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# SUMS OF MINIMA AND MAXIMA

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Abstract. Let $h_1,\cdots,h_n$ be positive integers. We study new sums

$$m(h_1,\cdots,h_n)=\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\min\left\{\frac{r_1}{h_1},\cdots,\frac{r_n}{h_n}\right\}$$

and

$$M(h_1,\cdots,h_n)=\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\max\left\{\frac{r_1}{h_1},\cdots,\frac{r_n}{h_n}\right\},$$

the first of which times $h_1\cdots h_n$ is the number of lattice points in a pyramid of dimension $n+1$. We show that

$$\frac{m(h_1,\cdots,h_n)}{(h_1-1)\cdots(h_n-1)}=1+\sum_{\emptyset\neq I\subseteq\{1,\cdots,n\}}(-1)^{|I|}\frac{m(\{h_i\}_{i\in I})}{\prod_{i\in I}(h_i-1)}$$

if $h_1,\cdots,h_n>1$, and that

$$\frac{M(h_1,\cdots,h_n)-h_1\cdots h_n+1}{(h_1+1)\cdots(h_n+1)}=\sum_{\emptyset\neq I\subseteq\{1,\cdots,n\}}(-1)^{|I|}\frac{M(\{h_i\}_{i\in I})}{\prod_{i\in I}(h_i+1)}.$$

The sums $m(h_1,h_2)$ and $M(h_1,h_2)$ are closely connected with the reciprocity law for Dedekind sums. The values of $m(h_1,h_2,h_3)$, $M(h_1,h_2,h_3)$ and $m(h_1,h_2,h_3,h_4)+M(h_1,h_2,h_3,h_4)$ are determined explicitly in the paper.

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## 1. Introduction

Let $x$ be any real number. By $\{x\}$ we mean the fractional part of $x$. Also we set $[x] = x - \{x\}$ and

$$(1.1)\qquad ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h$ and $k$ be positive integers. In 1892 R. Dedekind introduced the classical Dedekind sum

$$(1.2)\qquad s(h,k) = \sum_{r=0}^{k-1} \left(\left(\frac{r}{k}\right)\right) \left(\left(\frac{hr}{k}\right)\right).$$

This sum arose naturally in the study of the functional equation of Dedekind's eta function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})$$

where $\tau$ is in the upper half plane. When $h$ and $k$ are coprime, Dedekind was able to express $s(h,k) + s(k,h)$ as an explicit rational function in $h$ and $k$; the result is now known as the reciprocity law for Dedekind sums. (Cf. H. Rademacher and E. Grosswald [RG].) In 1931 Rademacher [R] proved Dedekind's reciprocity law by associating it with the sums

$$A(h,k) = \frac{1}{hk} \sum_{a=0}^{hk-1} \left[\frac{a}{h}\right]\left[\frac{a}{k}\right] \text{ and } B(h,k) = \sum_{r=0}^{k-1} \frac{r}{k}\left[\frac{hr}{k}\right] + \sum_{r=0}^{h-1} \frac{r}{h}\left[\frac{kr}{h}\right],$$

which have geometric interpretations.

For a finite sequence $\{h_i\}_{i=1}^n$ of positive integers, we define

$$(1.3)\quad A(h_1,\cdots,h_n) = A(\{h_i\}_{i=1}^n) = \frac{1}{H} \sum_{a=0}^{H-1} \left[\frac{h_1 a}{H}\right] \cdots \left[\frac{h_n a}{H}\right] \text{ where } H = h_1 \cdots h_n$$

and

$$(1.4)\qquad B(h_1,\cdots,h_n) = B(\{h_i\}_{i=1}^n) = \sum_{i=1}^{n} \sum_{r=0}^{h_i-1} \frac{r}{h_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{h_j r}{h_i}\right].$$

In addition we let $A(\emptyset) = 1$ and $B(\emptyset) = 0$. Inspired by Rademacher's work, we are led to study these two kinds of sums.

