

RESTRICTED SUMS OF SUBSETS OF \mathbb{Z}

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1. INTRODUCTION

Let

$$(1.1) \quad \{A_i\}_{i=1}^n$$

be a finite sequence of sets. If $a_1 \in A_1, \dots, a_n \in A_n$, and a_1, \dots, a_n are pairwise different, then we call $\{a_i\}_{i=1}^n$ a *system of distinct representatives* (abbreviated to SDR) of (1.1). Apparently (1.1) has an SDR provided that

$$(1.2) \quad |A_i| \geq i \quad \text{for all } i = 1, \dots, n.$$

If A_1, \dots, A_n are contained in a finite set $\{x_1, \dots, x_k\}$ with cardinality k , then (1.1) has as many SDR's as $\{A_i^*\}_{i=1}^n$ does where $A_i^* = \{1 \leq j \leq k : x_j \in A_i\} \subseteq \{1, \dots, k\}$.

Let A_1, \dots, A_n be finite subsets of an additive abelian group G . Their sumset is given by

$$(1.3) \quad A_1 + \dots + A_n = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n\}.$$

If we require the summands to be distinct, then we are led to the restricted sumset

$$(1.4) \quad \begin{aligned} & S(\{A_i\}_{i=1}^n) = S(A_1, \dots, A_n) \\ & = \left\{ \sum_{i=1}^n a_i : \{a_i\}_{i=1}^n \text{ forms an SDR of } \{A_i\}_{i=1}^n \right\}. \end{aligned}$$

Of course there are many other kinds of restricted sumsets. An interesting problem is to provide a nontrivial lower bound for the cardinality a

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restricted sumset of A_1, \dots, A_n . In the light of the fundamental theorem on finitely generated abelian groups, it suffices to work within the ring \mathbb{Z} of integers instead of a torsionfree abelian group G .

For a finite subset A of \mathbb{Z} , in 1995 M.B. Nathanson [N1] obtained the inequality

$$(1.5) \quad |n^{\wedge}A| \geq n|A| - n^2 + 1$$

and determined when equality holds. (By $n^{\wedge}A$ we mean $S(\{A_i\}_{i=1}^n)$ with $A_1 = \dots = A_n = A$.) Soon after this, Y. Bilu [B] gave the same result independently. Let p be a prime. In 1994 J. A. Dias da Silva and Y. O. Hamidoune [DH] proved the following generalization of a conjecture of P. Erdős and H. Heilbronn (cf. [EH] and [G]):

$$(1.6) \quad |n^{\wedge}A| \geq \min \{p, n|A| - n^2 + 1\} \quad \text{for any } A \subseteq \mathbb{Z}/p\mathbb{Z}.$$

By the so-called polynomial method, in 1996 N. Alon, M. B. Nathanson and I. Z. Ruzsa [ANR] got the following result: Let F be any field of characteristic p and A_1, \dots, A_n its subsets with $0 < |A_1| < \dots < |A_n| < \infty$, then

$$(1.7) \quad |S(\{A_i\}_{i=1}^n)| \geq \min \left\{ p, \sum_{i=1}^n |A_i| - \frac{n(n+1)}{2} + 1 \right\}.$$

Their method doesn't allow one to determine when the bound can be attained. Provided that A_1, \dots, A_n are finite subsets of \mathbb{Z} with $0 < |A_1| < \dots < |A_n|$, we then have

$$(1.8) \quad |S(\{A_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (|A_i| - i).$$

A purely combinatorial proof of this inequality was given by Hui-Qin Cao and Zhi-Wei Sun [CS], where the authors obtained some necessary conditions for the equality case.

Now we introduce our basic notations in this paper. For $A \subseteq \mathbb{Z}$ we put $-A = \{-x : x \in A\}$ and $a + A = A + a = \{a + x : x \in A\}$ for $a \in \mathbb{Z}$. An *arithmetic progression* A is a set of the form $\{a, a + d, \dots, a + kd\}$ where $a \in \mathbb{Z}$, d and k are positive integers, we use $d(A)$ to denote the (common) *difference* d of A . (A set having a single element is not considered as an arithmetic progression.) For the sake of convenience, AP will denote the class of all arithmetic progressions. For $a, b \in \mathbb{Z}$ we put

$$(a, b) = \{x \in \mathbb{Z} : a < x < b\}, \quad [a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}, \\ [a, b) = \{x \in \mathbb{Z} : a \leq x < b\}, \quad (a, b] = \{x \in \mathbb{Z} : a < x \leq b\}.$$

In this paper we study lower bounds for cardinalities of various restricted sumsets of subsets of \mathbb{Z} . We use the powerful techniques developed in [CS] to go further.

In the next section we will prove the following general result on linearly restricted sums of subsets of \mathbb{Z} .

Theorem 1.1. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} , and V a set of tuples in the form (s, t, μ, ν, w) where $1 \leq s, t \leq n$, $s \neq t$, $\mu, \nu \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $w \in \mathbb{Z}$. Set*

$$(1.9) \quad C = \{a_1 + \dots + a_n : a_i \in A_i, \text{ and } \mu a_i + \nu a_j \neq w \text{ if } (i, j, \mu, \nu, w) \in V\}.$$

If each $V_i = \{(s, t, \mu, \nu, w) \in V : i \in \{s, t\}\}$ has cardinality less than $|A_i|$, then

$$(1.10) \quad |C| \geq \sum_{i=1}^n |A_i| - 2|V| - n + 1 = 1 + \sum_{i=1}^n (|A_i| - |V_i| - 1) > 0.$$

Remark 1.1. If we replace $a_1 + \dots + a_n$ by $\lambda_1 a_1 + \dots + \lambda_n a_n$ in the definition (1.9) of C where $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^*$, then Theorem 1.1 remains valid. For, when $(i, j, \mu, \nu, w) \in V$, $a_i \in A_i$ and $a_j \in A_j$, we have

$$\mu a_i + \nu a_j = w \Leftrightarrow \lambda_i \lambda_j (\mu a_i + \nu a_j) = \lambda_i \lambda_j w \Leftrightarrow \mu' (\lambda_i a_i) + \nu' (\lambda_j a_j) = w'$$

where $\mu' = \lambda_j \mu$, $\nu' = \lambda_i \nu$ and $w' = \lambda_i \lambda_j w$.

Now we give several consequences of Theorem 1.1.

Corollary 1.1. *Let A_1, \dots, A_n be subsets of \mathbb{Z} which are nonempty and finite. Then*

$$(1.11) \quad |A_1 + \dots + A_n| \geq |A_1| + \dots + |A_n| - n + 1.$$

Proof. Just apply Theorem 1.1 with $V = \emptyset$. \square

Remark 1.2. Corollary 1.1 is a known result. Equality in (1.11) holds if and only if all those A_i with $|A_i| \geq 2$ are arithmetic progressions with the same difference. See Theorems 1.4 and 1.5 of [N2].

