,a,n\rangle\}\sim\{\underbrace{\langle1,a,n\rangle,\cdots,\langle1,a,n\rangle}_{m\text{ times}}\}\text{ and }\{\langle-m,a,n\rangle\}\sim-\{\underbrace{\langle1,a,n\rangle,\cdots,\langle1,a,n\rangle}_{m\text{ times}}\}.$$

So each $\mathcal{A}\in\mathrm{S}(\mathbb{Z})$ can be written as $\mathcal{A}_1\sqcup-\mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are in the form $\{(1,a_s,n_s)\}_{s=1}^k(k\geqslant0)$ which may be identified with

$$A=\{a_s(n_s)\}_{s=1}^k. \tag{5}$$

For system (5) of residue classes, if $w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|$ equals m for each $x \in \mathbb{Z}$, then as in [S5, S6] we call (5) an *exact m-cover* of $\mathbb{Z}$. (The study of covers of $\mathbb{Z}$ by residue classes was initiated by P. Erdös [E1], for problems and results in this area one may see R.K. Guy [G] and the introduction of the author [S5].) Notice that (5) is an exact m-cover of $\mathbb{Z}$ if and only if

$$\{a_s(n_s)\}_{s=1}^k \sim \{\underbrace{0(1), \cdots, 0(1)}_{m \text{ times}}\} \ \big(\text{i.e. } \{\langle 1, a_s, n_s\rangle\}_{s=1}^k \sqcup \{\langle -m, 0, 1\rangle\} \sim \emptyset\big).$$

Many known results concerning finite systems of residue classes with number weights can be expressed in the following form:

$$(6)\quad \{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^l \Rightarrow \sum_{s=1}^k \lambda_s f(a_s+n_s\mathbb{Z}) = \sum_{t=1}^l \mu_t f(b_t+m_t\mathbb{Z}).$$

Here are some examples of such results:

(a) (Erdös [E2]) $\{a_s(n_s)\}_{s=1}^k \sim \{0(1)\} \Longrightarrow \sum_{s=1}^k \frac{1}{n_s} = 1$.

(b) (B. Novák and Znám [NZ]) $\{a_s(n_s)\}_{s=1}^k \sim \{0(1)\} \Longrightarrow \sum_{s=1}^k \frac{z^{a_s}}{1-z^{n_s}} = \frac{1}{1-z}$ where z is any complex number with $|z| \neq 1$.

(c) (Porubský [P]) $\{\langle 1, a_s, n_s\rangle\}_{s=1}^k \sim \{\langle m, 0, 1\rangle\} \Longrightarrow \sum_{s=1}^k n_s^{n-1} B_n(\frac{a_s}{n_s}) = mB_n$ where $B_n(x)$ denotes the nth Bernoulli polynomial and $B_n = B_n(0)$.

(d) (Z. W. Sun [S3]) $\{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \emptyset \Longrightarrow \sum_{\substack{1\leqslant s\leqslant k\\ \alpha n_s\in\mathbb{Z}}} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} = 0$ where α is an arbitrary real number.

Let Ω be the family of all residue classes (i.e. $\Omega = \bigcup_{n\in\mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z}$). Then Ω forms a monoid with respect to the multiplication $\odot$ defined below:

$$(a + d\mathbb{Z}) \odot (r + n\mathbb{Z}) = a + rd + nd\mathbb{Z} \quad (a, r \in \mathbb{Z} \text{ and } d, n \in \mathbb{Z}^+).$$

For $a \in \mathbb{Z}$ and $d, n \in \mathbb{Z}^+$, clearly

$$\bigcup_{j=0}^{n-1} a + jd + nd\mathbb{Z} = a + d\mathbb{Z} \quad \text{and} \quad \{a + jd(nd)\}_{j=0}^{n-1} \sim \{a(d)\}.$$

Let M be an additive commutative monoid. The set of all maps $f : \Omega \to M$ is denoted by $\mathrm{F}(M)$, it can be viewed as a commutative monoid under the functional addition. A map $f : \Omega \to M$ is said to be *equivalent* if

$$(7)\qquad \sum_{j=0}^{n-1} f(a + jd + nd\mathbb{Z}) = f(a + d\mathbb{Z}) \quad \text{for any } a \in \mathbb{Z} \text{ and } d, n \in \mathbb{Z}^+.$$

(We may not have (7) even if $\sum_{r=0}^{n-1} f(r + n\mathbb{Z}) = f(\mathbb{Z})$ for all $n \in \mathbb{Z}^+$, for example $\sum_{r=0}^{n-1} \frac{2r+1}{2n^2} = \frac{2\cdot 0+1}{2\cdot 1^2}$ but $\frac{2\cdot 1+1}{2\cdot 4^2} + \frac{2\cdot 3+1}{2\cdot 4^2} \neq \frac{2\cdot 1+1}{2\cdot 2^2}$.) Those equivalent maps $f : \Omega \to M$ form a submonoid $\mathrm{E}(M)$ of $\mathrm{F}(M)$.

For any map ψ defined on $\mathbb{Z}$ we let $E^m\psi(x)=\psi(x+m)$ for any $m,x\in\mathbb{Z}$ and call $E=E^1$ the *shift operator*. Let S_1 and S_2 be sets. An operator $T: P(S_1)\to \mathrm{P}(S_2)$ is said to be commutable with E if $T(E(\psi))=E(T(\psi))$ for all $\psi\in \mathrm{P}(S_1)$. When T is commutable with E, if $\psi\in\mathrm{P}(S_1)$ is periodic mod n then so is $T(\psi)$ because $E^n(T(\psi))=T(E^n\psi)=T(\psi)$.

For any commutative monoids M and N, the set of all homomorphisms of M into N forms a commutative monoid naturally and we denote it by $\mathrm{Hom}(M,N)$; the set of those $T\in\mathrm{Hom}(\mathrm{P}(M),\mathrm{P}(N))$ commutable with E forms a submonoid of $\mathrm{Hom}(\mathrm{P}(M),\mathrm{P}(N))$ and we denote it by $\mathrm{Hom}'(\mathrm{P}(M),\mathrm{P}(N))$. If M and N are R-modules where R is a ring with identity, then the set of all R-module homomorphisms of M into N forms an R-module in a natural way and we denote it by $\mathrm{Hom}_R(M,N)$; the set of those $T\in\mathrm{Hom}_R(\mathrm{P}(M),\mathrm{P}(N))$ commutable with E forms a submodule of $\mathrm{Hom}_R(\mathrm{P}(M),\mathrm{P}(N))$ and we denote it by $\mathrm{Hom}'_R(\mathrm{P}(M),\mathrm{P}(N))$.

Let M and N be commutative monoids (or R-modules). In this paper we aim to determine all those $T\in\mathrm{Hom}'(\mathrm{P}(M),\mathrm{P}(N))$ (or $T\in\mathrm{Hom}'_R(\mathrm{P}(M),\mathrm{P}(N))$). For such an operator T and $\psi_1,\psi_2\in\mathrm{P}(M)$, $T(\psi_1+\psi_2)$ should depend on the smallest period of $\psi_1+\psi_2$, but usually we don't know the exact value of $n(\psi_1+\psi_2)$ even if $n(\psi_1)$ and $n(\psi_2)$ are given. This difficulty makes our goal more interesting and the task very challenging. As we will show in the next section, the problem is connected with $\mathrm{E}(N)$ closely. If R is a ring with identity and M is an R-module, then we can make $\mathrm{E}(R)$ form a ring with identity and $\mathrm{P}(M)$ form an $\mathrm{E}(R)$-module.

In Section 3 we are going to investigate $\mathrm{E}(\mathbb{C})$ thoroughly where $\mathbb{C}$ is the complex field, as an application we show the following central result which was announced by the author in [S2].

Theorem. *For $f\in\mathrm{F}(\mathbb{C})$, (6) holds if and only if f has the following form:*

$$f(a+n\mathbb{Z})=\frac{1}{n}\sum_{m=0}^{n-1}\psi\left(\frac{m}{n}\right)e^{2\pi i\frac{m}{n}a}\quad (a\in\mathbb{Z}\ \text{and}\ n\in\mathbb{Z}^+).$$

Now we state two more results in this paper:

I. *Let $\lambda_1,\cdots,\lambda_k\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, and $\xi_1,\cdots,\xi_k$ be distinct roots of unity. Then the smallest (positive) period of the arithmetical function $\psi(x)=\sum_{s=1}^k\lambda_s\xi_s^x$, coincides with $[n_1,\cdots,n_k]$ where n_s is the least $n\in\mathbb{Z}^+$ with $\xi_s^n=1$ (i.e., ξ_s is a primitive n_sth root of unity).*

II. *Let (5) be a system of residue classes with $n_1,\cdots,n_k$ squarefree and $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$. If $|\{1\leqslant s\leqslant k:(x+a_s,n_s)=1\}|=m$ for all $x\in\mathbb{Z}$, then $l\geqslant\min_{1\leqslant s\leqslant k-l}n_k/(n_s,n_k)$, furthermore*

$$\frac{l}{n_k}=\sum_{s=1}^{k-l}\frac{x_s}{(n_s,n_k)}\quad \textit{for some } x_1,\cdots,x_{k-l}\in\mathbb{N}=\{0,1,2,\cdots\}.$$

2. Algebraic structures concerning periodic arithmetical maps

Let us first look at

Example 1. Let M be a commutative monoid M considered as an additive one. Fix $\lambda \in M$. If $a, x \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ then we let

$$\lambda_{a+n\mathbb{Z}}(x) = (\lambda)_x(a+n\mathbb{Z}) = \begin{cases} \lambda & \text{if } x \in a+n\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Evidently $\lambda_{a+n\mathbb{Z}} : \mathbb{Z} \to M$ belongs to $\mathrm{P}(M)$ for any $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, and $(\lambda)_x : \Omega \to M$ lies in $\mathrm{E}(M)$ for each $x \in \mathbb{Z}$.

