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HALL'S THEOREM REVISITED

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Abstract. Let $A_1,\cdots,A_n$ $(n>1)$ be sets. By a simple graph-theoretic argument we show that any *set* of distinct representatives of $\{A_i\}_{i=1}^{n-1}$ can be extended to a set of distinct representatives of $\{A_i\}_{i=1}^{n}$ in more than $\min_{n\in I\subseteq\{1,\cdots,n\}}(|\bigcup_{i\in I}A_i|-|I|)$ ways. This yields a natural induction proof of the well-known theorem of P. Hall.

Let $A_1,\cdots,A_n$ be sets. If $a_1\in A_1,\cdots,a_n\in A_n$, and $a_1,\cdots,a_n$ are distinct, then we say that the sequence

$$\{A_i\}_{i=1}^{n} \tag{1}$$

has a *system of distinct representatives* (abbreviated to SDR) $\{a_i\}_{i=1}^{n}$.

A classical theorem of P. Hall [Ha] asserts that (1) has an SDR if and only if

$$\Big|\bigcup_{i\in I}A_i\Big|\geqslant|I|\qquad\text{for all } I\subseteq\{1,\cdots,n\}. \tag{2}$$

Hall's theorem and its restatements in other contexts have many applications throughout discrete mathematics; it is in some sense the fundamental combinatorial min-max relation. (For its connection with transversal matroids, the reader may see M. Aigner [A].) Many textbooks on combinatorics contain the proof given by P. R. Halmos and H. E. Vaughan [HV], who deduced the sufficiency by handling separately the case where $|\bigcup_{i\in I}A_i|>|I|$ for all nonempty $I\subset\{1,\cdots,n\}$ and the remaining case where strong induction on n is used. Surprisingly, it seems that no one has provided a proof of Hall's theorem by passing from $n-1$ sets to n sets.

Now we give

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Theorem. *Let $A_1,\cdots,A_n$ $(n>1)$ be subsets of a set X. Suppose that $\{a_i\}_{i=1}^{n-1}$ forms an SDR of $\{A_i\}_{i=1}^{n-1}$. Then for some $J\subseteq\{1,\cdots,n\}$ containing n there are exactly $|\bigcup_{j\in J}A_j|-|J|+1$ elements of X that combine with $a_1,\cdots,a_{n-1}$ to form an SDR of* (1) *with a_i representing A_i for $i\notin J$.*

Proof. Consider a directed graph G with vertices $1,\cdots,n$, which has an edge from i to j if and only if $i<n$ and $a_i\in A_j$. Let

$$J=\{1\leqslant j\leqslant n:\ \text{there exists a path in } G \text{ from } j \text{ to } n\}$$

and put $A=\bigcup_{j\in J}A_j$. For any $i=1,\cdots,n-1$,

$$\begin{aligned}a_i\in A &\iff a_i\in A_j \text{ for some } j\in J\\ &\iff \text{there is an edge in } G \text{ from } i \text{ to some } j\in J\\ &\iff G \text{ contains a path from } i \text{ to } n\\ &\iff i\in J.\end{aligned}$$

Thus $\{1\leqslant i<n: a_i\in A\}=J\setminus\{n\}$.

Set $B=A\setminus\{a_i: i\in J\setminus\{n\}\}$. Then $B\cap\{a_1,\cdots,a_{n-1}\}=\emptyset$ and $|B|=|A|-|J|+1$.

Let $a\in X$. If a and those a_i with $1\leqslant i<n$ can be rearranged to form an SDR of (1) with a_i representing A_i for $i\notin J$, then $a\in B$ since a represents A_j for some $j\in J$.

Conversely, if $a\in B$, then $a\in A_j$ for some $j\in J$. If $j=n$, then $a_n=a\in A_n$ and hence $\{a_i\}_{i=1}^n$ forms an SDR of $\{A_i\}_{i=1}^n$. If $j\neq n$, then G contains a path from j to n, say $j_0,j_1,\cdots,j_l$ where $j_0=j$ and $j_l=n$. Note that $I=\{j_0,\cdots,j_l\}\subseteq J$. Let $b_{j_0},b_{j_1},\cdots,b_{j_l}$ be $a,a_{j_0},\cdots,a_{j_{l-1}}$ respectively. Evidently $b_i\in A_i$ for all $i\in I$. Thus $\{b_i\}_{i=1}^n$ is an SDR of (1) where we set $b_i=a_i$ for $i\notin I$. This concludes the proof. □

Let $A_1,\cdots,A_n$ be finite sets. Concerning the number of SDR's of (1), in 1948 M. Hall [H] obtained the following lower bound providing (1) has an SDR:

$$f(m,n)=(m)_{\min\{m,n\}}=\begin{cases} m! & \text{if } m\leqslant n,\\ m(m-1)\cdots(m-n+1) & \text{if } m>n,\end{cases} \tag{3}$$

where $m=\min\{|A_1|,\cdots,|A_n|\}$. We emphasize that this bound can be used only if one has verified that

$$d=\min_{\emptyset\neq I\subseteq\{1,\cdots,n\}}\left(\left|\bigcup_{i\in I}A_i\right|-|I|\right)\geqslant 0.$$

For any SDR $\{a_i\}_{i=1}^n$ of (1) we call the *set* $\{a_1,\cdots,a_n\}$ of cardinality n a *set of distinct representatives* (abbreviated to s.d.r.) of (1). Distinct SDR's of (1) may yield the same set of distinct representatives. Obviously (1) has an s.d.r. if and only if (1) has an SDR. Our theorem shows that any s.d.r. of $\{A_i\}_{i=1}^{n-1}$ can be extended to an s.d.r. of (1) in at least $1 + d(n)$ ways where

$$d(n) = \min_{n\in I\subseteq\{1,\cdots,n\}} \left(\left| \bigcup_{i\in I} A_i \right| - |I| \right). \tag{4}$$

This immediately yields a proof of Hall's theorem by ordinary induction on the number of sets.

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