A CURIOUS IDENTITY INVOLVING BINOMIAL COEFFICIENTS

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Abstract

Let m be a nonnegative integer. For integers $0 \leqslant k \leqslant m$ and $n \geqslant 0$ we show the following curious identity

$$\begin{aligned}&(n+2(m-k)+1)\sum_{i=k}^{m}(-1)^i\binom{m+n+i}{m-i}\binom{2i}{k+i}\\ =&(-1)^k\sum_{i=0}^{m-k}\binom{m+n-k+i}{m-k-i}(-4)^i+(-1)^k n\binom{m+n-k}{n}.\end{aligned}$$

Equivalently, we have

$$(x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}-\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=(x-m)\binom{x}{m}.$$

1. Introduction

For integers $m>0$, $n\geqslant 0$ and r, let

$$T^n_{r(m)}=\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv r \ (\mathrm{mod}\ m)}}\binom{n}{k}.$$

Such sums were investigated and applied by the author and his twin brother Zhi-Hong Sun in [SS], [S1], [S2], [Su1] and [Su2]. In the study of the generating function of the sequence $\{T^n_{[n/2](m)}\}_{n=0}^{+\infty}$ where $[\cdot]$ is the greatest integer function, Zhi-Hong Sun posed the following conjecture:

If $m,n\in\mathbb{N}=\{0,1,2,\cdots\}$ and $m\leqslant n$, then

$$\sum_{i=0}^{m}(-1)^i\binom{n+i}{m-i}\left((m+n+1)\binom{2i}{i}-4^i\right)=(n-m)\binom{n}{m}. \tag{1.1}$$

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In other words, for any $m, n \in \mathbb{N}$ we have

$$\sum_{i=0}^{m}(-1)^i\binom{m+n+i}{m-i}\left((2m+n+1)\binom{2i}{i}-4^i\right)=n\binom{m+n}{n}. \tag{1.2}$$

The above conjecture is far from transparent and seems to be somewhat sophisticated. In this paper we will show an extension of the conjecture by means of generating functions, Chebyshev polynomials and double recursions.

For convenience we set

$$A_k(m,n)=\sum_{i=k}^{m}(-1)^i\binom{m+n+i}{m-i}\binom{2i}{k+i} \quad \text{for } k,m,n\in\mathbb{N} \text{ with } k\leqslant m \tag{1.3}$$

and

$$B(m,n)=\sum_{i=0}^{m}\binom{m+n+i}{m-i}(-4)^i \quad \text{for } m,n\in\mathbb{N}. \tag{1.4}$$

Our main result is as follows:

Theorem 1.1. *If $k, m, n \in \mathbb{N}$ and $k \leqslant m$, then*

$$(n+2(m-k)+1)A_k(m,n)-(-1)^kB(m-k,n)=(-1)^kn\binom{m+n-k}{n}. \tag{1.5}$$

We will provide recursions for $A_k(m,n)$ and $B(m,n)$ in the next section, and prove Theorem 1.1 in Section 3.

Recall that

$$\binom{x}{0}=1 \quad \text{and} \quad \binom{x}{n}=\frac{x(x-1)\cdots(x-n+1)}{n!} \quad \text{for } n=1,2,3,\cdots.$$

Theorem 1.1 has the following equivalent version.

Theorem 1.2. *For each $m = 0, 1, 2, \cdots$ we have*

$$\begin{aligned}&(x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}\\&=\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i+(x-m)\binom{x}{m}.\end{aligned} \tag{1.6}$$

2. Double Recursions for $A_k(m,n)$ and $B(m,n)$

Lemma 2.1. *Let $k,m\in\mathbb{N}$ and $k\leqslant m$. Then $A_k(m,0)=(-1)^m$.*

Proof. For $i\in\mathbb{Z}$ with $k\leqslant i\leqslant m$, clearly

$$\binom{m+i}{m-i}\binom{2i}{k+i}=\frac{(m+i)!}{(m-i)!(2i)!}\times\frac{(2i)!}{(k+i)!(i-k)!}=\binom{m+i}{k+i}\binom{m-k}{m-i}.$$

So

$$\begin{aligned}(-1)^m A_k(m,0)&=\sum_{i=k}^{m}(-1)^{m-i}\binom{m+i}{m-i}\binom{2i}{k+i}\\&=\sum_{i=k}^{m}(-1)^{m-i}\binom{m-k}{m-i}\binom{(m-k)+k+i}{k+i}.\end{aligned}$$

This is the coefficient of x^{k+m} in the power series of

$$(1-x)^{m-k}\sum_{n=0}^{+\infty}\binom{m-k+n}{n}x^n=(1-x)^{m-k}\cdot\frac{1}{(1-x)^{m-k+1}}=\frac{1}{1-x}\quad(|x|<1).$$