This example suggests that periodic arithmetical maps might be related to equivalent maps.

Lemma 1. *Let M and N be additive commutative monoids. Let τ be a map of M into $\mathrm{E}(N)$ and define the operator $S_\tau : \mathrm{P}(M) \to \mathrm{P}(N)$ as follows:*

$$S_\tau(\psi)(x) = \sum_{r=0}^{n(\psi)-1} \tau(\psi(x-r))(r+n(\psi)\mathbb{Z}) \quad \text{for } \psi \in \mathrm{P}(M) \text{ and } x \in \mathbb{Z}.$$

Then

(i) *For any period $n \in \mathbb{Z}^+$ of $\psi \in \mathrm{P}(M)$ we have*

$$S_\tau(\psi)(x) = \sum_{r=0}^{n-1} \tau(\psi(x-r))(r+n\mathbb{Z}) \quad \text{for all } x \in \mathbb{Z}. \tag{9}$$

(ii) *The operator S_τ is commutable with E.*

(iii) $S_\tau \in \mathrm{Hom}(\mathrm{P}(M), \mathrm{P}(N))$ *if* $\tau \in \mathrm{Hom}(M, \mathrm{E}(N))$.

Proof. Let us first prove (i) and (ii). Suppose that $\psi \in P(M)$ is periodic mod n. Then $n_0 = n(\psi)$ divides n. For any $x \in \mathbb{Z}$ we have

$$\begin{aligned}\sum_{r=0}^{n-1} \tau(\psi(x-r))(r+n\mathbb{Z}) &= \sum_{u=0}^{n_0-1} \sum_{v=0}^{n/n_0-1} \tau(\psi(x-(u+vn_0)))(u+vn_0+n\mathbb{Z}) \\ &= \sum_{u=0}^{n_0-1} \sum_{v=0}^{n/n_0-1} \tau(\psi(x-u)) \left(u+vn_0+\frac{n}{n_0}n_0\mathbb{Z}\right) \\ &= \sum_{u=0}^{n_0-1} \tau(\psi(x-u))(u+n_0\mathbb{Z}) = S_\tau(\psi)(x)\end{aligned}$$

and

$$\begin{aligned}S_\tau(E\psi)(x) &= \sum_{r=0}^{n-1} \tau(E\psi(x-r))(r+n\mathbb{Z}) = \sum_{r=0}^{n-1} \tau(\psi(x+1-r))(r+n\mathbb{Z}) \\ &= S_\tau(\psi)(x+1) = E(S_\tau(\psi))(x).\end{aligned}$$

Now let $\tau\in\mathrm{Hom}(M,\mathrm{E}(N))$. We come to show that $S_\tau\in\mathrm{Hom}(\mathrm{P}(M),\mathrm{P}(N))$. Apparently

$$S_\tau(0)(x)=\tau(0)(0+1\mathbb{Z})=0\quad\text{for all } x\in\mathbb{Z}.$$

If $\psi_1,\psi_2\in\mathrm{P}(M)$ have periods $n_1,n_2\in\mathbb{Z}^+$ respectively, then ψ_1,ψ_2 and $\psi_1+\psi_2$ are periodic mod n where $n=[n_1,n_2]$ and so

$$\begin{aligned}S_\tau(\psi_1+\psi_2)(x)&=\sum_{r=0}^{n-1}\tau((\psi_1+\psi_2)(x-r))(r+n\mathbb{Z})\\&=\sum_{r=0}^{n-1}(\tau(\psi_1(x-r))+\tau(\psi_2(x-r)))(r+n\mathbb{Z})\\&=\sum_{r=0}^{n-1}\tau(\psi_1(x-r))(r+n\mathbb{Z})+\sum_{r=0}^{n-1}\tau(\psi_2(x-r))(r+n\mathbb{Z})\\&=S_\tau(\psi_1)(x)+S_\tau(\psi_2)(x)\end{aligned}$$

for every integer x. Thus $S_\tau\in\mathrm{Hom}(\mathrm{P}(M),\mathrm{P}(N))$. □

Now we give

Theorem 1. *Let M and N be additive commutative monoids. For any $\lambda\in M$ and $T\in\mathrm{Hom}'(\mathrm{P}(M),\mathrm{P}(N))$ we let*

$$\sigma_T(\lambda)(a+n\mathbb{Z})=T(\lambda_{n\mathbb{Z}})(a)\quad\text{for } a\in\mathbb{Z} \text{ and } n\in\mathbb{Z}^+. \tag{10}$$

Then $\sigma:T\mapsto\sigma_T$ gives an isomorphism of the monoid $\mathrm{Hom}'(\mathrm{P}(M),\mathrm{P}(N))$ *onto* $\mathrm{Hom}(M,\mathrm{E}(N))$.

Proof. Fix $T\in\mathrm{Hom}'(\mathrm{P}(M),\mathrm{P}(N))$. Let $\lambda\in M$. For each $n\in\mathbb{Z}^+$, $\lambda_{n\mathbb{Z}}\in\mathrm{P}(M)$ is periodic mod n and hence so is $T(\lambda_{n\mathbb{Z}})\in\mathrm{P}(N)$. Clearly

$$T(\lambda_{n\mathbb{Z}})(m)=E^m(T(\lambda_{n\mathbb{Z}}))(0)=T(E^m\lambda_{n\mathbb{Z}})(0)=T(\lambda_{-m+n\mathbb{Z}})(0)$$

for all $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$. For any $d,n\in\mathbb{Z}^+$ and $a\in\mathbb{Z}$ we have

$$\begin{aligned}&\sum_{j=0}^{n-1}\sigma_T(\lambda)(a+jd+nd\mathbb{Z})=\sum_{j=0}^{n-1}T(\lambda_{nd\mathbb{Z}})(a+jd)\\&=\sum_{j=0}^{n-1}T(\lambda_{-(a+jd)+nd\mathbb{Z}})(0)=T\bigg(\sum_{j=0}^{n-1}\lambda_{-(a+jd)+nd\mathbb{Z}}\bigg)(0)\\&=T(\lambda_{-a+d\mathbb{Z}})(0)=T(\lambda_{d\mathbb{Z}})(a)=\sigma_T(\lambda)(a+d\mathbb{Z}).\end{aligned}$$

Therefore $\sigma_T:\lambda\mapsto\sigma_T(\lambda)$ is a map from M into $\mathrm{E}(N)$.

Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Clearly $\sigma_T(0)(a+n\mathbb{Z}) = T(0_{n\mathbb{Z}})(a) = 0$. If $\lambda, \mu \in M$ then

$$\begin{aligned}\sigma_T(\lambda+\mu)(a+n\mathbb{Z}) &= T((\lambda+\mu)_{n\mathbb{Z}})(a) = T(\lambda_{n\mathbb{Z}} + \mu_{n\mathbb{Z}})(a)\\ &= T(\lambda_{n\mathbb{Z}})(a) + T(\mu_{n\mathbb{Z}})(a) = \sigma_T(\lambda)(a+n\mathbb{Z}) + \sigma_T(\mu)(a+n\mathbb{Z}).\end{aligned}$$

Thus $\sigma_T \in \mathrm{Hom}(M, \mathrm{E}(N))$.