Corollary 1.2. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} such that $|A_i| \geq |J_i|$ for all $i = 1, \dots, n$ where $J_i = \{1 \leq j \leq n : A_i \cap A_j \neq \emptyset\}$. Then*

$$(1.12) \quad |S(\{A_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (|A_i| - |J_i|).$$

Proof. Put $V = \{(i, j, 1, -1, 0) : 1 \leq i < j \leq n \text{ \& } A_i \cap A_j \neq \emptyset\}$. Then

$$|V_i| = |\{1 \leq j \leq n : j \neq i \text{ \& } A_i \cap A_j \neq \emptyset\}| = |J_i \setminus \{i\}| < |A_i| \quad \text{for } i \in [1, n].$$

Applying Theorem 1.1 we immediately get the desired inequality. \square

Corollary 1.3. *Let Λ, A_1, \dots, A_n be finite subsets of \mathbb{Z} such that*

$$|A_i| > \sum_{j \neq i} |(A_i + A_j) \cap \Lambda| \quad \text{for all } i = 1, \dots, n.$$

Let $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^$ and*

$$L = \{\lambda_1 a_1 + \dots + \lambda_n a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i + a_j \notin \Lambda \text{ if } i \neq j\}.$$

Then

$$\sum_{i=1}^n |A_i| - |L| \leq 2 \sum_{1 \leq i < j \leq n} |(A_i + A_j) \cap \Lambda| + n - 1 \leq (n|\Lambda| + 1)(n - 1).$$

Proof. Set

$$V = \{(i, j, 1, 1, \lambda) : 1 \leq i < j \leq n \text{ \& } \lambda \in (A_i + A_j) \cap \Lambda\}.$$

Then

$$|V| = \sum_{1 \leq i < j \leq n} |(A_i + A_j) \cap \Lambda| \leq \binom{n}{2} |\Lambda|,$$

and $|V_i| = \sum_{j \neq i} |(A_i + A_j) \cap \Lambda|$ for $i = 1, \dots, n$. Thus the required result follows from Theorem 1.1 and Remark 1.1. \square

Corollary 1.4. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} , and*

$$S = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i \neq \mu_{ij} a_j + \nu_{ij} \text{ if } i \neq j\},$$

where $\mu_{ij} \in \mathbb{Z}^$ and $\nu_{ij} \in \mathbb{Z}$. If $|A_i| \geq 2n - 1$ for all $i = 1, \dots, n$, then*

$$|S| \geq \sum_{i=1}^n |A_i| - 2n^2 + n + 1.$$

Proof. Let $V = \{(i, j, 1, -\mu_{ij}, \nu_{ij}) : 1 \leq i, j \leq n \text{ \& } i \neq j\}$. If $1 \leq i \leq n$ then $|V_i| = n - 1 + (n - 1) = 2n - 2$. Clearly $2|V| + n - 1 = 2(n^2 - n) + n - 1 = 2n^2 - n - 1$. So it suffices to apply Theorem 1.1. \square

Remark 1.3. For $1 \leq i < j \leq n$ let $\mu_{ij} = 1$, $\mu_{ji} = -1$ and $\nu_{ij} = \nu_{ji} = 0$. Then the set S given in Corollary 1.4 becomes $\{\sum_{i=1}^n a_i : a_i \in A_i \text{ and all the } a_i^2 \text{ are distinct}\}$.

Corollary 1.5. *For each $i = 1, \dots, n$ let $A_i \subseteq \mathbb{Z}$ and $3 \leq |A_i| < \infty$. Then the set*

$$\{a_1 + \dots + a_n : a_i \in A_i, a_i \neq a_{i+1} \text{ if } i < n, \text{ and } a_n \neq a_1\}$$

has cardinality at least $\sum_{i=1}^n |A_i| - 3n + 1$.

Proof. Let $V = \{(i, i+1, 1, -1, 0) : i \in [1, n)\} \cup \{(n, 1, 1, -1, 0)\}$. Then $|V| = n$, and $|V_i| = 2 < |A_i|$ for all $i \in [1, n]$. So the desired result follows immediately from Theorem 1.1.

Let F be a field of characteristic p where p is a prime, and A_1, \dots, A_n its finite subsets satisfying (1.2). Then Theorem 3.2 of [ANR] essentially asserts that

$$|S(\{A_i\}_{i=1}^n)| \geq \min \left\{ p, 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j) \right\}.$$

In the last section we will show the following general result by our combinatorial method.

Theorem 1.2. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with (1.2) and $|A_1| \leq \dots \leq |A_n|$. Then*

$$(1.13) \quad |S(\{A_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j).$$

In the equality case, $\bigcup_{i=1}^m A_i = A_m$ if m lies in

$$(1.14) \quad M = \{1 \leq i \leq n : |A_i| - i < |A_j| - j \text{ for all } j \in (i, n]\},$$

providing $|A_i| > i$ for all $i \in [1, n]$ the set $\bigcup_{i=1}^n A_i = A_n$ lies in AP with the only exceptions as follows:

- (i) $n = 1$ or $|A_n| = n + 1$;
- (ii) $n = 2$, $|A_1| \in \{3, 4\}$ and A_2 is of the form

$$(1.15) \quad \{x_1, x_2, x_3, x_4\} \quad \text{with } x_1 < x_2 < x_3 < x_4 \text{ and } x_4 - x_3 = x_2 - x_1;$$

(iii) $n > 1$, $|A_{n-1}| = n$, A_{n-1} and $A_n \setminus A_{n-1}$ belong to AP, and $d(A_{n-1}) = d(A_n \setminus A_{n-1})$.

Remark 1.4. Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with $k_i = |A_i| \geq i$ for all $i \in [1, n]$. Providing $k_s > k_{s+1}$ for some $s \in [1, n)$, we still have inequality

have $x_j = j - 1 + d_j \geq i + d_i$. If $x_i < i - 1 + d_i$ then by substituting $i - 1 + d_i \in A_i$ for x_i we would get a sum larger than $x_1 + \cdots + x_n$, thus $x_i = i - 1 + d_i$. By the above,

$$\max S(\{A_i\}_{i=1}^n) = \sum_{i=1}^n x_i = \sum_{i=1}^n (i - 1 + d_i) = \frac{n(n-1)}{2} + \sum_{i=1}^n d_i.$$

Obviously

$$\min S(\{A_i\}_{i=1}^n) = 0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

So we also have

$$S(\{A_i\}_{i=1}^n) \subseteq \frac{n(n-1)}{2} + \left[0, \sum_{i=1}^n d_i\right].$$

Therefore

$$S(\{A_i\}_{i=1}^n) = \left[\frac{n(n-1)}{2}, \frac{n(n-1)}{2} + \sum_{i=1}^n d_i\right]$$

and hence $|S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n d_i$.

Remark 1.5. Example 1.1 was realized by Alon, Nathanson and Ruzsa [ANR], but they did not go into details. Let k_1, \dots, k_n and A_1, \dots, A_n be as in Example 1. For $i = 1, \dots, n$ put $A_i^* = \{a + jd : j \in [0, k_i]\}$ where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^*$. By Example 1.1,

$$|S(\{A_i^*\}_{i=1}^n)| = |S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j^*| - j).$$

As for the exceptions (i) and (ii), here we give

Example 1.2. Let A be a finite subset of \mathbb{Z} with $|A| \geq n \geq 1$, and A_1, \dots, A_n subsets of \mathbb{Z} with $\bigcup_{i=1}^n A_i = A_n = A$. Suppose that $|A_i| - i \geq |A_n| - n$ for all $i = 1, \dots, n$ (i.e. the set M defined by (1.14) only contains n). If $\{a_i\}_{i=1}^n$ is an SDR of $\{A_i\}_{i=1}^n$, then $\{a_1, \dots, a_n\}$ is a subset of A with cardinality n . If $S \subseteq A$ and $|S| = n$, then for each $i \in [1, n]$ we have

$$|S \cap A_i| \geq |S| - |A \setminus A_i| = n - (|A_n| - |A_i|) \geq i,$$

therefore $\{S \cap A_i\}_{i=1}^n$ has an SDR $\{a_i\}_{i=1}^n$ and hence $S = \{a_1, \dots, a_n\}$. Thus $S(\{A_i\}_{i=1}^n) = n \wedge A$, (1.13) is equivalent to (1.5), and the equality case of (1.13) is as same as that of (1.5). A result of Nathanson says that

$|n^{\wedge}A| = n|A| - n^2 + 1$ if and only if $n \in \{1, |A| - 1, |A|\}$, or $A \in \text{AP}$, or $n = 2$ and A can be written in the form (1.15). (See Section 3 of [N1] and Section 1.3 of [N2].) Thus, if $n = 1$ or $|A| = n + 1$, whether $A \in \text{AP}$ or not, the two sides of (1.13) are always equal, this corresponds to the exception (i). In the case $n = 2$, if $A_2 = A$ is of the form (1.15), then $|A_1| \in \{|A_2| - 1, |A_2|\} = \{3, 4\}$ and

$$\begin{aligned} |S(\{A_i\}_{i=1}^2)| &= |2^{\wedge}A| = 2|A| - 2^2 + 1 = 5 \\ &= 1 + \min\{|A_1| - 1, |A_2| - 2\} + |A_2| - 2 \end{aligned}$$

though we may not have $A_2 = A \in \text{AP}$.