For a positive integer $h$ we put

$$(1.5)\qquad D_h = \{0 \leqslant x < 1 : hx \in \mathbb{Z}\} = \left\{\frac{r}{h} : r = 0, 1, \cdots, h-1\right\}.$$

If $k$ is also a positive integer, then

$$D_h \cap D_k = \{0 \leqslant x < 1 : hx, kx \in \mathbb{Z}\} = \{0 \leqslant x < 1 : (h,k)x \in \mathbb{Z}\} = D_{(h,k)}$$

because the greatest common divisor $(h,k)$ can be written in the form $hu+kv$ with $u, v \in \mathbb{Z}$. For a vector $\vec{x} = \langle x_1, \cdots, x_n \rangle$ in the Euclid space $\mathbb{R}^n$, we let

$$\min \vec{x} = \min\{x_1, \cdots, x_n\} \quad \text{and} \quad \max \vec{x} = \max\{x_1, \cdots, x_n\}. \tag{1.6}$$

For a finite sequence $\{h_i\}_{i=1}^n$ of positive integers, we define

$$m(h_1, \cdots, h_n) = m(\{h_i\}_{i=1}^n) = \sum_{\vec{x} \in D_{h_1} \times \cdots \times D_{h_n}} \min \vec{x} \tag{1.7}$$

and

$$M(h_1, \cdots, h_n) = M(\{h_i\}_{i=1}^n) = \sum_{\vec{x} \in D_{h_1} \times \cdots \times D_{h_n}} \max \vec{x}. \tag{1.8}$$

For convenience, we set $m(\emptyset) = 1$ and $M(\emptyset) = 0$ additionally.

The sums $m(h_1, \cdots, h_n)$ and $M(h_1, \cdots, h_n)$ have not been studied before. The only known related result is the following one: If $0 \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{h-1} < 1$ then

$$\sum_{r_1=0}^{h-1} \cdots \sum_{r_n=0}^{h-1} \max\{x_{r_1}, \cdots, x_{r_n}\} = \sum_{r=0}^{h-1} x_r \left((r+1)^n - r^n\right).$$

This was first obtained K. Nagasaka and J. S. Shiue [NS], and then reproved by T. W. Leung [L] in 1994. The identity is almost apparent since $|\{\langle r_1, \cdots, r_n \rangle \in \{0, 1, \cdots, r\}^n : \max\{r_1, \cdots, r_n\} = r\}| = (r+1)^n - r^n$.

In this paper we mainly investigate the sums $m(h_1, \cdots, h_n)$ and $M(h_1, \cdots, h_n)$ and their relations to $A(h_1, \cdots, h_n)$ and $B(h_1, \cdots, h_n)$. These four sums are different from higher dimensional Dedekind sums introduced by D. Zagier [Z]. Throughout the paper an empty product takes the value 1 while an empty sum vanishes. When positive integers $h_1, \cdots, h_n$ are given, for $I \subseteq [1,n] = \{1, \cdots, n\}$ we let $\bar{I} = [1,n] \setminus I$.

Now we introduce our results.

**Theorem 1.1.** *Let $h_1, \cdots, h_n$ be positive integers. Let $P$ denote the integral point $(0, \cdots, 0, H)$ in the Euclid space $\mathbb{R}^{n+1}$ where $H = h_1 \cdots h_n$. For $I \subseteq [1,n]$ let $P_I = (x_1^{(I)}, \cdots, x_n^{(I)}, 0)$ where $x_i^{(I)} = h_i$ or $0$ according to whether $i \in I$ or not. Denote by $N(h_1, \cdots, h_n)$ the number of lattice points $(x_1, \cdots, x_n, x_{n+1})$ with $0 < x_{n+1} < H$ in the pyramid with vertices $P$ and $P_I$ $(I \subseteq [1,n])$. Then*

$$m(h_1, \cdots, h_n) = A(h_1, \cdots, h_n) = N(h_1, \cdots, h_n)/H. \tag{1.9}$$

How can we determine the values of $m(h_1, \cdots, h_n)$ and $M(h_1, \cdots, h_n)$? In the case $n = 2$ we do this in Section 3 and relate it to the reciprocity law for Dedekind sums. For the case $n = 3$, we have the following result.