We assert that $S_{\sigma_T} = T$. Let $\psi \in \mathrm{P}(M)$ be periodic mod n and x be an integer. Then

$$\sum_{r=0}^{n-1} \psi(x-r)_{-r+n\mathbb{Z}}(a) = \sum_{\substack{r=0\\ n\mid a+r}}^{n-1} \psi(x-r) = \psi(x+a) = E^x\psi(a) \text{ for all } a \in \mathbb{Z}$$

and hence

$$\begin{aligned}S_{\sigma_T}(\psi)(x) &= \sum_{r=0}^{n-1} \sigma_T(\psi(x-r))(r+n\mathbb{Z}) = \sum_{r=0}^{n-1} T(\psi(x-r)_{n\mathbb{Z}})(r)\\ &= \sum_{r=0}^{n-1} T(\psi(x-r)_{-r+n\mathbb{Z}})(0) = T\left(\sum_{r=0}^{n-1} \psi(x-r)_{-r+n\mathbb{Z}}\right)(0)\\ &= T(E^x\psi)(0) = E^x(T(\psi))(0) = T(\psi)(x).\end{aligned}$$

Let $\tau \in \mathrm{Hom}(M, \mathrm{E}(N))$. Then $S_\tau \in \mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(N))$ by Lemma 1. For any $\lambda \in M$, $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\sigma_{S_\tau}(\lambda)(a+n\mathbb{Z}) &= S_\tau(\lambda_{n\mathbb{Z}})(a) = \sum_{r=0}^{n-1} \tau(\lambda_{n\mathbb{Z}}(a-r))(r+n\mathbb{Z})\\ &= \sum_{\substack{r=0\\ n\mid a-r}}^{n-1} \tau(\lambda)(r+n\mathbb{Z}) + \sum_{\substack{r=0\\ n\nmid a-r}}^{n-1} \tau(0)(r+n\mathbb{Z}) = \tau(\lambda)(a+n\mathbb{Z}).\end{aligned}$$

So $\sigma_{S_\tau} = \tau$.

In view of the above, map $\sigma : T \mapsto \sigma_T$ of $\mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(N))$ into $\mathrm{Hom}(M, \mathrm{E}(N))$ is bijective and its inverse is the map $S : \tau \mapsto S_\tau$ from $\mathrm{Hom}(M, \mathrm{E}(N))$ into $\mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(N))$.

It is easy to see that $\sigma : \mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(N)) \to \mathrm{Hom}(M, \mathrm{E}(N))$ is a monoid homomorphism. So the two monoids $\mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(N))$ and $\mathrm{Hom}(M, \mathrm{E}(N))$ are isomorphic via the map σ. □

With the help of Theorem 1 we can present

Theorem 2. *Let M be an R-module where R is a ring with identity. For any $f \in \mathrm{E}(M)$ we define $T_f : \mathrm{P}(R) \to \mathrm{P}(M)$ by letting*

$$(11)\quad T_f(\psi)(x) = \sum_{r=0}^{n-1} \psi(x-r) f(r+n\mathbb{Z}) \quad \textit{for } x \in \mathbb{Z} \textit{ and } \psi \in \mathrm{P}(R) \textit{ with period } n.$$

Then $T : f \mapsto T_f$ *gives an isomorphism of the additive abelian group* $\mathrm{E}(M)$ *onto* $\mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$*. If* R *is commutative then the* R*-modules* $\mathrm{E}(M)$ *and* $\mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$ *are module isomorphic via the map* T*.*

Proof. Let $f \in \mathrm{E}(M)$ and set $\tau_f(\lambda) = \lambda f$ for $\lambda \in R$. Then $\tau_f \in \mathrm{Hom}_R(R, \mathrm{E}(M))$ and $T_f = S_{\tau_f} \in \mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$.

Evidently map $\tau : f \mapsto \tau_f$ gives a homomorphism of abelian group $\mathrm{E}(M)$ into $\mathrm{Hom}_R(R, \mathrm{E}(M))$ and

$$\mathrm{Ker}\tau = \{f \in \mathrm{E}(M) : \ \tau_f = 0\} \subseteq \{f \in \mathrm{E}(M) : \ \tau_f(1) = 0\} = \{0\}.$$

If $h \in \mathrm{Hom}_R(R, \mathrm{E}(M))$ and $\lambda \in R$, then $h(\lambda) = h(\lambda \cdot 1) = \lambda(h(1)) = \tau_{h(1)}(\lambda)$. So the additive groups $\mathrm{E}(M)$ and $\mathrm{Hom}_R(R, \mathrm{E}(M))$ are isomorphic via the map τ.

For any $H \in \mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$, we have $f = \sigma_H(1) \in \mathrm{E}(M)$, also $\sigma_H = \tau_f \in \mathrm{Hom}_R(R, \mathrm{E}(M))$ and $H = S_{\sigma_H} = S_{\tau_f}$. It follows that the abelian group $\mathrm{Hom}_R(R, \mathrm{E}(M))$ is isomorphic to $\mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$.

Combining the above we obtain that $T : f \mapsto T_f = S_{\tau_f}$ gives an isomorphism of the additive group $\mathrm{E}(M)$ onto $\mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$.

When R is commutative, if $\lambda \in R$, $f \in \mathrm{E}(M)$ and $\psi \in \mathrm{P}(R)$ then $T_{\lambda f}(\psi) = \lambda T_f(\psi)$, therefore the map $T : f \mapsto T_f$ is an R-module isomorphism of $\mathrm{E}(M)$ onto $\mathrm{Hom}'_R(\mathrm{P}(R), \mathrm{P}(M))$. $\square$

Example 2. Let M be a commutative monoid. For any integer m operator E^m : $\mathrm{P}(M) \to \mathrm{P}(M)$ is in $\mathrm{End}'(\mathrm{P}(M)) = \mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(M))$ and the corresponding $\sigma_{E^m} \in \mathrm{Hom}(M, \mathrm{E}(M))$ is given by $\sigma_{E^m}(\lambda) = (\lambda)_{-m}$ where $\lambda \in M$. Let R be a ring with identity 1. We can view R as an R-module. When M forms an R-module, for any $\lambda \in M$ and $m \in \mathbb{Z}$ we have $T_{(\lambda)_m}(\psi)(x) = (E^{-m}\psi(x))\lambda$ where $\psi \in \mathrm{P}(R)$ and $x \in \mathbb{Z}$. Clearly $e_m = (1)_m \in \mathrm{E}(R)$ and $T_{e_m} = E^{-m}$ for all $m \in \mathbb{Z}$, in particular T_e is the identical map of $\mathrm{P}(R)$ onto itself where $e = e_0$ lies in $\mathrm{E}(R)$.

Let R be a ring. For $f, g \in \mathrm{F}(R)$, we define the *convolution* $f * g \in \mathrm{F}(R)$ by

$$f * g(a + n\mathbb{Z}) = \sum_{r=0}^{n-1} f(r + n\mathbb{Z})g(a - r + n\mathbb{Z}) \quad (a \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+). \tag{12}$$

Example 3. Let R be a ring. If $\lambda, \mu \in R$ and $m, n \in \mathbb{Z}$, then $(\lambda)_m, (\mu)_n \in \mathrm{E}(R)$ by Example 1, we can easily verify that $(\lambda)_m * (\mu)_n = (\lambda\mu)_{m+n} \in \mathrm{E}(R)$. Thus, when R has identity 1, $e_m * e_n = e_{m+n}$ for all $m, n \in \mathbb{Z}$.

Example 4. For $\alpha \in \mathbb{Q} \cap [0, 1)$ we define $\rho_\alpha \in \mathrm{F}(R)$ by

$$\rho_\alpha(a + n\mathbb{Z}) = \frac{1}{n} \sum_{\substack{m=0 \\ m/n=\alpha}}^{n-1} e^{2\pi i \frac{m}{n} a} = \begin{cases} \frac{1}{n} e^{2\pi i \alpha a} & \text{if } \alpha n \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

When $\alpha, \beta \in \mathbb{Q} \cap [0,1)$, $a \in \mathbb{Z}$ and $d, n \in \mathbb{Z}^+$, we have

$$\sum_{j=0}^{n-1} \rho_\alpha(a + jd + nd\mathbb{Z}) = \frac{1}{nd} \sum_{\substack{m=0 \\ m/(nd)=\alpha}}^{nd-1} \sum_{j=0}^{n-1} e^{2\pi i \frac{m}{nd}(a+jd)}$$

$$= \frac{1}{nd} \sum_{\substack{m=0 \\ m/(nd)=\alpha}}^{nd-1} e^{2\pi i \frac{ma}{nd}} \sum_{j=0}^{n-1} e^{2\pi i \frac{m}{n} j} = \frac{1}{d} \sum_{\substack{m=0 \\ m/(nd)=\alpha \\ n|m}}^{nd-1} e^{2\pi i \frac{ma}{nd}}$$

$$= \frac{1}{d} \sum_{\substack{r=0 \\ r/d=\alpha}}^{d-1} e^{2\pi i \frac{r}{d} a} = \rho_\alpha(a + d\mathbb{Z}),$$

and

$$\rho_\alpha * \rho_\beta(a + n\mathbb{Z}) = \sum_{\substack{r=0 \\ \alpha n, \beta n \in \mathbb{Z}}}^{n-1} \frac{1}{n} e^{2\pi i \alpha r} \cdot \frac{1}{n} e^{2\pi i \beta(a-r)}$$

$$= \frac{1}{n^2} e^{2\pi i \beta a} \sum_{\substack{r=0 \\ \alpha n, \beta n \in \mathbb{Z}}}^{n-1} e^{2\pi i(\alpha-\beta)r} = \begin{cases} \rho_\alpha(a + n\mathbb{Z}) & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

So $\{\rho_\alpha\}_{\alpha \in \mathbb{Q} \cap [0,1)}$ is an orthogonal system of functions in $\mathrm{E}(\mathbb{C})$ with respect to convolution $*$.