So $(-1)^m A_k(m,0)=1$. This ends the proof. $\square$

Lemma 2.2. *For $m\in\mathbb{N}$ we have $B(m,0)=(-1)^m(2m+1)$.*

Proof. For $n=0,1,2,\cdots$ the nth Chebyshev polynomial $U_n(x)$ of the second kind is defined by

$$\sin((n+1)\theta)=\sin\theta\cdot U_n(\cos\theta).$$

It is well-known that

$$U_n(x)=\sum_{j=0}^{[n/2]}(-1)^j\binom{n-j}{j}(2x)^{n-2j}.$$

In view of the above,

$$\begin{aligned}2m+1&=\lim_{\theta\to0}\frac{\sin((2m+1)\theta)}{\sin\theta}=\lim_{\theta\to0}U_{2m}(\cos\theta)=U_{2m}(\cos 0)=U_{2m}(1)\\&=\sum_{j=0}^{m}(-1)^j\binom{2m-j}{j}2^{2m-2j}\\&=\sum_{i=0}^{m}(-1)^{m-i}\binom{m+i}{m-i}4^i=(-1)^m B(m,0).\end{aligned}$$

We are done. $\square$

Lemma 2.3. *Let $k, m, n \in \mathbb{N}$ and $k \leqslant m$. Then*

$$A_k(m+1, n+1) = A_k(m+1, n) + A_k(m, n+1)$$

and

$$B(m+1, n+1) = B(m+1, n) + B(m, n+1).$$

Proof. For any $a_0, a_1, \cdots, a_{m+1} \in \mathbb{Z}$ we have

$$\sum_{i=0}^{m+1} \binom{m+1+n+1+i}{m+1-i} a_i = \sum_{i=0}^{m+1} \binom{m+1+n+i}{m+1-i} a_i + \sum_{i=0}^{m} \binom{m+n+1+i}{m-i} a_i.$$

So the desired equalities follow. □

Theorem 2.1. *Let $k, m, n \in \mathbb{N}$ and $k \leqslant m$. Then we have*

$$\begin{cases} A_k(m,0) = (-1)^m \\ A_k(k,n) = (-1)^k \\ A_k(m+1, n+1) = A_k(m+1, n) + A_k(m, n+1) \end{cases} \tag{2.1}$$

and

$$\begin{cases} B(m,0) = (-1)^m(2m+1) \\ B(0,n) = 1 \\ B(m+1, n+1) = B(m+1, n) + B(m, n+1). \end{cases} \tag{2.2}$$

Proof. In view of Lemmas 2.1—2.3, it suffices to check equalities $A_k(k,n) = (-1)^k$ and $B(0,n) = 1$, which can be easily seen. This concludes the proof. □

3. Proofs of Theorems 1.1 and 1.2

Lemma 3.1. *For $k, m, n \in \mathbb{N}$ with $k \leqslant m$, we have*

$$A_k(m,n) + A_k(m+1,n) = (-1)^k \binom{m-k+n}{m-k+1} \tag{3.1}$$

and

$$(-1)^m A_k(m,n) = \sum_{i=0}^{m-k} (-1)^i \binom{n+i-1}{i}. \tag{3.2}$$

Proof. i) We fix $k \in \mathbb{N}$ and use induction on mn to show (3.1).

If $n = 0$ or $m = k$ then (3.1) holds, for,

$$A_k(m,0) + A_k(m+1,0) = (-1)^m + (-1)^{m+1} = 0 = (-1)^k \binom{m-k}{m-k+1}$$

and

$$\begin{aligned}A_k(k,n) + A_k(k+1,n) =& (-1)^k + \sum_{i=k}^{k+1} (-1)^i \binom{k+1+n+i}{k+1-i}\binom{2i}{k+i} \\ =& (-1)^k + (-1)^k(2k+n+1) + (-1)^{k+1}\binom{2k+2}{2k+1} \\ =& (-1)^k(1+2k+n+1-2k-2) = (-1)^k\binom{n}{1}.\end{aligned}$$

Clearly both $m(n+1)$ and $(m+1)n$ are less than $(m+1)(n+1)$. Assume that

$$A_k(m,n+1) + A_k(m+1,n+1) = (-1)^k \binom{m-k+n+1}{m-k+1}$$

and

$$A_k(m+1,n) + A_k((m+1)+1,n) = (-1)^k \binom{m+1-k+n}{m+1-k+1}.$$

With the help of Lemma 2.3,

$$\begin{aligned}& A_k(m+1,n+1) + A_k((m+1)+1,n+1) \\ =& A_k(m+1,n) + A_k(m,n+1) + (A_k(m+2,n) + A_k(m+1,n+1) \\ =& (A_k(m+1,n) + A_k(m+2,n)) + (A_k(m,n+1) + A_k(m+1,n+1)) \\ =& (-1)^k \binom{m-k+n+1}{m-k+2} + (-1)^k \binom{m-k+n+1}{m-k+1} \\ =& (-1)^k \binom{m+1-k+n+1}{m+1-k+1}.\end{aligned}$$

In view of the above, we have proved (3.1).

ii) Observe that

$$\begin{aligned}(-1)^m A_k(m,n) - (-1)^k A_k(k,n) &= \sum_{k \leqslant l < m} \left((-1)^{l+1} A_k(l+1,n) - (-1)^l A_k(l,n)\right) \\ &= \sum_{k \leqslant l < m} (-1)^{l+1} \left(A_k(l+1,n) + A_k(l,n)\right).\end{aligned}$$