For the equality case of (1.13), Example 1.2 shows that the necessary conditions given by Theorem 1.2 are also sufficient in the case $M = \{n\}$.

From Theorem 1.2 we have

Corollary 1.6. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with $|A_1| \leq \dots \leq |A_n|$ and $\min_{1 \leq i \leq n} (|A_i| - i) = 0$. Put $m = \max\{1 \leq i \leq n : |A_i| = i\}$. Suppose that the two sides of (1.13) are equal. Then $A_n \setminus A_m \in \text{AP}$ unless we have one of the following:*

- (i') $m \in \{n - 1, n\}$ or $|A_n| = n + 1$;
- (ii') $m = n - 2$, $|A_{n-1}| \in \{n + 1, n + 2\}$ and $A_n \setminus A_{n-2}$ is of the form (1.15);
- (iii') $m < n - 1$, $|A_{n-1}| = n$, $A_{n-1} \setminus A_m$ and $A_n \setminus A_{n-1}$ lie in AP, and $d(A_{n-1} \setminus A_m) = d(A_n \setminus A_{n-1})$.

Proof. Write $M = \{m_1, \dots, m_l\}$ where $m_0 = 0 < m_1 < \dots < m_l = n$. Clearly $m_1 = m$. For any $j \in [1, l]$ set $A_i^* = A_{m_j}$ for all $i \in (m_{j-1}, m_j]$. By Theorem 1.2, $A_i \subseteq A_{m_j}$ for all $i = 1, \dots, m_j$. In the light of Example 1.2, any $m_j - m_{j-1}$ distinct elements of A_{m_j} can be arranged to form an SDR of $\{A_i\}_{m_{j-1} < i \leq m_j}$. So

$$\begin{aligned} S(\{A_i\}_{i=1}^n) &= S(\{A_i^*\}_{i=1}^n) \\ &= \left\{ \sum_{x \in A_m} x + \sum_{m < i \leq n} a_i : a_i \in A_i^* \setminus A_m, \text{ all the } a_i \text{ are distinct} \right\} \\ &= \sum_{x \in A_m} x + S(\{A_i^* \setminus A_m\}_{i \in (m, n]}) \end{aligned}$$

where we regard $S(\emptyset)$ as $\{0\}$. Observe that

$$\begin{aligned} \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j) &= \sum_{j=1}^l \sum_{m_{j-1} < i \leq m_j} (|A_{m_j}| - m_j) \\ &= \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_j^*| - j) = \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_j^* \setminus A_m| - (j - m)). \end{aligned}$$

Thus

$$\begin{aligned} |S(\{A_i^* \setminus A_m\}_{i \in (m, n]})| &= |S(\{A_i\}_{i=1}^n)| \\ &= 1 + \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_j^* \setminus A_m| - (j - m)). \end{aligned}$$

If $i \in (m, n]$, then $|A_i| - i > |A_m| - m = 0$ and hence $|A_i^* \setminus A_m| = |A_i^*| - m > i - m$.

Below we assume that $m \neq n$. Let us apply Theorem 1.2 to the sets $A_{m+1}^* \setminus A_m, \dots, A_n^* \setminus A_m$. If $A_n^* \setminus A_m = A_n \setminus A_m \notin \text{AP}$, then we are lead to the exceptions corresponding to (i)–(iii) in Theorem 1.2. Obviously

$$|(m, n]| = 1 \Leftrightarrow m = n - 1, \quad \text{and} \quad |A_n^* \setminus A_m| = (n - m) + 1 \Leftrightarrow |A_n| = n + 1.$$

In the case $n - m = 2$, $A_n^* \setminus A_m = A_n \setminus A_{n-2}$ and

$$|A_{n-1}^* \setminus A_m| \in |A_n^* \setminus A_m| + \{0, -1\} \Leftrightarrow |A_{n-1}^*| \in |A_n| + \{0, -1\} \Leftrightarrow n - 1 \notin M,$$

if $|A_n^* \setminus A_m| = |A_n \setminus A_{n-2}| = 4$ then $|A_n| = |A_{n-2}| + 4 = n + 2$ and

$$|A_{n-1}^* \setminus A_m| \in \{3, 4\} \Leftrightarrow |A_{n-1}| \in |A_n| + \{0, -1\} = \{n + 1, n + 2\}.$$

When $n - m > 1$, we have

$$\begin{aligned} &|A_{n-1}^* \setminus A_m| = n - m \ \& \ (A_n^* \setminus A_m) \setminus (A_{n-1}^* \setminus A_m) \in \text{AP} \\ \Leftrightarrow &|A_{n-1}^*| = n, \ A_{n-1}^* \neq A_n^* = A_n \ \& \ A_n^* \setminus A_{n-1}^* \in \text{AP} \\ \Leftrightarrow &n - 1 \in M, \ |A_{n-1}| = n \ \& \ A_n \setminus A_{n-1} \in \text{AP} \\ \Leftrightarrow &|A_{n-1}| = n \ \& \ A_n \setminus A_{n-1} \in \text{AP}. \end{aligned}$$

In view of these notes, we have (i') or (ii') or (iii') if $A_n \setminus A_m \notin \text{AP}$. \square

Remark 1.6. Clearly (i), (ii) and (iii) correspond to (i'), (ii') and (iii') with $m = 0$ and $A_0 = \emptyset$. The proof of Corollary 1.6 shows that in equality case of (1.13) those A_m with $m \in M$ are vital.

Let A_1, \dots, A_n be finite subsets of \mathbb{Z} satisfying (1.2). Theorem 1.2, together with Example 1.1, Remark 1.5 and Corollary 1.6, shows that we have completely determined the set $\bigcup_{i=1}^n A_i = A_n$ in the equality case of (1.13).

Corollary 1.7. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with (1.2) and $|A_1| \leq \dots \leq |A_n|$. Then*

$$(1.16) \quad |S(\{A_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (|A_i| + h_i - n)$$

where

$$(1.17) \quad h_i = |\{|A_j| : 1 \leq j \leq n \ \& \ |A_j| > |A_i|\}|.$$

Furthermore, when the lower bound in (1.16) is reached, $A_i \subseteq A_m$ for all $i = 1, \dots, m$ if $|A_m| < |A_{m+1}| - 1$ or $m = n$, also $|A_l| < \dots < |A_n|$ where l is the least index with $|A_l| < |A_{l+1}| - 1$ or $l = n$, providing $\min\{n, |A_1| - 1, \dots, |A_n| - n\} \geq 2$ we have $A_n \in \text{AP}$ unless A_n is of the form (1.15).

Proof. Let $k_i = |A_i|$ for $i \in [1, n]$. When $i \in [1, n)$, if $k_i = k_{i+1}$ then $h_i = h_{i+1}$, if $k_i \leq k_{i+1} - 1$ then $h_i = h_{i+1} + 1$, thus $k_i + h_i \leq k_{i+1} + h_{i+1}$, and $k_i + h_i < k_{i+1} + h_{i+1}$ if and only if $k_i < k_{i+1} - 1$. For $i \in [1, n]$, if $j \in [i, n]$ then $k_i + h_i - n \leq k_j + h_j - n \leq k_j - j$, so $k_i + h_i - n \leq d_i = \min_{i \leq j \leq n} (k_j - j)$. Thus (1.16) holds by Theorem 1.2.