Lemma 2. *Let R be a ring, and $f, g, h \in \mathrm{F}(R)$. Then $(f * g) * h = f * (g * h)$, also $f * g \in \mathrm{E}(R)$ if $f, g \in \mathrm{E}(R)$.*

Proof. Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. It is easy to check that

$$(f * g) * h(a + n\mathbb{Z}) = f * (g * h)(a + n\mathbb{Z}).$$

If $f, g \in \mathrm{E}(R)$ and $d \in \mathbb{Z}^+$, then

$$\begin{aligned}
\sum_{j=0}^{n-1} f * g(a + jd + nd\mathbb{Z}) &= \sum_{j=0}^{n-1} \sum_{m=0}^{nd-1} f(m + nd\mathbb{Z}) g(a + jd - m + nd\mathbb{Z}) \\
&= \sum_{m=0}^{nd-1} f(m + nd\mathbb{Z}) \sum_{j=0}^{n-1} g(a - m + jd + nd\mathbb{Z}) \\
&= \sum_{r=0}^{d-1} \sum_{s=0}^{n-1} f(r + sd + nd\mathbb{Z}) g(a - (r + sd) + d\mathbb{Z}) \\
&= \sum_{r=0}^{d-1} f(r + d\mathbb{Z}) g(a - r + d\mathbb{Z}) = f * g(a + d\mathbb{Z}).
\end{aligned}$$

This ends the proof. □

Lemma 3. *Let M be an additive commutative monoid and $\{f_\alpha\}_{\alpha\in S}$ a family of maps in* $\mathrm{F}(M)$ *such that* $\{\alpha\in S: f_\alpha(a+n\mathbb{Z})\neq 0\}$ *is finite for any $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$. Define the map* $\sum_{\alpha\in S} f_\alpha\in\mathrm{F}(M)$ *by* $(\sum_{\alpha\in S} f_\alpha)(a+n\mathbb{Z})=\sum_{\alpha\in S} f_\alpha(a+n\mathbb{Z})$.

(i) *If $f_\alpha\in\mathrm{E}(M)$ for all $\alpha\in S$, then $\sum_{\alpha\in S} f_\alpha\in\mathrm{E}(M)$.*

(ii) *If M is a ring R and g is in $\mathrm{F}(R)$, then*

$$\{\alpha\in S:\ f_\alpha * g(a+n\mathbb{Z})\neq 0\}\quad \textit{and}\quad \{\alpha\in S:\ g*f_\alpha(a+n\mathbb{Z})\neq 0\}$$

are finite for all $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$, moreover

$$\left(\sum_{\alpha\in S} f_\alpha\right)*g=\sum_{\alpha\in S}(f_\alpha*g)\quad \textit{and}\quad g*\sum_{\alpha\in S} f_\alpha=\sum_{\alpha\in S}(g*f_\alpha)\,.$$

Proof. i) Let $a\in\mathbb{Z}$ and $d,n\in\mathbb{Z}^+$. Then the set

$$S'=\{\alpha\in S: f_\alpha(a+jd+nd\mathbb{Z})\neq 0 \text{ for some } j\in R(n)\}$$

is finite. As $f_\alpha\in\mathrm{E}(M)$ for all $\alpha\in S$, S' contains $\{\alpha\in S: f_\alpha(a+d\mathbb{Z})\neq 0\}$. Thus

$$\begin{aligned}&\sum_{j=0}^{n-1}\sum_{\alpha\in S} f_\alpha(a+jd+nd\mathbb{Z})=\sum_{j=0}^{n-1}\sum_{\alpha\in S'} f_\alpha(a+jd+nd\mathbb{Z})\\ =&\sum_{\alpha\in S'}\sum_{j=0}^{n-1} f_\alpha(a+jd+nd\mathbb{Z})=\sum_{\alpha\in S'} f_\alpha(a+d\mathbb{Z})=\sum_{\alpha\in S} f_\alpha(a+d\mathbb{Z}).\end{aligned}$$

ii) Let $a\in\mathbb{Z}$, $n\in\mathbb{Z}^+$ and

$$S_*=\{\alpha\in S: f_\alpha(r+n\mathbb{Z})\neq 0 \text{ for some } r\in R(n)\}.$$

Then S_* is finite, and for $\alpha\in S\setminus S_*$ we have $f_\alpha*g(a+n\mathbb{Z})=0=g*f_\alpha(a+n\mathbb{Z})$. Thus both $\{\alpha\in S: f_\alpha*g(a+n\mathbb{Z})\neq 0\}$ and $\{\alpha\in S: g*f_\alpha(a+n\mathbb{Z})\neq 0\}$ are subsets of the finite set S_*. Observe that

$$\begin{aligned}\left(\sum_{\alpha\in S} f_\alpha\right)*g(a+n\mathbb{Z})=&\sum_{r=0}^{n-1}\sum_{\alpha\in S_*} f_\alpha(r+n\mathbb{Z})g(a-r+n\mathbb{Z})\\ =&\sum_{\alpha\in S_*} f_\alpha*g(a+n\mathbb{Z})=\sum_{\alpha\in S} f_\alpha*g(a+n\mathbb{Z}).\end{aligned}$$

Similarly

$$\left(g*\sum_{\alpha\in S} f_\alpha\right)(a+n\mathbb{Z})=\sum_{\alpha\in S_*} g*f_\alpha(a+n\mathbb{Z})=\sum_{\alpha\in S} g*f_\alpha(a+n\mathbb{Z}).$$

We are done. □

Example 5. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, clearly

$$\{\alpha \in \mathbb{Q} \cap [0,1) : \rho_\alpha(a+n\mathbb{Z}) \neq 0\} = \{\alpha \in \mathbb{Q} \cap [0,1) : \alpha n \in \mathbb{Z}\} = \left\{\frac{m}{n} : m \in R(n)\right\}.$$

Since $\rho_\alpha \in \mathrm{E}(\mathbb{C})$ for all $\alpha \in \mathbb{Q} \cap [0,1)$, $\check{\psi} = \sum_{\alpha \in \mathbb{Q}\cap[0,1)} \psi(\alpha)\rho_\alpha \in \mathrm{E}(\mathbb{C})$ where ψ is any map from $\mathbb{Q} \cap [0,1)$ into $\mathbb{C}$. Note that

$$\check{\psi}(a+n\mathbb{Z}) = \frac{1}{n}\sum_{m=0}^{n-1} \psi\left(\frac{m}{n}\right) e^{2\pi i \frac{m}{n} a} \quad \text{for } a \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+. \tag{14}$$

Clearly $e = \check{I}$ where $I(x) = 1$ for all $x \in \mathbb{Q} \cap [0,1)$. For any functions $\psi, \chi : \mathbb{Q} \cap [0,1) \to \mathbb{C}$, by Lemma 3 and Example 4 we have

$$\check{\psi} * \check{\chi} = \sum_{\alpha \in \mathbb{Q}\cap[0,1)} \psi(\alpha) \sum_{\beta \in \mathbb{Q}\cap[0,1)} \chi(\beta)(\rho_\alpha * \rho_\beta) = \sum_{\alpha \in \mathbb{Q}\cap[0,1)} \psi(\alpha)\chi(\alpha)\rho_\alpha. \tag{15}$$

Let M be an R-module where R is a ring with identity. For $f \in \mathrm{E}(R)$ and $\psi \in \mathrm{P}(M)$ we define $f \circ \psi \in \mathrm{P}(M)$ by

$$f \circ \psi(x) = \sum_{r=0}^{n-1} f(r+n\mathbb{Z})\psi(x-r) \quad \text{where } n \in \mathbb{Z}^+ \text{ is a period of } \psi. \tag{16}$$

(Note that $f \circ \psi = S_\tau(\psi)$ where $\tau = \tau_f \in \mathrm{Hom}(M, \mathrm{E}(M))$ is given by $\tau_f(x)(a+n\mathbb{Z}) = f(a+n\mathbb{Z})x$ ($x \in M,\ a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$) and S_τ is as in Lemma 1.)

Theorem 3. *Let R be a ring. Then*

(i) $\mathrm{F}(R)$ *forms a ring with subring* $\mathrm{E}(R)$ *under the natural addition + and the convolution $*$. When R is commutative, so is* $\mathrm{F}(R)$*; if* $\mathrm{E}(R)$ *is commutative then so is R.*

(ii) *Suppose that R has identity* 1*. Then* $\mathrm{F}(R)$ *has identity* $e \in \mathrm{E}(R)$*. Furthermore, for any R-module M,* $\mathrm{P}(M)$ *forms an* $\mathrm{E}(R)$*-module with respect to the natural addition + and the scalar multiplication ∘.*

Proof. i) Since R is an additive abelian group, so is $\mathrm{F}(R)$. By Lemmas 2 and 3 we have the associative law and the distributive laws. Thus $\mathrm{F}(R)$ forms a ring. In view of Lemma 2, $\mathrm{E}(R)$ is a subring of $\mathrm{F}(R)$.