**Theorem 1.2.** *Let $h,k,l$ be positive integers. Then*

$$
\begin{aligned}
m(h,k,l) =&\frac{hkl}{4}-\frac{hk+hl+kl}{6}+\frac{h+k+l-1}{8}\\
&+\frac{h+k-2hk}{24l}+\frac{h+l-2hl}{24k}+\frac{k+l-2kl}{24h}\\
&+\frac{(h-1)(k,l)^2}{24kl}+\frac{(k-1)(h,l)^2}{24hl}+\frac{(l-1)(h,k)^2}{24hk}
\end{aligned}
\tag{1.10}
$$

*and*

$$
\begin{aligned}
M(h,k,l) =&\frac{3}{4}hkl-\frac{hk+hl+kl}{6}-\frac{h+k+l+1}{8}\\
&+\frac{h+k+2hk}{24l}+\frac{h+l+2hl}{24k}+\frac{k+l+2kl}{24h}\\
&-\frac{(h+1)(k,l)^2}{24kl}-\frac{(k+1)(h,l)^2}{24hl}-\frac{(l+1)(h,k)^2}{24hk}.
\end{aligned}
\tag{1.11}
$$

In terms of those $m(\{h_i\}_{i\in I})$ or $M(\{h_i\}_{i\in I})$ with $I\subset[1,n]$, we can determine $m(h_1,\cdots,h_n)+M(h_1,\cdots,h_n)$ in general, and $m(h_1,\cdots,h_n)$ and $M(h_1,\cdots,h_n)$ in the case $2\nmid n$. Namely we have the following theorem.

**Theorem 1.3.** *Let $h_1,\cdots,h_n$ be positive integers. Then*

$$
\begin{aligned}
&m(h_1,\cdots,h_n)+M(h_1,\cdots,h_n)\\
=&h_1\cdots h_n-\sum_{I\subset[1,n]}m(\{h_i\}_{i\in I})\\
=&\prod_{i=1}^{n}(h_i-1)-\sum_{I\subset[1,n]}(-1)^{n-|I|}M(\{h_i\}_{i\in I}).
\end{aligned}
\tag{1.12}
$$

*We also have*

$$
\begin{aligned}
m(h_1,\cdots,h_n)&=\sum_{I\subseteq[1,n]}(-1)^{|I|}m(\{h_i\}_{i\in I})\prod_{j\in\bar I}(h_j-1)\\
&=\sum_{\emptyset\ne I\subseteq[1,n]}(-1)^{|I|-1}M(\{h_i\}_{i\in I})\prod_{j\in\bar I}h_j
\end{aligned}
\tag{1.13}
$$

*and*

$$
\begin{aligned}
M(h_1,\cdots,h_n)=&h_1\cdots h_n-1+\sum_{I\subseteq[1,n]}(-1)^{|I|}M(\{h_i\}_{i\in I})\prod_{j\in\bar I}(h_j+1)\\
=&\sum_{\emptyset\ne I\subseteq[1,n]}(-1)^{|I|-1}m(\{h_i\}_{i\in I})\prod_{j\in\bar I}h_j.
\end{aligned}
\tag{1.14}
$$

*Remark* 1.1. Let $h_1, \cdots, h_n$ be positive integers. If $I = \{1 \leqslant i \leqslant n : h_i = 1\} \neq \emptyset$, then $m(h_1, \cdots, h_n) = 0$ and $M(h_1, \cdots, h_n) = M(\{h_j\}_{j \in \bar{I}})$. In the case $h_1 = \cdots = h_n = h$, we have

$$m(\{h_i\}_{i=1}^n) = A(\{h_i\}_{i=1}^n) = \frac{1}{h^n} \sum_{a=0}^{h^n-1} \left[ \frac{a}{h^{n-1}} \right]^n = \frac{1}{h} \sum_{q=0}^{h-1} q^n = \frac{B_{n+1}(h) - B_{n+1}(0)}{h(n+1)}$$

where $B_{n+1}(x)$ is the Bernoulli polynomial of degree $n+1$.