Clearly $k_1 + h_1 = \dots = k_l + h_l$ by the above, and $d_1 = \dots = d_l$ since $k_1 - 1 \geq \dots \geq k_l - l$. When $k_i + h_i - n = d_i$ for all $i = 1, \dots, n$, for each $m \in [1, n)$ we have

$$m \in M \Leftrightarrow d_m < d_{m+1} \Leftrightarrow k_m + h_m < k_{m+1} + h_{m+1} \Leftrightarrow k_m < k_{m+1} - 1,$$

so $l \in M$ and $k_l + h_l - n = d_l = k_l - l$, therefore $h_l = n - l$ and $|A_l| < \dots < |A_n|$. Conversely, if $|A_l| < \dots < |A_n|$, then $k_l - l \leq \dots \leq k_n - n$ and hence $d_i = k_i - i = k_i + h_i - n$ for all $i \in [l, n]$. So $k_i + h_i - n = d_i$ for all $i \in [1, n]$ if and only if $k_l < \dots < k_n$.

Suppose that the two sides of (1.16) are equal. Then the two sides of (1.13) are equal, and $k_l < \dots < k_n$ by the above. In view of Theorem 1.2, $\bigcup_{i=1}^m A_i = A_m$ provided that $k_m < k_{m+1} - 1$ or $m = n$. If $n \geq 2$ and $d_1 = \min_{1 \leq i \leq n} (k_i - i) \geq 2$, then either $A_n \in \text{AP}$, or $n = 2$ and A_2 can be written in the form (1.15).

Remark 1.7. In the case $A_1 = \dots = A_n = A$, we have $h_1 = \dots = h_n = 0$ and Corollary 1.7 reduces to Theorem 2 of Nathanson [N1]. When $|A_1| < \dots < |A_n|$, Corollary 1.7 is a slight improvement to the main theorem of Cao and Sun [CS].

Corollary 1.8. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with (1.2). Then*

$$(1.18) \quad |S(\{A_i\}_{i=1}^n)| \geq \sum_{i=1}^n |A_i| - n^2 + 1.$$

Providing $2 \leq n \leq |A_n| - 2$ and $|A_n| \neq 4$, the two sides are equal if and only if $A_1 = \dots = A_n \in \text{AP}$.

Proof. For each $i = 1, \dots, n$, the number h_i given by (1.17) does not depend on the order of A_1, \dots, A_n . Thus, if we rearrange the order of A_1, \dots, A_n , both sides of (1.16) keep unchanged. Suppose that $|A_{\sigma(1)}| \leq \dots \leq |A_{\sigma(n)}|$ where σ is a permutation on $\{1, \dots, n\}$. If $|A_{\sigma(i)}| < i$, then

$$[i, n] \subseteq \{1 \leq j \leq n : |A_j| \geq i\} \subseteq \{\sigma(j) : j \in (i, n)\}$$

which is impossible. So $|A_{\sigma(i)}| \geq i$ for all $i \in [1, n]$. By Corollary 1.7, (1.16) holds and hence (1.18) follows. If both sides of (1.18) are equal, then $h_i = 0$ for all $i = 1, \dots, n$ and hence $|A_1| = \dots = |A_n|$, as $\bigcup_{i=1}^n A_i = A_n$ by Corollary 1.7 we must have $A_1 = \dots = A_n$. Now it suffices to apply the Nathanson result. \square

For the equality case of (1.13), let us look at one more example.

Example 1.3. Let k and n be integers with $k > n > 1$. Let A_1, \dots, A_{n-1} be subsets of $A_n = [0, k-1]$ with $A_1 = [0, k-n] \setminus \{k-n-1\}$ and $|A_{i+1}| - |A_i| \in \{0, 1\}$ for all $i \in (1, n)$. We assert that

$$S = S(\{A_i\}_{i=1}^n) = \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} \right] \setminus \left\{ kn - \frac{n(n+1)}{2} - 1 \right\}$$

and hence

$$|S| = kn - n^2 = 1 + (|A_1| - 1) + (n-1)(k-n) = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j).$$

Since $M = \{1, n\}$, by the arguments in the proof of Corollary 1.6, we may assume that $A_2 = \dots = A_n$ without any loss of generality.

In the case $k = n+1$, clearly $A_1 = \{1\}$ and $A_i = [0, n]$ for $i \in (1, n]$, setting $A = [0, n] \setminus \{1\}$ we then have

$$\begin{aligned} S &= 1 + (n-1) \wedge A = 1 + \left\{ \sum_{x \in A} x - a : a \in A \right\} = \sum_{i=1}^n i - A \\ &= \frac{n(n+1)}{2} - ([0, n] \setminus \{1\}) = \left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2} \right] \setminus \left\{ \frac{n(n+1)}{2} - 1 \right\} \\ &= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} \right] \setminus \left\{ kn - \frac{n(n+1)}{2} - 1 \right\}. \end{aligned}$$

Below we verify the assertion on the condition $k > n + 1$. By Example 1.1,

$$\begin{aligned} S \subseteq S([0, k - n], A_2, \dots, A_n) &= \frac{n(n-1)}{2} + \left[0, \sum_{i=1}^n (k - n)\right] \\ &= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2}\right] \end{aligned}$$

and S contains

$$\begin{aligned} S([0, k - n - 2], A_2, \dots, A_n) &= \frac{n(n-1)}{2} + [0, k - n - 2 + (n-1)(k - n)] \\ &= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} - 2\right]. \end{aligned}$$

Observe that

$$\max S = k - n + (k - n - 1) + \dots + (k - 1) = kn - \frac{n(n+1)}{2}.$$

Now it suffices to show that $kn - n(n+1)/2 - 1 \notin S$. On the contrary, we can write

$$kn - \frac{n(n+1)}{2} - 1 = k - n + (k - i_1) + \dots + (k - i_{n-1})$$

where $1 \leq i_1 < \dots < i_{n-1} \leq k$ and $n \notin \{i_1, \dots, i_{n-1}\}$. Apparently

$$i_1 + \dots + i_{n-1} = \frac{n(n+1)}{2} + 1 - n, \text{ i.e. } \sum_{j=1}^{n-1} (i_j - j) = 1.$$

So $i_t - t = 1$ for some $t \in [1, n)$, and $i_j = j$ for all $j \in [1, n) \setminus \{t\}$. As $i_{n-1} \neq n$ we have $t < n - 1$ and hence $i_t = t + 1 = i_{t+1}$. This contradicts the fact that $i_t < i_{t+1}$.

Let A_1, \dots, A_{n-1} be subsets of $A_n = [0, k_n - 1]$ with the two sides of (1.13) equal. Set $A'_i = \{k_n - 1 - x : x \in A_i\}$ for $i = 1, \dots, n$. Then

$$|S(\{A'_i\}_{i=1}^n)| = |S(\{A_i\}_{i=1}^n)| = 1 + \min_{i \leq j \leq n} (|A'_j| - j).$$

If $\min A_1 + \max A_1 \geq k_n$, then $\min A'_1 + \max A'_1 = 2(k_n - 1) - \min A_1 - \max A_1 < k_n$. So, to discuss the equality case of (1.13) with $A_n \in \text{AP}$, we may simply take $A_n = [0, k_n - 1]$ and assume that $\min A_1 + \max A_1 < k_n$.

Now we pose a conjecture which essentially determines the equality case of (1.13).