If R is commutative, then for any $f, g \in \mathrm{F}(R)$, $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we have

$$f*g(a+n\mathbb{Z}) = \sum_{r=0}^{n-1} f(r+n\mathbb{Z})g(a-r+n\mathbb{Z}) = \sum_{s=0}^{n-1} g(s+n\mathbb{Z})f(a-s+n\mathbb{Z}) = g*f(a+n\mathbb{Z}),$$

therefore $\mathrm{F}(R)$ is commutative. On the other hand, if $\mathrm{E}(R)$ is commutative, then so is R because by Example 3 for each $\lambda, \mu \in R$, $m_1, m_2 \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\lambda\mu =&(\lambda\mu)_{m_1+m_2}(m_1+m_2+n\mathbb{Z}) = (\lambda)_{m_1} * (\mu)_{m_2}(m_1+m_2+n\mathbb{Z})\\ =&(\mu)_{m_2} * (\lambda)_{m_1}(m_1+m_2+n\mathbb{Z}) = \mu\lambda.\end{aligned}$$

ii) Suppose that R has identity 1. By Examples 1 and 2, $e = e_0 = (1)_0 \in \mathrm{E}(R) \subseteq \mathrm{F}(R)$. It is clear that $e * f = f = f * e$ for all $f \in \mathrm{F}(R)$.

Let M be arbitrary R-module. Then $\mathrm{P}(M)$ forms an additive abelian group. Let $f, g \in \mathrm{E}(R)$ and $\psi, \chi \in \mathrm{P}(M)$. Obviously $(f+g)\circ\psi = f\circ\psi + g\circ\psi$. For any $x \in M$ the map $\tau_f(x) : a + n\mathbb{Z} \mapsto f(a+n\mathbb{Z})x$ lies in $\mathrm{E}(M)$. Clearly $\tau_f \in \mathrm{Hom}(M, \mathrm{E}(M))$ and hence $S_{\tau_f} \in \mathrm{End}'(\mathrm{P}(M)) = \mathrm{Hom}'(\mathrm{P}(M), \mathrm{P}(M))$ by Lemma 1. Thus

$$f\circ(\psi+\chi) = S_{\tau_f}(\psi+\chi) = S_{\tau_f}(\psi) + S_{\tau_f}(\chi) = f\circ\psi + f\circ\chi.$$

Let $n \in \mathbb{Z}^+$ be a period of ψ, then for each $x \in \mathbb{Z}$ we have

$$e\circ\psi(x) = \sum_{r=0}^{n-1} e(r+n\mathbb{Z})\psi(x-r) = \psi(x)$$

and

$$\begin{aligned}(f*g)\circ\psi(x) =& \sum_{r=0}^{n-1} f*g(r+n\mathbb{Z})\psi(x-r)\\ =&\sum_{r=0}^{n-1}\sum_{s=0}^{n-1} f(s+n\mathbb{Z})g(r-s+n\mathbb{Z})\psi(x-r)\\ =&\sum_{s=0}^{n-1} f(s+n\mathbb{Z})\sum_{r=0}^{n-1} g(r-s+n\mathbb{Z})\psi(x-r)\\ =&\sum_{s=0}^{n-1} f(s+n\mathbb{Z})\sum_{t=0}^{n-1} g(t+n\mathbb{Z})\psi(x-s-t) = f\circ(g\circ\psi)(x).\end{aligned}$$

Thus $\mathrm{P}(M)$ forms an $\mathrm{E}(R)$-module. The proof is ended. □

3. Equivalent maps and their applications

A subset D of $\mathbb{Z}^+$ is said to be *divisor-closed* if any (positive) divisor of an element of D belongs to D. We set

$$[0,1)_D = \{0 \leqslant \alpha < 1 :\ \alpha n \in \mathbb{Z} \text{ for some } n \in D\}.$$

Theorem 4. *Let $D \subseteq \mathbb{Z}^+$ be divisor-closed. For a function $f : \bigcup_{n\in D} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ the following statements are equivalent:*

(a) *For all $a \in \mathbb{Z}$ and $d, n \in \mathbb{Z}^+$ with $nd \in D$,*

$$\sum_{j=0}^{n-1} f(a + jd + nd\mathbb{Z}) = f(a + d\mathbb{Z}).$$

(b) *There exists a function $\psi : [0,1)_D \to \mathbb{C}$ such that*

$$f(a + n\mathbb{Z}) = \frac{1}{n}\sum_{m=0}^{n-1} \psi\left(\frac{m}{n}\right) e^{2\pi i \frac{m}{n} a} \quad \text{for any } a \in \mathbb{Z} \text{ and } n \in D. \tag{17}$$

(c) *There is a function $g : \bigcup_{n\in D} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ such that*

$$f(a + n\mathbb{Z}) = \frac{1}{n}\sum_{m|n}\sum_{d|m} \mu\left(\frac{m}{d}\right) \sum_{r=0}^{d-1} g\left(\frac{m}{d} r + m\mathbb{Z}\right) e^{2\pi i \frac{r}{d} a} \tag{18}$$

holds for all $a \in \mathbb{Z}$ and $n \in D$ where μ denotes the Möbius function.

Proof. (a)⇒(b). For any $n \in D$ we set

$$g(m + n\mathbb{Z}) = \sum_{r=0}^{n-1} f(r + n\mathbb{Z}) e^{-2\pi i \frac{m}{n} r} \quad \text{for each } m = 0, 1, \cdots, n-1.$$

It is well-known that

$$f(a + n\mathbb{Z}) = \frac{1}{n}\sum_{m=0}^{n-1} g(m + n\mathbb{Z}) e^{2\pi i \frac{m}{n} a} \qquad \text{for all } a = 0, 1, \cdots, n-1.$$

Now we show that $g(m + n\mathbb{Z})$ $(m \in R(n))$ only depends on the rational $m/n \in [0,1)_D$, i.e. $g(m+n\mathbb{Z}) = g(\frac{m}{d} + \frac{n}{d}\mathbb{Z})$ where $d = (m, n)$ and hence $\frac{m/d}{n/d}$ is the reduced form of $\frac{m}{n}$. In fact, for each $a \in \mathbb{Z}$ we have

$$\begin{aligned}
&\frac{d}{n}\sum_{k=0}^{\frac{n}{d}-1} g\left(k + \frac{n}{d}\mathbb{Z}\right) e^{2\pi i \frac{k}{n/d} a} = f\left(a + \frac{n}{d}\mathbb{Z}\right) = \sum_{j=0}^{d-1} f\left(a + j\frac{n}{d} + \frac{n}{d} d\mathbb{Z}\right) \\
=& \sum_{\substack{r=0 \\ r \in a + \frac{n}{d}\mathbb{Z}}}^{n-1} \frac{1}{n}\sum_{k=0}^{n-1} g(k + n\mathbb{Z}) e^{2\pi i \frac{k}{n} r} = \frac{1}{n}\sum_{k=0}^{n-1} g(k + n\mathbb{Z}) \sum_{j=0}^{d-1} e^{2\pi i \frac{k}{n}(a + j\frac{n}{d})} \\
=& \frac{1}{n}\sum_{k=0}^{n-1} g(k + n\mathbb{Z}) e^{2\pi i \frac{k}{n} a} \sum_{j=0}^{d-1} e^{2\pi i \frac{k}{d} j} = \frac{d}{n}\sum_{\substack{k=0 \\ d|k}}^{n-1} g(k + n\mathbb{Z}) e^{2\pi i \frac{k}{n} a} \\
=& \frac{d}{n}\sum_{l=0}^{\frac{n}{d}-1} g(dl + n\mathbb{Z}) e^{2\pi i \frac{l}{n/d} a},
\end{aligned}$$

so $g(k+\frac{n}{d}\mathbb{Z})=g(dk+n\mathbb{Z})$ for any $k\in R(\frac{n}{d})$, in particular $g(\frac{m}{d}+\frac{n}{d}\mathbb{Z})=g(m+n\mathbb{Z})$ for all $m\in R(n)$.