The following is a remarkable consequence of Theorems 1.2 and 1.3.

**Corollary 1.4.** *Let $h, k, l, n$ be positive integers. Then $m(h, k, l, n) + M(h, k, l, n)$ equals*

$$\begin{aligned} & hkln - \frac{hkl + hkn + hln + kln}{4} - \frac{h+k+l+n}{8} \\ & + \frac{hk + hl + kl}{12n} + \frac{hk + hn + kn}{12l} + \frac{hl + hn + ln}{12k} + \frac{kl + kn + ln}{12h} \\ & - \frac{h+k}{24} \cdot \frac{(l,n)^2}{ln} - \frac{h+l}{24} \cdot \frac{(k,n)^2}{kn} - \frac{h+n}{24} \cdot \frac{(k,l)^2}{kl} \\ & - \frac{k+l}{24} \cdot \frac{(h,n)^2}{hn} - \frac{k+n}{24} \cdot \frac{(h,l)^2}{hl} - \frac{l+n}{24} \cdot \frac{(h,k)^2}{hk}. \end{aligned} \tag{1.15}$$

For the average $A(h_1, \cdots, h_n)$ we have the following result.

**Theorem 1.5.** *Let $h_1, \cdots, h_n$ be positive integers. Then*

$$A(h_1, \cdots, h_n) = \prod_{i=1}^n (h_i - 1) + \sum_{\emptyset \neq I \subseteq [1,n]} (-1)^{|I|} \sum_{r=0}^{(h_i)_{i \in I} - 1} \frac{r}{(h_i)_{i \in I}} \prod_{j \in \bar{I}} \left[ \frac{h_j r}{(h_i)_{i \in I}} \right] \tag{1.16}$$

*where $(h_i)_{i \in I}$ denotes the greatest common divisor of those $h_i$ with $i \in I$.*

*Remark* 1.2. Let $h, k, l$ be positive integers. By Theorem 1.5 we immediately have

$$A(h,k) + B(h,k) = (h-1)(k-1) + \frac{(h,k)-1}{2} \tag{1.17}$$

and

$$\begin{aligned} & A(h,k,l) + B(h,k,l) - (h-1)(k-1)(l-1) + \frac{(h,k,l)-1}{2} \\ & = \sum_{r=0}^{(h,k)-1} \frac{r}{(h,k)} \left[ \frac{lr}{(h,k)} \right] + \sum_{r=0}^{(h,l)-1} \frac{r}{(h,l)} \left[ \frac{kr}{(h,l)} \right] + \sum_{r=0}^{(k,l)-1} \frac{r}{(k,l)} \left[ \frac{hr}{(k,l)} \right]. \end{aligned} \tag{1.18}$$

**Theorem 1.6.** *Let $h_1,\cdots,h_n$ be pairwise coprime positive integers. Then*

$$A(h_1,\cdots,h_n)+B(h_1,\cdots,h_n)=\prod_{i=1}^n(h_i-1), \tag{1.19}$$

$$m(h_1,\cdots,h_n)=-\sum_{I\subseteq[1,n]}(-1)^{|I|}B(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}(h_j-1) \tag{1.20}$$