Conjecture 1.1. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with $|A_1| \leq \dots \leq |A_n|$, $k_i = |A_i| > i$ for $i \in [1, n]$ and $\bigcup_{i=1}^m A_i = A_m$ for all $m \in M$. Suppose that $A_n = [0, k_n - 1]$ and $\min A_1 + \max A_1 < k_n$. If the two sides of (1.13) are equal, then $A_m = [0, k_m - 1]$ for all $m \in M$, unless*

$$(1.19) \quad M = \{1, n\}, \quad k_n - k_1 = n \text{ and } A_1 = [0, k_1] \setminus \{k_1 - 1\}.$$

Though we are unable to solve this conjecture, we have found evidence to support it through computer calculations.

2. PROOF OF THEOREM 1.1

We use induction on n . In the case $n = 1$, the inequality is obvious since $C = A_1$ and $V_1 = V = \emptyset$. So we proceed to the induction step.

Let $n > 1$ and assume the assertion for smaller values of n . Set $a = \min A_n$ and

$$V' = \{(s, t, \mu, \nu, w) \in V : 1 \leq s, t \leq n - 1\}.$$

For each $i = 1, \dots, n-1$ let A'_i consist of those $a_i \in A_i$ for which $\mu a_i + \nu a \neq w$ if $(i, n, \mu, \nu, w) \in V$, and $\mu a + \nu a_i \neq w$ if $(n, i, \mu, \nu, w) \in V$. Apparently

$$|A'_i| \geq |A_i| - |\{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\}|,$$

and thus

$$\begin{aligned} V'_i &= \{(s, t, \mu, \nu, w) \in V' : i \in \{s, t\}\} \\ &= V_i \setminus \{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\} \end{aligned}$$

has cardinality not more than $|V_i| + |A'_i| - |A_i| < |A'_i|$. Let

$$C' = \{a_1 + \dots + a_{n-1} : a_i \in A'_i, \text{ and } \mu a_i + \nu a_j \neq w \text{ if } (i, j, \mu, \nu, w) \in V'\}.$$

By the induction hypothesis,

$$|C'| \geq 1 + \sum_{i=1}^{n-1} (|A'_i| - |V'_i| - 1) \geq 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) > 0.$$

Write $\max C' = \sum_{i=1}^{n-1} a'_i$ where $a'_1 \in A'_1, \dots, a'_{n-1} \in A'_{n-1}$, and $\mu a'_i + \nu a'_j \neq w$ if $(i, j, \mu, \nu, w) \in V'$. Let A'_n consist of those $a_n \in A_n$ for which

$\mu a'_i + \nu a_n \neq w$ if $(i, n, \mu, \nu, w) \in V$, and $\mu a_n + \nu a'_i \neq w$ if $(n, i, \mu, \nu, w) \in V$. Note that $a \in A'_n$ and $|A'_n| \geq |A_n| - |V_n| > 0$. Clearly

$$(C' + a) \cup (a'_1 + \cdots + a'_{n-1} + A'_n) \subseteq C$$

and

$$\max(C' + a) = a'_1 + \cdots + a'_{n-1} + a = \min(a'_1 + \cdots + a'_{n-1} + A'_n).$$

Therefore

$$\begin{aligned} |C| &\geq |C' + a| + |a'_1 + \cdots + a'_{n-1} + A'_n| - 1 = |C'| + |A'_n| - 1 \\ &\geq 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) + |A_n| - |V_n| - 1 = 1 + \sum_{i=1}^n (|A_i| - |V_i| - 1). \end{aligned}$$

Since $\sum_{i=1}^n |V_i| = 2|V|$, we are done.

3. SEVERAL LEMMAS

We first check the exception (iii) given in Theorem 1.2.

Lemma 3.1. *Let A_1, \dots, A_n ($n > 1$) be finite subsets of \mathbb{Z} such that $|A_i| > i$ for all $i \in [1, n]$, $|A_{n-1}| = n < |A_n| - 1$ and $\bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n$. Then the two sides of (1.13) are equal if and only if $A_{n-1}, A_n \setminus A_{n-1} \in \text{AP}$ and $d(A_{n-1}) = d(A_n \setminus A_{n-1})$.*

Proof. Let $S = S(\{A_i\}_{i=1}^n)$ and $k_i = |A_i|$ for all $i = 1, \dots, n$. Write $A_{n-1} = \{x_1, \dots, x_n\}$ and $A_n \setminus A_{n-1} = \{y_1, \dots, y_{k_n - k_{n-1}}\}$ where $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_{k_n - k_{n-1}}$. Since $k_i - i \geq 1 = k_{n-1} - (n-1)$ for all $i \in [1, n-1]$, $S(\{A_i\}_{i=1}^{n-1}) = (n-1) \wedge A_{n-1}$ as we pointed out in Example 1.2. Thus

$$\begin{aligned} S &= \bigcup_{i=1}^n \{x_1 + \cdots + x_n - x_i + y : y \in \{x_i, y_1, \dots, y_{k_n - k_{n-1}}\}\} \\ &= x_1 + \cdots + x_n + (\{0\} \cup \{y_j - x_i : i \in [1, n], j \in [1, k_n - k_{n-1}]\}) \end{aligned}$$

and hence $|S| = 1 + |(A_n \setminus A_{n-1}) - A_{n-1}|$ where we let $A - B = A + (-B) = \{a - b : a \in A, b \in B\}$ for $A, B \subseteq \mathbb{Z}$. By a known result (cf. Lemma 1.3 and Theorem 1.5 of [N2]), for any finite subsets A and B of \mathbb{Z} with $|A| \geq 2$ and $|B| \geq 2$, $|A + B| = |A| + |B| - 1$ if and only if $A, B \in \text{AP}$ and $d(A) = d(B)$. So

$$\begin{aligned} |S| &= 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (k_j - j) = 1 + (n-1)(k_{n-1} - (n-1)) + k_n - n = k_n \\ &\iff |(A_n \setminus A_{n-1}) - A_{n-1}| = k_n - 1 = |A_n \setminus A_{n-1}| + |-A_{n-1}| - 1 \\ &\iff x_{i+1} - x_i = y_{j+1} - y_j \text{ for all } i \in [1, n) \text{ and } j \in [1, k_n - k_{n-1}). \end{aligned}$$

We are done. \square

The following lemma is an improvement to Lemma 2 of [CS].

Lemma 3.2. *Let A_1 and A_2 be finite subsets of \mathbb{Z} with $|A_1| \geq 3$, $A_1 \subset A_2$, $\min A_1 = \min A_2$, $\max A_1 \neq \max A_2$ and $|S(A_1, A_2)| = |A_1| + |A_2| - 2$. Then $A_2 \in \text{AP}$ unless $|A_1| = 3$ and A_2 can be written in the form (1.15).*

Proof. Let $A_1 = \{a_1, \dots, a_k\}$ and $A_2 = \{b_1, \dots, b_l\}$ where $a_1 < \dots < a_k$ and $b_1 < \dots < b_l$. By the proof of Lemma 2 of [CS], $a_i \in \{b_i, b_{i+1}\}$ for all $i \in [1, k]$,

$$S(A_1, A_2) = \{a_1 + b_2, \dots, a_1 + b_{l-1}, a_1 + b_l, \dots, a_k + b_l\},$$

and $A_2 \in \text{AP}$ if $a_3 < b_{l-1}$.

Suppose that $a_3 = b_{l-1}$. Then $k = 3$ since $a_3 \leq a_k < b_l$. As $a_1 + b_{l-1} < a_2 + b_{l-1} < a_2 + b_l$, we must have $a_2 + b_{l-1} = a_1 + b_l$, i.e. $b_l - b_{l-1} = a_2 - a_1$. If $a_3 = b_3$, then $l = 4$, $a_2 = b_2$ and hence $b_4 - b_3 = b_2 - b_1$, so A_2 is of the form (1.15). Below we let $a_3 = b_4$. Then $l = 5$ and $b_5 - b_4 = a_2 - a_1$. As $a_1 + b_4 < a_3 + b_2 = b_4 + b_2 \leq a_2 + b_4 = a_1 + b_5$, we must have $a_2 = b_2 < b_3$. Observe that

$$a_1 + b_3 < a_2 + b_3 < a_2 + b_4 = a_1 + b_5 < a_3 + b_3 < a_3 + b_5.$$

So $a_2 + b_3 = a_1 + b_4$ and $a_3 + b_3 = a_2 + b_5$, therefore $A_2 \in \text{AP}$. \square

We now present a lemma reflecting some symmetry.