(b)⇔(c). Let g be any function of $\bigcup_{n\in D}\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{C}$. Let $a\in\mathbb{Z}$ and $n\in D$. If $m\in\mathbb{Z}^+$ divides n, then

$$\sum_{\substack{k=0\\(k,n)=\frac{n}{m}}}^{n-1} g\left(\frac{k}{n/m}+\frac{n}{n/m}\mathbb{Z}\right)e^{2\pi i\frac{k}{n}a}=\sum_{\substack{u=0\\(\frac{n}{m}u,n)=\frac{n}{m}}}^{m-1} g\left(\frac{nu/m}{n/m}+m\mathbb{Z}\right)e^{2\pi i\frac{nu/m}{n}a}$$

$$=\sum_{u=0}^{m-1}\sum_{d\mid(u,m)}\mu(d)g(u+m\mathbb{Z})e^{2\pi i\frac{u}{m}a}=\sum_{d\mid m}\mu(d)\sum_{\substack{u=0\\d\mid u}}^{m-1}g(u+m\mathbb{Z})e^{2\pi i\frac{u}{m}a}$$

$$=\sum_{d\mid m}\mu(d)\sum_{v=0}^{m/d-1}g(dv+m\mathbb{Z})e^{2\pi i\frac{dv}{m}a}=\sum_{d\mid m}\mu\left(\frac{m}{d}\right)\sum_{r=0}^{d-1}g\left(\frac{m}{d}r+m\mathbb{Z}\right)e^{2\pi i\frac{r}{d}a}.$$

Thus

$$\sum_{k=0}^{n-1}g\left(\frac{k}{(k,n)}+\frac{n}{(k,n)}\mathbb{Z}\right)e^{2\pi i\frac{k}{n}a}=\sum_{m\mid n}\sum_{d\mid m}\mu\left(\frac{m}{d}\right)\sum_{r=0}^{d-1}g\left(\frac{m}{d}r+m\mathbb{Z}\right)e^{2\pi i\frac{r}{d}a}.$$

From this we see that (b) and (c) are equivalent.

(b)⇒(a). Let $\psi:[0,1)_D\to\mathbb{C}$ be a function satisfying (17). Then f is the restriction of $\sum_{\alpha\in[0,1)_D}\psi(\alpha)\rho_\alpha$ on $\bigcup_{n\in D}\mathbb{Z}/n\mathbb{Z}$. So (a) holds by Example 5.

The proof of Theorem 4 is now complete. □

Remark. In the case $D=\mathbb{Z}^+$, Theorem 4 was announced by the author [S2] in 1989.

Let D be a divisor-closed subset of $\mathbb{Z}^+$, and $f:\bigcup_{n\in D}\mathbb{Z}/n\mathbb{Z}\to\mathbb{C}$ a function satisfying part (a) of Theorem 4. Then there exists a function $\psi:[0,1)_D\to\mathbb{C}$ for which (17) holds and hence

$$\psi\left(\frac{m}{n}\right)=\sum_{\substack{k=0\\n\mid k-m}}^{n-1}\psi\left(\frac{k}{n}\right)=\frac{1}{n}\sum_{k=0}^{n-1}\psi\left(\frac{k}{n}\right)\sum_{r=0}^{n-1}e^{2\pi i\frac{k-m}{n}r}=\sum_{r=0}^{n-1}f(r+n\mathbb{Z})e^{-2\pi i\frac{m}{n}r}$$

for all $n\in D$ and $m\in R(n)$, this unique ψ will be denoted by $\hat{f}$. Note that f can be extended to the equivalent function $\sum_{\alpha\in[0,1)_D}\hat{f}(\alpha)\rho_\alpha$.

All those functions $\psi:\mathbb{Q}\cap[0,1)\to\mathbb{C}$ form a commutative ring under the functional addition and the functional multiplication, we denote this ring by $\mathrm{Q}(\mathbb{C})$.

Corollary 1. *The ring* $\mathrm{E}(\mathbb{C})$ *is isomorphic to* $\mathrm{Q}(\mathbb{C})$ *via the map* $f\mapsto\hat{f}$ *whose inverse is the map* $\psi\mapsto\check{\psi}$.

Proof. For $\psi\in\mathrm{Q}(\mathbb{C})$ and $f\in\mathrm{E}(C)$, clearly $\check{\psi}=f$ if and only if $\hat{f}=\psi$. Thus the map $T:\mathrm{Q}(\mathbb{C})\to\mathrm{E}(\mathbb{C})$ given by $T(\psi)=\check{\psi}$ is bijective and its inverse is the map $f\mapsto\hat{f}$. For $\psi,\chi\in\mathrm{Q}(\mathbb{C})$, apparently $(\psi+\chi)\check{}=\check{\psi}+\check{\chi}$, also $\check{I}=e$ and $(\psi\chi)\check{}=\check{\psi}*\check{\chi}$ by Example 5. So the rings $\mathrm{Q}(\mathbb{C})$ and $\mathrm{E}(\mathbb{C})$ are isomorphic via the map T. □

Now we give some applications of equivalent maps.

Theorem 5. *Let M be an R-module where R is a ring with identity.*

(i) *For $\psi_1, \cdots, \psi_k \in \mathrm{P}(R)$,*

$$\psi_1(x) + \cdots + \psi_k(x) \in \mathrm{Ann}(M) \quad \text{for all } x \in \mathbb{Z} \tag{19}$$

if and only if

$$\sum_{s=1}^{k} T_f(\psi_s) = 0 \quad \text{for each } f \in \mathrm{E}(M) \tag{20}$$

where $\mathrm{Ann}(M)$ *denotes the annihilator*

$$\bigcap_{x \in M} \mathrm{Ann}(x) = \{a \in R : \ ax = 0 \text{ for every } x \in M\}.$$

(ii) *A map $f \in \mathrm{F}(M)$ is equivalent, if and only if*

$$\sum_{s=1}^{k} \lambda_s f(a_s + n_s\mathbb{Z}) = 0 \quad \text{for all } \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^{k} \in \mathrm{K}(R) \tag{21}$$

(i.e., we have $\sum_{s=1}^{k} \lambda_s f(a_s+n_s\mathbb{Z}) = \sum_{t=1}^{l} \mu_t f(b_t+m_t\mathbb{Z})$ whenever $\{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^{k}$ and $\{\langle \mu_t, b_t, m_t \rangle\}_{t=1}^{l}$ are equivalent systems in $\mathrm{S}(R)$*).*

Proof. i) When (19) is valid, by Theorem 2 for any $f \in \mathrm{E}(M)$ and $x \in \mathbb{Z}$ we have

$$\sum_{s=1}^{k} T_f(\psi_s)(x) = T_f\left(\sum_{s=1}^{k} \psi_s\right)(x) = \sum_{r=0}^{n-1}\left(\sum_{s=1}^{k} \psi_s(x-r)\right) f(r+n\mathbb{Z}) = 0$$

where $n \in \mathbb{Z}^+$ is any period of $\psi_1 + \cdots + \psi_k$. If (20) holds, $x \in \mathbb{Z}$ and $\lambda \in M$, then $(\lambda)_0 \in \mathrm{E}(M)$ by Example 1, and hence

$$\sum_{s=1}^{k} \psi_s(x)\lambda = \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} \psi_s(x-r)(\lambda)_0(r+n_s\mathbb{Z}) = \sum_{s=1}^{k} T_{(\lambda)_0}(\psi_s)(x) = 0$$

where $n_1, \cdots, n_k$ are periods of $\psi_1, \cdots, \psi_k$ respectively. Therefore (20) also implies (19). This proves part (i).

ii) If $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{S}(R)$, then $\mathcal{A}_1 \sim \mathcal{A}_2 \Leftrightarrow \mathcal{A}_1 \sqcup -\mathcal{A}_2 \sim \emptyset \Leftrightarrow \mathcal{A}_1 \sqcup -\mathcal{A}_2 \in \mathrm{K}(R)$. As $\{\langle 1, a+jd, nd \rangle\}_{j=0}^{n-1} \sim \{\langle 1, a, d \rangle\}$ for any $d, n \in \mathbb{Z}^+$ and $a \in R(d)$, (21) implies that $f \in \mathrm{E}(M)$.

Now let $\mathcal{A} = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^{k} \in \mathrm{K}(R)$. For $s = 1, \cdots, k$ let $\psi_s \in \mathrm{P}(R)$ be given by $\psi_s(x) = \lambda_s e_{-x}(a_s + n_s\mathbb{Z})$. Then $\psi_1 + \cdots + \psi_k = 0$ since $\mathcal{A} \sim \emptyset$. If $f \in \mathrm{E}(M)$, then by part (i) we have

$$\sum_{s=1}^{k} \lambda_s f(a_s + n_s\mathbb{Z}) = \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} \psi_s(-r) f(r+n_s\mathbb{Z}) = \sum_{s=1}^{k} T_f(\psi_s)(0) = 0.$$

This concludes the proof. □

Remark. Part (ii) of Theorem 5 was announced by the author [S2] in the case $M = R = \mathbb{C}$. It implies the following result obtained by the author [S1] in a quite different way.