Applying (2.1) and (3.1) we then obtain that

$$\begin{aligned}(-1)^m A_k(m,n) =&(-1)^k(-1)^k + (-1)^k \sum_{k\leqslant l<m} (-1)^{l+1}\binom{l-k+n}{l-k+1}\\ =&\sum_{i=0}^{m-k}(-1)^i\binom{n+i-1}{i}.\end{aligned}$$

This ends the proof. □

Remark 3.1. Let $n\in\mathbb{N}$. It is well-known that

$$\frac{1}{(1-x)^n} = \sum_{i=0}^{+\infty}\binom{n-1+i}{i}x^i \quad \text{for } x\in(-1,1).$$

An identity of Shi-Jie Zhu (cf. (4.3.8) of [B]) asserts that

$$\sum_{i=0}^{l}\binom{n-1+i}{i} = \binom{l+n}{l} \quad \text{for } l=0,1,2,\cdots.$$

Thus, $A_k(m,n)$ (with $k,m\in\mathbb{N}$ and $k\leqslant m$) is interesting in view of (3.2).

Proof of Theorem 1.1. Fix $k\in\mathbb{N}$. Below we use induction on mn to show that (1.5) holds for $m\geqslant k$ and $n\geqslant 0$.

If $m\geqslant k$ and $n=0$, then (1.5) is valid, for

$$\begin{aligned}&(2(m-k)+1)A_k(m,0)-(-1)^k B_k(m-k,0)\\ =&(2(m-k)+1)(-1)^m-(-1)^k(-1)^{m-k}(2(m-k)+1)=0.\end{aligned}$$

In the case $m=k$ and $n\in\mathbb{N}$, (1.5) also holds because

$$(n+1)A_k(k,n)-(-1)^k B(0,n) = (n+1)(-1)^k-(-1)^k\times 1 = (-1)^k n.$$

Now let $m,n\in\mathbb{N}$ and $m\geqslant k$. Put $m'=m+1$ and $n'=n+1$. Assume that

$$(n'+2(m-k)+1)A_k(m,n')-(-1)^k B(m-k,n') = (-1)^k n'\binom{m-k+n'}{n'}$$

and

$$(n+2(m'-k)+1)A_k(m',n)-(-1)^k B(m'-k,n) = (-1)^k n\binom{m'-k+n}{n}.$$

Then

$$\begin{aligned}
&(n'+2(m'-k)+1)A_k(m',n')-(-1)^kB(m'-k,n')\\
=&A_k(m',n')+(n+2(m-k)+3)(A_k(m,n')+A_k(m',n))\\
&-(-1)^k(B(m-k,n')+B(m'-k,n))\\
=&A_k(m',n')+A_k(m,n')+(n'+2(m-k)+1)A_k(m,n')-(-1)^kB(m-k,n')\\
&+(n+2(m'-k)+1)A_k(m',n)-(-1)^kB(m'-k,n)\\
=&(-1)^k\binom{m-k+n'}{m-k+1}+(-1)^kn\binom{m'-k+n}{n}+(-1)^kn'\binom{m-k+n'}{n'}\\
=&(-1)^kn'\left(\binom{m-k+n'}{n}+\binom{m-k+n'}{n'}\right)=(-1)^kn'\binom{m'-k+n'}{n'}.
\end{aligned}$$

By the above, we have proved Theorem 1.1. □

Proof of Theorem 1.2. By Theorem 1.1, for any $k,n=0,1,2,\cdots$, we have

$$\begin{aligned}
&(n+2m+1)\sum_{i=k}^{k+m}(-1)^i\binom{k+m+n+i}{k+m-i}\binom{2i}{k+i}\\
=&(-1)^k\sum_{i=0}^{m}\binom{m+n+i}{m-i}(-4)^i+(-1)^kn\binom{m+n}{n},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(n+2m+1)\sum_{j=0}^{m}(-1)^j\binom{2k+m+n+j}{m-j}\binom{2k+2j}{2k+j}\\
=&\sum_{i=0}^{m}\binom{m+n+i}{m-i}(-4)^i+n\binom{m+n}{m}.
\end{aligned}$$

Write

$$P(x,y)=\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}=\sum_{i=0}^{m}P_i(x)y^i$$

and

$$Q(x)=\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i+(x-m)\binom{x}{m}.$$

Then, for any given integer $t\geqslant m$, the polynomial equation

$$\sum_{i=0}^{m}(t+m+1)P_i(t)y^i=P(t,y)=Q(t)$$

has solutions $y=0,2,4,\cdots$, hence $0=(t+m+1)P_0(t)-Q(t)=P_1(t)=\cdots=P_m(t)$. Therefore

$$(x+m+1)P_0(x)=Q(x)\ \text{ and } P_i(x)=0 \text{ for } 1\leqslant i\leqslant m.$$

It follows that $(x+m+1)P(x,y)=Q(x)$. We are done. □

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