Lemma 3.3. *Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with $A_1 = \dots = A_m \subseteq A_{m+1} = \dots = A_n$ and $0 < |A_m| - m \leq |A_n| - n$ where $m \in [1, n]$. Define the dual sequence $\{B_j\}_{j=1}^{|A_n|-n}$ of $\{A_i\}_{i=1}^n$ as follows:*

$$B_i = A_n \setminus A_m \text{ for each } i \in [1, |A_n| - n - (|A_m| - m)]$$

and

$$B_j = A_n \text{ for all } j \in (|A_n| - n - (|A_m| - m), |A_n| - n].$$

Then $|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{|A_n|-n})|$ and

$$\sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j) = \sum_{i=1}^{|A_n|-n} \min_{i \leq j \leq n} (|B_j| - j).$$

Proof. Let $k_m = |A_m|$ and $k_n = |A_n|$. Suppose that $A_m = \{x_1, \dots, x_{k_m}\}$ and $A_n \setminus A_m = \{y_1, \dots, y_{k_n - k_m}\}$. Then $S(\{A_i\}_{i=1}^n)$ consists of integers in

the form $\sum_{i \in I} x_i + \sum_{j \in J} y_j$ where $I \subseteq [1, k_m]$, $J \subseteq [1, k_n - k_m]$, $|I| + |J| = n$ and $|I| \geq m$, in other words the elements of $S(\{A_i\}_{i=1}^n)$ are integers of the form

$$\sum_{i=1}^{k_m} x_i - \sum_{i \in \bar{I}} x_i + \sum_{j=1}^{k_n - k_m} y_j - \sum_{j \in \bar{J}} y_j$$

where

$$\bar{I} \subseteq [1, k_m], \bar{J} \subseteq [1, k_n - k_m], |\bar{I}| + |\bar{J}| = k_m + (k_n - k_m) - n = k_n - n$$

and $|\bar{J}| \geq k_n - k_m - (n - m) = k_n - n - (k_m - m)$. Thus

$$S(\{A_i\}_{i=1}^n) = \sum_{x \in A_n} x - S(\{B_i\}_{i=1}^{k_n - n})$$

and so $|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{k_n - n})|$. Clearly $\sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j) = m(k_m - m) + (n - m)(k_n - n)$. Also,

$$\begin{aligned} & \sum_{i=1}^{k_n - n} \min_{i \leq j \leq n} (|B_j| - j) - (k_m - m)(|A_n| - (k_n - n)) \\ &= (k_n - n - (k_m - m))(|A_n \setminus A_m| - (k_n - n - (k_m - m))) \\ &= (n - m)(k_n - n) + (m - n)(k_m - m). \end{aligned}$$

This concludes the proof. \square

Let $A_1 \subseteq A_2 \subseteq \mathbb{Z}$, $|A_1| = 3$ and $|A_2| = 4$. Then the dual sequence of $\{A_i\}_{i=1}^2$ is the sequence A_2, A_2 . Thus the example (given by Nathanson) with $|2^{\wedge} A_2| = 2|A_2| - 2^2 + 1$ and $A_2 \notin \text{AP}$, induces the exception (ii) in Theorem 1.2.

4. REDUCTION OF THEOREM 1.2

Let A_1, \dots, A_n be finite subsets of \mathbb{Z} with (1.2) and $|A_1| \leq \dots \leq |A_n|$. Put $d_i = \min_{i \leq j \leq n} (|A_j| - j)$ and $k'_i = d_i + i$ for $i = 1, \dots, n$. Clearly $k'_n = |A_n|$ and $k'_i < k'_{i+1}$ for all $i \in [1, n)$. As $k'_i \leq |A_i|$, we can choose a subset A'_i of A_i with $|A'_i| = k'_i$. Obviously $A'_n = A_n$ and $\sum_{i=1}^n |A'_i| \leq \sum_{i=1}^n |A_i|$. By the Theorem of Cao and Sun [CS], we have

$$|S(\{A_i\}_{i=1}^n)| \geq |S(\{A'_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (k'_i - i) = 1 + \sum_{i=1}^n d_i.$$

So (1.13) holds. If equality is valid in (1.13), then

$$|S(\{A'_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n (k'_i - i),$$

hence by the Theorem of [CS] we have $\bigcup_{i=1}^m A'_i = A'_m \subseteq A_m$ for any m in the set

$$\begin{aligned} M &= \{1 \leq i < n : k'_i < k'_{i+1} - 1\} \cup \{n\} = \{1 \leq i \leq n : d_i < d_{i+1}\} \cup \{n\} \\ &= \{1 \leq i \leq n : |A_i| - i < |A_j| - j \text{ for all } j \in (i, n)\}. \end{aligned}$$

For any $i = 1, \dots, n$, if $a_i \in A_i$ then we can select $A'_i \subseteq A_i$ so that $a_i \in A'_i$ and $|A'_i| = k'_i$. Thus, in equality case of (1.13) we have $\bigcup_{i=1}^m A_i \subseteq A_m$ for all $m \in M$.

Let $1 \leq i \leq n$. Then

$$k'_i > i \iff d_i > 0 \iff |A_j| > j \text{ for all } j \in [i, n].$$

Thus

$$|A_i| > i \text{ for all } i \in [1, n] \iff |A'_i| > i \text{ for all } i \in [1, n].$$

Recall that $A'_n = A_n$. When $n = 2$ and $A'_2 = A_2$ is of the form (1.15), clearly

$$|A_1| \in \{3, 4\} \iff |A_1| - 1 \geq |A_2| - 2 \iff d_1 = 2 \iff k'_1 = 3.$$

In the case $n > 1$ and $|A_n| > n$, we have

$$|A_{n-1}| = n \iff d_{n-1} = 1 \iff k'_{n-1} = n,$$

thus $A_{n-1} = A'_{n-1}$ providing $|A_{n-1}| = n$ or $k'_{n-1} = n$.

In view of the above and Lemma 3.1, Theorem 1.2 can be reduced to the following

Theorem 4.1. *Let A_1, \dots, A_n be subsets of \mathbb{Z} with $|A_1| < \dots < |A_n| < \infty$ and $|A_i| > i$ for all $i = 1, \dots, n$. If*

$$(4.1) \quad |S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n (|A_i| - i),$$

then $A_n \in \text{AP}$ unless we have (i) or (iii), or (ii) with $|A_1| = 3$.

Remark 4.1. Let k be a positive integer. By the previous reasoning, if Theorem 4.1 holds for those subsets A_1, \dots, A_n of \mathbb{Z} with $|A_1| + \dots + |A_n| \leq k$, then so does Theorem 1.2.

5. PROOF OF THEOREM 4.1

We proceed by induction on $k = \sum_{i=1}^n |A_i|$. Apparently $k \geq |A_1| > 1$.

If $k = 2$, then $n = 1$ and $|A_1| = 2$. In the case $n = 1$, both (4.1) and (i) hold.