*and*

$$M(h_1,\cdots,h_n)=\sum_{I\subseteq[1,n]}B(\{h_i\}_{i\in I}). \tag{1.21}$$

By (1.9), (1.10) and Theorem 1.6 (or (1.18)), we immediately have

**Corollary 1.7.** *Let $h,k,l$ be positive integers with $(h,k)=(h,l)=(k,l)=1$. Then*

$$\begin{aligned}B(h,k,l)=&\sum_{r=0}^{h-1}\frac{r}{h}\left[\frac{kr}{h}\right]\left[\frac{lr}{h}\right]+\sum_{r=0}^{k-1}\frac{r}{k}\left[\frac{hr}{k}\right]\left[\frac{lr}{k}\right]+\sum_{r=0}^{l-1}\frac{r}{l}\left[\frac{hr}{l}\right]\left[\frac{kr}{l}\right]\\ =&\frac{3}{4}hkl-\frac{5}{6}(hk+hl+kl)+\frac{7}{8}(h+k+l-1)\\ &+\frac{2hk-h-k}{24l}+\frac{2hl-h-l}{24k}+\frac{2kl-k-l}{24h}-\frac{h^2+k^2+l^2-h-k-l}{24hkl}.\end{aligned} \tag{1.22}$$

In the next section we will investigate general relations among the symmetric sums

$$m(h_1,\cdots,h_n),\ M(h_1,\cdots,h_n),\ A(h_1,\cdots,h_n)\text{ and }B(h_1,\cdots,h_n).$$

In Section 3 we are going to compute their values in the cases $n=2,3$ and prove Corollary 1.4. We conjecture that if $h_1,\cdots,h_n$ are pairwise coprime then the four sums are rational functions in $h_1,\cdots,h_n$.

## 2. Relations among $m(\{h_i\}_{i=1}^n)$, $M(\{h_i\}_{i=1}^n)$, $A(\{h_i\}_{i=1}^n)$ and $B(\{h_i\}_{i=1}^n)$

*Proof of Theorem 1.1.* For any $x_1,\cdots,x_n\in\mathbb{Z}$ and $h\in\{1,\cdots,H-1\}$, the point $Q$ with coordinates $(x_1,\cdots,x_n,H-h)$ in $\mathbb{R}^{n+1}$ belongs to the pyramid if and only if there are $l_1\in(0,h_1],\cdots,l_n\in(0,h_n]$ such that

$$\frac{x_1}{l_1}=\cdots=\frac{x_n}{l_n}=\frac{h}{H}.$$

Thus

$$\begin{aligned} N(h_1,\cdots,h_n) &= \sum_{0<h<H} \left| \left\{ \langle x_1,\cdots,x_n\rangle \in \mathbb{Z}^n : 0 < x_i \leqslant \frac{h}{H} h_i \text{ for } i \in [1,n] \right\} \right| \\ &= \sum_{0<h<H} \left[\frac{h_1 h}{H}\right] \cdots \left[\frac{h_n h}{H}\right] = HA(h_1,\cdots,h_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} N(h_1,\cdots,h_n) &= \sum_{x_1=1}^{h_1} \cdots \sum_{x_n=1}^{h_n} \left| \left\{ h \in \mathbb{Z} : H > h \geqslant \frac{H}{h_i} x_i \text{ for } i \in [1,n] \right\} \right| \\ &= \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \left| \left\{ h \in \mathbb{Z} : H > h \geqslant H - \frac{H}{h_i} r_i \text{ for } i \in [1,n] \right\} \right| \\ &= \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \left| \left\{ a \in \mathbb{Z} : 0 < a \leqslant \min_{1\leqslant i\leqslant n} \frac{H}{h_i} r_i \right\} \right| = Hm(h_1,\cdots,h_n). \end{aligned}$$

So (1.9) holds. □

The following lemma is well known, it can be verified directly.