Corollary 2. *Let M be an R-module and F a map of two complex variables into M such that $\{\langle \frac{x+r}{n}, ny\rangle : r \in R(n)\} \subseteq \mathrm{Dom}(F)$ for all $\langle x,y\rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$. Then*

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y) \quad \text{for any } \langle x,y\rangle \in \mathrm{Dom}(F) \text{ and } n \in \mathbb{Z}^+, \tag{22}$$

if and only if we have

$$\sum_{s=1}^{k} \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \text{for all } \langle x,y\rangle \in \mathrm{Dom}(F) \tag{23}$$

whenever two systems $\mathcal{A} = \{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k$ and $\mathcal{B} = \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^l$ in $\mathrm{S}(R)$ are equivalent.

Proof. Since $\{\langle 1, r, n\rangle\}_{r=0}^{n-1} \sim \{\langle 1,0,1\rangle\}$ for $n = 1,2,3,\cdots$, the sufficiency is apparent.

Now we assume (22) and let $\langle x,y\rangle \in \mathrm{Dom}(F)$. Set $f(a+n\mathbb{Z}) = F(\frac{x+a}{n}, ny)$ for $n \in \mathbb{Z}^+$ and $a \in R(n)$. Then for any $d, n \in \mathbb{Z}^+$ and $a \in R(d)$ we have

$$\sum_{j=0}^{n-1} f(a+jd+nd\mathbb{Z}) = \sum_{j=0}^{n-1} F\left(\frac{(x+a)/d+j}{n}, n(dy)\right) = F\left(\frac{x+a}{d}, dy\right) = f(a+d\mathbb{Z}).$$

So $f \in \mathrm{E}(M)$. Applying Theorem 5(ii) we get the desired result. □

Remark. The recent paper [S7] contains a slight generalization of Corollary 2. The functional equation (22) is satisfied by lots of maps in terms of well-known special functions (see [S8]).

Notice that the Theorem stated in Section 1 follows from Theorem 4 and Theorem 5(ii).

Theorem 6. *Let $n_1, \cdots, n_k \in \mathbb{Z}^+$ and $f \in \mathrm{E}(\mathbb{C})$. Then*

$$\sum_{r=0}^{n_s-1} f(r+n_s\mathbb{Z}) e^{2\pi i \frac{a}{n_s} r} \neq 0 \quad \text{for all } a \in \mathbb{Z} \text{ and } s = 1, \cdots, k, \tag{24}$$

if and only if for any $\psi_1 \in \mathrm{P}(\mathbb{C})$ periodic mod n_1, $\cdots$, *$\psi_k \in \mathrm{P}(\mathbb{C})$ periodic* mod n_k *we have*

$$\psi_1 + \cdots + \psi_k = 0 \iff f\circ\psi_1 + \cdots + f\circ\psi_k = 0. \tag{25}$$

Proof. Let s be among $1, \cdots, k$ and w_s be an n_sth root of unity. For each $t = 1, \cdots, k$ define $\psi_{st} \in \mathrm{P}(\mathbb{C})$ by $\psi_{st}(x) = \delta_{st} w_s^{-x}$ where $\delta_{st} = 1$ if $s = t$, and 0 otherwise. If $\sum_{t=1}^k \psi_{st} = 0 \iff \sum_{t=1}^k f\circ\psi_{st} = 0$, then

$$\sum_{r=0}^{n_s-1} f(r+n_s\mathbb{Z}) w_s^r = w_s^x \sum_{r=0}^{n_s-1} f(r+n_s\mathbb{Z}) w_s^{-(x-r)} = w_s^x \sum_{t=1}^{k} f\circ\psi_{st}(x) \neq 0$$

for some $x \in \mathbb{Z}$ because $\sum_{t=1}^{k} \psi_{st} = \psi_{ss} \neq 0$. This proves the sufficiency.

Let $\psi_1, \cdots, \psi_k \in \mathrm{P}(\mathbb{C})$ have periods $n_1, \cdots, n_k$ respectively. If $\psi_1 + \cdots + \psi_k = 0$, then

$$\sum_{s=1}^{k} f \circ \psi_s = f \circ \sum_{s=1}^{k} \psi_s = f \circ 0 = 0.$$

Now assume that $\psi_1 + \cdots + \psi_k \neq 0$. By Theorem 5(i), $\sum_{s=1}^{k} g \circ \psi_s \neq 0$ for some $g \in \mathrm{E}(\mathbb{C})$. If $N = [n_1, \cdots, n_k]$ and $x \in \mathbb{Z}$, then

$$\begin{aligned}\sum_{s=1}^{k} g \circ \psi_s(x) &= \sum_{s=1}^{k} \sum_{a=0}^{n_s-1} \sum_{\alpha \in \mathbb{Q} \cap [0,1)} \hat{g}(\alpha) \rho_\alpha(a + n_s\mathbb{Z}) \psi_s(x-a) \\ &= \sum_{s=1}^{k} \sum_{\substack{\alpha \in [0,1) \\ \alpha n_s \in \mathbb{Z}}} \frac{\hat{g}(\alpha)}{n_s} \sum_{a=0}^{n_s-1} e^{2\pi i \alpha a} \psi_s(x-a) \\ &= \sum_{\substack{\alpha \in [0,1) \\ \alpha N \in \mathbb{Z}}} \hat{g}(\alpha) \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^{k} \frac{1}{n_s} \sum_{r=0}^{n_s-1} \psi_s(r) e^{2\pi i \alpha (x-r)}.\end{aligned}$$

So there exists an $\alpha \in \mathbb{Q} \cap [0,1)$ such that

$$c = \sum_{s \in I} \frac{1}{n_s} \sum_{r=0}^{n_s-1} \psi_s(r) e^{-2\pi i \alpha r} \neq 0 \quad \text{where } I = \{1 \leqslant s \leqslant k : \alpha n_s \in \mathbb{Z}\}.$$

For any $s \in I$ we have $\sum_{r=0}^{n_s-1} f(r + n_s\mathbb{Z}) e^{-2\pi i \alpha r} = \hat{f}(\alpha)$. Therefore

$$\begin{aligned}\bar{c} &= \sum_{s \in I} \frac{1}{n_s} \sum_{r=0}^{n_s-1} f \circ \psi_s(r) e^{-2\pi i \alpha r} = \sum_{s \in I} \frac{1}{n_s} \sum_{r=0}^{n_s-1} \sum_{a=0}^{n_s-1} f(a + n_s\mathbb{Z}) \psi_s(r-a) e^{-2\pi i \alpha r} \\ &= \sum_{s \in I} \frac{1}{n_s} \sum_{a=0}^{n_s-1} f(a + n_s\mathbb{Z}) e^{-2\pi i \alpha a} \sum_{r'=0}^{n_s-1} \psi_s(r') e^{-2\pi i \alpha r'} = c\hat{f}(\alpha).\end{aligned}$$

On the other hand, if $x \in \mathbb{Z}$ then

$$\bar{c} e^{2\pi i \alpha x} = \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} f \circ \psi_s(r) \rho_\alpha(x - r + n_s\mathbb{Z}) = \rho_\alpha \circ \sum_{s=1}^{k} f \circ \psi_s(x).$$

Suppose (24) and choose a $j \in I$. Then $\hat{f}(\alpha) = \sum_{r=0}^{n_j-1} f(r + n_j\mathbb{Z}) e^{-2\pi i \alpha r} \neq 0$. By the above, $\bar{c} \neq 0$ and hence $\sum_{s=1}^{k} f \circ \psi_s \neq 0$. This ends the proof. □

Let $\psi_s(x) = \lambda_s e^{2\pi i \alpha_s x}$ for $s = 1, \cdots, k$ where $\lambda_1, \cdots, \lambda_k \in \mathbb{C}^*$, and $\alpha_1 = a_1/n_1, \cdots, \alpha_k = a_k/n_k$ are distinct reduced rationals in $[0,1)$. Suppose that

$\psi_0 = -(\psi_1 + \cdots + \psi_k)$ has a period $n_0 \in \mathbb{Z}^+$ not divisible by $N = [n_1, \cdots, n_k]$. Then $n_t \nmid n_0$ (i.e. $\alpha_t n_0 \notin \mathbb{Z}$) for some $1 \leqslant t \leqslant k$. For any $s = 1, \cdots, k$ with $\alpha_t n_s \in \mathbb{Z}$, clearly

$$\frac{1}{n_s}\sum_{r=0}^{n_s-1}\psi_s(r)e^{-2\pi i\alpha_t r} = \frac{\lambda_s}{n_s}\sum_{r=0}^{n_s-1}e^{2\pi i(\alpha_s-\alpha_t)r} = \lambda_s\delta_{st}.$$

Since $\psi_0 + \psi_1 + \cdots + \psi_k = 0$, we have

$$0 = \sum_{s=0}^{k}\rho_{\alpha_t}\circ\psi_s(0) = \sum_{\substack{s=0\\ \alpha_t n_s\in\mathbb{Z}}}^{k}\frac{1}{n_s}\sum_{r=0}^{n_s-1}\psi_s(r)e^{-2\pi i\alpha_t r} = \sum_{\substack{s=1\\ \alpha_t n_s\in\mathbb{Z}}}^{k}\lambda_s\delta_{st} = \lambda_t \neq 0.$$

The contradiction shows that N must be the least (positive) period of $\psi_1+\cdots+\psi_k$. (When $n_1 < \cdots < n_k$, this result was observed by the author [S4] in 1991.)