Below we let $k > 2$ and $n \geq 2$, and assume that the result holds if $|A_1| + \cdots + |A_n| < k$. Now let $|A_1| + \cdots + |A_n| = k$. For all $i \in [1, n]$ we set

$$(5.1) \quad k_i = |A_i| \quad \text{and} \quad d_i = \min_{i \leq j \leq n} (k_j - j) = k_i - i.$$

Obviously $1 \leq d_1 \leq \cdots \leq d_n$. Put

$$(5.2) \quad a = \min \bigcup_{i=1}^n A_i, \quad I = \{1 \leq i \leq n : a \in A_i\}, \quad r = \min I \quad \text{and} \quad t = \max I.$$

For $i \in I$ let

$$(5.3) \quad A'_i = \begin{cases} A_i \setminus \{a\} & \text{if } i \neq r, \\ \{a\} & \text{if } i = r; \end{cases}$$

and for $i \in \bar{I} = [1, n] \setminus I$ set

$$(5.4) \quad A'_i = \begin{cases} A_i \setminus \{a_i\} & \text{if } r < i < t \text{ and } i \notin M, \\ A_i & \text{otherwise,} \end{cases}$$

where a_i is an arbitrary element of A_i . Write $k'_i = |A'_i|$ for $i \in [1, n] \setminus \{r\}$. Then $1 < k'_1 < \cdots < k'_{r-1} < k_r \leq k'_{r+1} < \cdots < k'_n$ and $\sum_{i \neq r} k'_i < \sum_{i=1}^n k_i = k$. For $i \in [1, n] \setminus \{r\}$ we set

$$(5.5) \quad d'_i = \begin{cases} k'_i - i & \text{if } i < r, \\ k'_i - (i - 1) & \text{if } i > r. \end{cases}$$

Let $S = S(\{A_i\}_{i=1}^n)$, and assume that (4.1) holds. By the Theorem of [CS] and its proof, $\bigcup_{i=1}^m A_i = A_m$ for all $m \in M$, and

$$|S(\{A'_i\}_{i \neq r})| = \sum_{i \neq r} k'_i - \frac{n(n-1)}{2} + 1 = 1 + \sum_{i \neq r} d'_i.$$

Also, $t = n$ and $(r, t) \cap \bar{I} \cap M = \emptyset$ (see (12) and (14) of [CS]), therefore $d'_i = d_i$ for all $i \in [1, n] \setminus \{r\}$.

Clearly $b = \max \bigcup_{i=1}^n A_i \neq a$ (otherwise $|A_n| = |\{a\}| < n$), $-b = \min \bigcup_{i=1}^n (-A_i)$ and

$$|S(\{-A_i\}_{i=1}^n)| = |S| = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|-A_j| - j).$$

Like the fact that $a \in A_t = A_n$ we should also have $-b \in -A_n$. Thus $b \in A_n \setminus \{a\}$.

Let s denote the least index such that $b \in A_s$. By p. 166 of [CS], there exists an $l \in [r, n]$ such that $k_l - l = k_r - r$ (i.e. $d_r = \dots = d_l$), and $l = s = r < n$ is impossible.

From now on we assume that none of (i)—(iii) (in Theorem 1.2) holds. Then $k_n > n+1$. If $k_{n-1} = n$, then $n-1 \in M$ and $\bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n$, thus (iii) holds by Lemma 3.1. Now that (iii) fails, we must have $k_{n-1} > n$.

We claim that $A_n^* = A_n \setminus \{a\} \in \text{AP}$. For this conclusion, it suffices to work under the condition $A_n^* \notin \text{AP}$.

Case 1. $r < n-1$.

Apparently $n > 2$, $k'_n = k_n - 1 > n = (n-1) + 1$ and $k'_{n-1} \geq k_{n-1} - 1 > n-1 = (n-2) + 1$. As $A'_n = A_n^* \notin \text{AP}$, by the induction hypothesis, $n-1 = 2$, $r = 1$, $k'_2 = 3$ and $A'_3 = A_3 \setminus \{a\}$ is of the form (1.15). Note that $n = 3 < k_{n-1} = k_2 < k_3 = k'_3 + 1 = 5$ and hence $k_2 = 4$. If $k_1 > 2$, then $k_1 = 3$ and $M = \{3\}$, hence $S = 3 \wedge A_3$ and $A_3 \in \text{AP}$ by Example 1.2. Thus $k_1 = 2$, $k_2 = 4$ and $k_3 = 5$. Observe that $|S| = 1 + (2-1) + (4-2) + (5-3) = 6$. If $1 \leq i < j \leq 4$, then x_i or x_j lies in A_2 (since $A_2 \subseteq A_3$ and $k_3 - k_2 = 1$), therefore $a + x_i + x_j \in S$. Thus S contains the following 5 integers:

$$a + x_1 + x_2, a + x_1 + x_3, a + x_1 + x_4 = a + x_2 + x_3, a + x_2 + x_4, a + x_3 + x_4.$$

Suppose that $A_1 = \{a, x_i\}$ where $1 \leq i \leq 4$. If $i \in \{3, 4\}$, then both $x_4 + x_3 + x_1$ and $x_4 + x_3 + x_2$ belong to S , this contradicts the fact that $|S| = 6 < 5 + 2$. So $i \in \{1, 2\}$, and S consists of the above 5 integers and the number $x_i + x_3 + x_4$. Apparently S also contains $x_1 + x_2 + x_3$ and $x_1 + x_2 + x_4$. Since $a + x_2 + x_3 < x_1 + x_2 + x_3 < x_1 + x_2 + x_4 < x_i + x_3 + x_4$, we must have $x_1 + x_2 + x_3 = a + x_2 + x_4$ and $x_1 + x_2 + x_4 = a + x_3 + x_4$. Thus $x_4 - x_3 = x_1 - a = x_3 - x_2$ and hence $A_n = A_3 \in \text{AP}$.

Case 2. $A_{n-1} \subset A_n^*$.

Clearly $k_1 < \dots < k_{n-1} < k_n^* = |A_n^*| = k_n - 1$. Let S^* denote the set $S(A_1, \dots, A_{n-1}, A_n^*)$. Then $a + \min S(\{A_i\}_{i=1}^{n-1}) = \min S \notin S^*$. So $|S^*| \leq |S| - 1 = \sum_{i=1}^n (k_i - i)$ and hence $|S^*| = |S| - 1 = 1 + \sum_{i=1}^{n-1} (k_i - i) + (k_n^* - n)$.

Recall that $k_n^* = k_n - 1 > k_{n-1} \geq n + 1$. By the induction hypothesis, $n = 2$, $k_1 = 3$, A_2^* has the form (1.15) and hence $k_2 = 5$. For any two distinct elements x and y of A_2^* we have $x + y \in S^*$ since one of them belongs to A_1 . All the $1 + (3 - 1) + (4 - 2) = 5$ elements of S^* are as follows:

$$x_1 + x_2, x_1 + x_3, x_1 + x_4 = x_2 + x_3, x_2 + x_4, x_3 + x_4.$$

As $|a + A_1| = 3$, $\max(a + A_1) < x_1 + x_4$ and $|S| = 1 + (3 - 1) + (5 - 2) = 6$, we must have

$$S = (a + A_1) \cup \{x_i + x_4 : i = 1, 2, 3\}.$$

Evidently $x_4 \in A_1$ and $x_1 + x_3 = a + x_4$ since $x_1 + x_3 \in a + A_1$, also $x_3 \in A_1$ and $x_1 + x_2 = a + x_3$ since $x_1 + x_2 \in a + A_1$. So $x_4 - x_3 = x_1 - a = x_3 - x_2$ and hence $A_n^* = A_2^* \in \text{AP}$.

Case 3. $r = n - 1$, or $r = n$ and $A_{n-1} = A_n^*$.

Let $\bar{r} = n$ if $r = n - 1$, and $\bar{r} = n - 1$ if $r = n$. Clearly $A'_{\bar{r}} = A_n^*$ and $k'_{\bar{r}} = |A_n^*| = k_n - 1 > n = (n - 1) + 1$.

Let us handle the case $n = 2$. Note that $k_1 = k_{n-1} > n = 2$. If $A_1 = A_2^*$, then $\min(-A_1) = \min(-A_2)$ and $\max(-A_1) < \max(-A_2) = -a$, hence $-A_2 \in \text{AP}$ (i.e. $A_2 \in \text{AP}$) by Lemma 3.2 since (ii) fails. When $r = 1$, we have $\min A_1 = \min A_2$, if $s = 2$ (i.e. $\max A_1 \neq \max A_2$) then $A_2 \in \text{AP}$ by Lemma 3.2. In the case $r = s = 1$, we have $l > 1$ because $l = r = s < n$ is impossible, hence $k_1 = k_2 - 1$ since $k_r - r = k_l - l$, thus $S = 2^{\wedge} A_2$ and $A_2 \in \text{AP}$ by Example 1.2. (Recall that (ii) fails.)

Let $n - 1 = 2$, $k_1 = k'_1 = 3$ and $A'_{\bar{r}}$ have the form (1.15). Observe that $n = 3 < k_{n-1} = k_2 \leq k_3 - 1 = |A_3^*| = |A'_{\bar{r}}| = 4$. So $M = \{3\}$ and hence $A_3 \in \text{AP}$ by Example 1.2.

Now we assume that $n > 2$, and $n \neq 3$ or $k'_1 \neq 3$ or $A'_{\bar{r}}$ is not of the form (1.15). As $A'_{\bar{r}} = A_n^* \notin \text{AP}$, by the induction hypothesis, $k_{n-2} = k'_{n-2} = n - 1$, also $A_{n-2} = A'_{n-2}$ and $A_n^* \setminus A_{n-2} = A'_{\bar{r}} \setminus A'_{n-2}$ form arithmetic progressions with the same difference d . Since $k_{n-2} = n - 1 < n < k_{n-1}$, we have $n - 2 \in M$ and hence $\bigcup_{i=1}^{n-2} A_i = A_{n-2} \subseteq A_n^*$. Let $A_{n-1}^* = A_{n-1} \setminus \{a\}$, $k_{n-1}^* = |A_{n-1}^*|$ and $S^* = S(A_1, \dots, A_{n-2}, A_{n-1}^*, A_n^*)$. Then

$$1 < k_1 < \dots < k_{n-2} = n - 1 < k_{n-1}^* \leq k_n^* < k_n,$$

$$d_n^* = k_n^* - n = k_n - 1 - n = d_n - 1 > 0,$$

$$d_{n-1}^* = \min\{k_{n-1}^* - (n - 1), k_n^* - n\} = k_{n-1} - n = d_{n-1} - 1 > 0,$$

$$d_i^* = \min\{k_i - i, \dots, k_{n-2} - (n - 2), d_{n-1}^*\} = 1 = d_i \text{ for } i \in [1, n - 2].$$

Write $A_{n-2} = \{x_1, \dots, x_{n-1}\}$ and $A_n^* \setminus A_{n-2} = \{y_1, \dots, y_{k_n-1-(n-1)}\}$ where $x_1 < \dots < x_{n-1}$ and $y_1 < \dots < y_{k_n-n}$. In view of Example 1.2, $S(\{A_i\}_{i=1}^{n-2}) = (n-2) \wedge A_{n-2} = \{x-x_i : 1 \leq i \leq n-1\}$ where $x = \sum_{i=1}^{n-1} x_i$. As $A_{n-1}^* \subseteq A_n^*$ all elements of S^* have the form $x-x_i+y_j+z$ where $1 \leq i \leq n-1$, $1 \leq j \leq k_n-n$ and $z \in \{x_i, y_1, \dots, y_{k_n-n}\} \setminus \{y_j\}$, they are all greater than $x-x_{n-1}+y_1+a$. If $x-x_{n-1}+y_2+a = x-x_i+y_j+z$ where i, j, z are as above, then $j=1$ and $z=x_i$ since $a+y_2 < \min\{x_i+y_2, y_1+y_2\}$, hence $-x_{n-1}+y_2+a = -x_i+y_1+x_i = y_1$ and $x_{n-1}-a = y_2-y_1 = d = x_{n-1}-x_{n-2}$; this is impossible. So $x-x_{n-1}+y_1+a, x-x_{n-1}+y_2+a \notin S^*$. However, both $x-x_{n-1}+y_1+a$ and $x-x_{n-1}+y_2+a$ lie in S , for, $a \in A_{n-1}$ if $r = n-1$, and $y_1, y_2 \in A_{n-1}$ if $A_{n-1} = A_n^*$. Therefore

$$|S^*| \leq |S| - 2 = 1 + \sum_{i=1}^n d_i - 2 = 1 + \sum_{i=1}^n d_i^*.$$

If $A_{n-1} = A_n^*$, then $k_{n-1}^* = k_{n-1} > n$. Since $A_n^* \notin \text{AP}$, by the induction hypothesis we have either

$$(i^*) \quad k_n - 1 = k_n^* = n + 1 \text{ and hence } k_{n-1} = n + 1,$$

or

(iii*) $|A_{n-1}^*| = n$ (whence $r = n-1$), and A_{n-1}^* and $A_n \setminus A_{n-1} = A_n^* \setminus A_{n-1}^*$ form arithmetic progressions with the same difference.

Assume (i*). Let $B_1 = \dots = B_{n-2} = A_{n-2}$ and $B_{n-1} = B_n = A_n$. As $M = \{n-2, n\}$, by the idea in Example 1.2 or the proof of Corollary 1.6, $S = S(\{B_i\}_{i=1}^n)$ and $|S(\{B_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|B_j| - j)$. The dual sequence of $\{B_i\}_{i=1}^n$ is the sequence $A_n \setminus A_{n-2}, A_n$ with $|A_n \setminus A_{n-2}| = n+2-(n-1) = 3$, $|A_n| = n+2 > 4$ and $|A_n \setminus A_{n-2}| + |A_n| < (n+1) + k_n \leq k = k_1 + \dots + k_n$. In view of Lemma 3.3 and the induction hypothesis, we have $A_n \in \text{AP}$.

Now we consider the case (iii*). Clearly $k_{n-1} = n+1$ and $k_n - k_{n-1} \geq 2$, so $n-1 \in M$ and $A_{n-2} \subset A_{n-1} \subset A_n$. Write $A_{n-1} = \{a, x_1, \dots, x_{n-1}, y_j\}$ where $1 \leq j \leq k_n - n$. Then $A_n \setminus A_{n-1} = \{y_1, \dots, y_{k_n-n}\} \setminus \{y_j\}$. Since $d(A_{n-1}^*) = d(A_n \setminus A_{n-1}) \geq d$, we must have $y_j \in \{x_1 - d, x_{n-1} + d\}$. Now that $d(A_n \setminus A_{n-1}) = d(A_{n-1}^*) = d$, j must be 1 or $k_n - n$. If $y_1 \in A_{n-1}$ (i.e. $j=1$), then $y_1 + d = y_2 \neq x_1$ and hence $y_1 = x_{n-1} + d$, thus $A_n^* = \{x_1, \dots, x_{n-1}, y_1, \dots, y_{k_n-n}\} \in \text{AP}$. If $y_{k_n-n} \in A_{n-1}$ (i.e. $j = k_n - n$), then $y_{k_n-n} - d = y_{k_n-n-1} \neq x_{n-1}$ and hence $y_{k_n-n} = x_1 - d$, thus $A_n^* = \{y_1, \dots, y_{k_n-n}, x_1, \dots, x_{n-1}\} \in \text{AP}$.

By the above, we do have $A_n \setminus \{a\} \in \text{AP}$ in either case. As $-b = \min \bigcup_{i=1}^n (-A_i)$, by analogy $-A_n \setminus \{-b\} \in \text{AP}$. Because $k_n > n+1 \geq 3$,

and $A_n \setminus \{\min A_n\}$ and $A_n \setminus \{\max A_n\}$ are both in AP, the set A_n must form an arithmetic progression.

The induction step is now complete and the proof of Theorem 4.1 is ended.

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