Corollary 3. *Let $\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^k \in \mathrm{S}(\mathbb{C})$. Then for any $f \in \mathrm{E}(\mathbb{C})$ satisfying (24), $\mathcal{A} \sim \emptyset$ if and only if*

$$\sum_{s=1}^{k}\lambda_s f(x + a_s + n_s\mathbb{Z}) = 0 \quad \text{for all } x \in \mathbb{Z}. \tag{26}$$

Proof. Let $x \in \mathbb{Z}$ and $\psi_s(x) = \lambda_s e(x + a_s + n_s\mathbb{Z})$ for $s = 1, \cdots, k$. Clearly $w_{\mathcal{A}}(-x) = \sum_{s=1}^k \psi_s(x)$ and $\lambda_s f(x + a_s + n_s\mathbb{Z}) = f\circ\psi_s(x)$. So the desired result follows from Theorem 6. □

Remark. In 1989 the author announced Corollary 3 as Theorem 4 of [S2].

Example 6. Let $h \in \mathbb{Z}$ and define $\varphi_h : \Omega \to \mathbb{C}$ in the following way:

$$\varphi_h(a + n\mathbb{Z}) = \begin{cases} \frac{1}{\varphi(n)} & \text{if } (h+a, n) = 1,\\ 0 & \text{otherwise,}\end{cases} \tag{27}$$

where φ is Euler's totient function. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, using the Ramanujan sum (cf. [HW]) we find that

$$\sum_{r=0}^{n-1}\varphi_h(r + n\mathbb{Z})e^{-2\pi i\frac{a}{n}r} = \frac{1}{\varphi(n)}\sum_{\substack{j=0\\ (j,n)=1}}^{n-1}e^{-2\pi i\frac{a}{n}(j-h)} = e^{2\pi i h\frac{a}{n}}\frac{\mu(n/(a,n))}{\varphi(n/(a,n))},$$

which only depends on the rational a/n. So $\varphi_h = \check{\psi} \in \mathrm{E}(\mathbb{C})$ where $\psi(\alpha) = e^{2\pi i\alpha h}\frac{\mu(d(\alpha))}{\varphi(d(\alpha))}$ for $\alpha \in \mathbb{Q}\cap[0,1)$, and $d(\alpha)$ denotes the denominator of α (which is the least $l \in \mathbb{Z}^+$ such that $l\alpha \in \mathbb{Z}$). If $n \in \mathbb{Z}^+$ is squarefree, then $\widehat{\varphi_h}(a/n) = \psi(a/n) \neq 0$

for all $a \in R(n)$. Let $n_1, \cdots, n_k \in \mathbb{Z}^+$ be all squarefree, and $\mathcal{A} = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^k \in \mathrm{S}(\mathbb{C})$. Then by Corollary 3 we have

$$\begin{aligned} \mathcal{A} \sim \emptyset \iff & \sum_{s=1}^{k} \lambda_s \varphi_h(x + a_s + n_s\mathbb{Z}) = 0 \text{ for any } x \in \mathbb{Z} \\ \iff & \sum_{\substack{s=1 \\ (y+a_s, n_s)=1}}^{k} \frac{\lambda_s}{\varphi(n_s)} = 0 \text{ for all } y \in \mathbb{Z}. \end{aligned}$$

This result was also announced by the author in [S2]. Suppose that

$$|\{1 \leqslant s \leqslant k : (x + a_s, n_s) = 1\}| = m \quad \text{for any } x \in \mathbb{Z}.$$

Then $\sum_{s=1}^{k} \varphi(n_s)\varphi_0(x + a_s + n_s\mathbb{Z}) - m\varphi_0(x + \mathbb{Z}) = 0$ for all $x \in \mathbb{Z}$, hence $\mathcal{A}' = \{\langle \varphi(n_s), a_s, n_s \rangle\}_{s=1}^k \sim \{\langle m, 0, 1 \rangle\}$ and $w_{\mathcal{A}'}$ has period $n_0 = 1$. Thus, by Theorem 1 of [S3], for any integer $d > 1$ dividing one of $n_1, \cdots, n_k$, we have

$$|\{a_s + d\mathbb{Z} : 1 \leqslant s \leqslant k \ \&\ n_s \equiv 0 \ (\mathrm{mod}\ n_s)\}| \geqslant \min_{\substack{0 \leqslant s \leqslant k \\ d \nmid n_s}} \frac{d}{(d, n_s)} \tag{28}$$

and so $|\{1 \leqslant s \leqslant k : d \mid n_s\}|$ is not less than the least prime divisor $p(d)$ of d. Assume that $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$ where $1 \leqslant l < k$. Then $l \geqslant \min_{1 \leqslant s \leqslant k-l} n_k/(n_s, n_k) \geqslant p(n_k)$. For any $r \in R(n_k)$ divisible by none of $\frac{n_k}{(n_1,n_k)}, \cdots, \frac{n_k}{(n_{k-l},n_k)}$, clearly $\frac{r}{n_k} n_s \in \mathbb{Z} \iff k - l < s \leqslant k$, thus

$$\begin{aligned} 0 = & m\rho_{r/n_k}(\mathbb{Z}) = \sum_{s=1}^{k} \varphi(n_s)\rho_{r/n_k}(a_s + n_s\mathbb{Z}) \\ = & \sum_{k-l<s\leqslant k} \frac{\varphi(n_s)}{n_s} e^{2\pi i \frac{r}{n_k} a_s} = \frac{\varphi(n_k)}{n_k} \sum_{k-l<s\leqslant k} e^{2\pi i \frac{a_s}{n_k} r}. \end{aligned}$$

In view of Lemma 9 of [S5], there are $x_1, \cdots, x_{k-l} \in \mathbb{N}$ such that $l = \sum_{k-l<s\leqslant k} 1 = \sum_{s=1}^{k-l} \frac{n_k}{(n_s,n_k)} x_s$.

References

[E1] P. Erdös, *On integers of the form $2^k + p$ and some related problems*, Summa Brasil. Math. **2** (1950), 113–123. MR 13, 437.

[E2] P.Erdös, *Remarks on number theory IV: Extremal problems in number theory I*, Mat. Lapok **13** (1962), 241–243. MR 33#4020.

[G] R. K. Guy, *Unsolved Problems in Number Theory* (2nd, ed.), Springer-Verlag, New York, 1994, Sections A19, B21, E23, F13, F14.

[HW] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, Oxford, 1981, Ch. XVI.

[NZ] B. Novák, and Š. Znám, *Disjoint covering systems*, Amer. Math. Monthly **81** (1974), 42–45. MR 48#10958.

[P] Š. Porubský, *On m times covering systems of congruences*, Acta Arith. **29** (1976), 159–169. MR 53#2884.

[S1] Z. W. Sun, *Systems of congruences with multipliers*, Nanjing Univ. J. Math. Biquarterly **6** (1989), no. 1, 124–133. Zbl. M. 703.11002, MR 90m:11006.

[S2] Z. W. Sun, *Several results on systems of residue classes*, Adv. in Math. (China) **18** (1989), no. 2, 251–252.

[S3] Z. W. Sun, *An improvement to the Znám-Newman result*, Chinese Quart. J. Math. **6** (1991), no. 3, 90–96.

[S4] Z. W. Sun, *On a generalization of a conjecture of Erdös*, J. Nanjing Univ. Natur. Sci. **27** (1991), no. 1, 8–15. MR 92f:11008.

[S5] Z. W. Sun, *Covering the integers by arithmetic sequences* II, Trans. Amer. Math. Soc. **348** (1996), 4279–4320. MR 97c:11011.

[S6] Z. W. Sun, *Exact m-covers and the linear form* $\sum_{s=1}^{k} x_s/n_s$, Acta Arith. **81** (1997), 175–198. MR 98h:11019.

[S7] Z. W. Sun, *Products of binomial coefficients modulo* p^2, Acta Arith. **97** (2001), 87–98.

[S8] Z. W. Sun, *On covering equivalence*, in: "Analytic Number Theory, the Joint Proceedings of the China-Japan Number Theory Conference (Beijing, 1999) and the RIMS Analytic Number Theory Conference (Kyoto, 1999)" (C. H. Jia and K. Matsumoto, Eds.), Kluwer, Dordrecht, in press.

[Z1] Š.Znám, *Vector-covering systems of arithmetic sequences*, Czech. Math. J. **24** (1974), 455–461. MR 50#4520.

[Z2] Š. Znám, *Vector-covering systems with a single triple of equal moduli*, Czech. Math. J. **34** (1984), 343–348. MR 85k:11007.