**Lemma 2.1.** *Let $S$ be a finite set, and $f, g$ be functions from the power set of $S$ to an additive abelian group. Then*

$$f(I) = \sum_{J\subseteq I} (-1)^{|J|} g(J) \qquad \textit{for all } I \subseteq S \tag{2.1}$$

*if and only if*

$$g(I) = \sum_{J\subseteq I} (-1)^{|J|} f(J) \qquad \textit{for all } I \subseteq S. \tag{2.2}$$

*Remark* 2.1. If $h_1,\cdots,h_n > 1$ are integers, then by (1.13) we have

$$f(I) = \sum_{J\subseteq I} (-1)^{|J|} f(J) \qquad \text{for all } I \subseteq [1,n] \tag{2.3}$$

where $f(I) = m(\{h_i\}_{i\in I})/\prod_{i\in I}(h_i - 1)$.

**Lemma 2.2.** *Let $h_1,\cdots,h_n$ be positive integers and $c_1,\cdots,c_n$ complex numbers. Then*

$$\frac{1}{H}\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left(x_i + c_i - \left[\frac{h_i a}{H}\right]\right) = \sum_{I\subseteq[1,n]} \frac{\prod_{j\in\bar{I}} x_j}{h_I} \sum_{a=0}^{h_I-1}\prod_{i\in I}\left(c_i - \left[\frac{h_i a}{h_I}\right]\right)$$

*where* $H=h_1\cdots h_n$ *and* $h_I=\prod_{i\in I}h_i$.

*Proof.* For $I\subseteq[1,n]$, clearly

$$\begin{aligned}\frac{1}{H}\sum_{a=0}^{H-1}\prod_{i\in I}\left(c_i-\left[\frac{h_ia}{H}\right]\right)=&\frac{1}{h_Ih_{\bar I}}\sum_{a=0}^{h_Ih_{\bar I}-1}\prod_{i\in I}\left(c_i-\left[\frac{\lfloor a/h_{\bar I}\rfloor}{h_I/h_i}\right]\right)\\=&\frac{1}{h_I}\sum_{q=0}^{h_I-1}\prod_{i\in I}\left(c_i-\left[\frac{h_iq}{h_I}\right]\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{H}\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left(x_i+c_i-\left[\frac{h_ia}{H}\right]\right)=&\frac{1}{H}\sum_{a=0}^{H-1}\sum_{I\subseteq[1,n]}\prod_{i\in I}\left(c_i-\left[\frac{h_ia}{H}\right]\right)\cdot\prod_{j\in\bar I}x_j\\=&\sum_{I\subseteq[1,n]}\frac{\prod_{j\in\bar I}x_j}{H}\sum_{a=0}^{H-1}\prod_{i\in I}\left(c_i-\left[\frac{h_ia}{H}\right]\right)\\=&\sum_{I\subseteq[1,n]}\frac{\prod_{j\in\bar I}x_j}{h_I}\sum_{a=0}^{h_I-1}\prod_{i\in I}\left(c_i-\left[\frac{h_ia}{h_I}\right]\right).\end{aligned}$$

We are done. □

*Proof of Theorem 1.3.* Let $H=h_1\cdots h_n$. Observe that

$$\begin{aligned}Hm(h_1,\cdots,h_n)=&\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\left|\left\{a\in\mathbb{Z}:0\leqslant a<\min_{1\leqslant i\leqslant n}\frac{Hr_i}{h_i}\right\}\right|\\=&\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left|\left\{r_i\in\mathbb{Z}:\frac{a}{H}<\frac{r_i}{h_i}<1\right\}\right|\\=&\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left(h_i-1-\left[\frac{h_ia}{H}\right]\right).\end{aligned}$$

Also,

$$\begin{aligned}H-M(h_1,\cdots,h_n)=&\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\min_{1\leqslant i\leqslant n}\frac{h_i-r_i}{h_i}\\=&\frac{1}{H}\sum_{x_1=1}^{h_1}\cdots\sum_{x_n=1}^{h_n}\left|\left\{a\in\mathbb{Z}:0\leqslant a<\min_{1\leqslant i\leqslant n}\frac{Hx_i}{h_i}\right\}\right|\\=&\frac{1}{H}\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left|\left\{x_i\in\mathbb{Z}:\frac{a}{H}<\frac{x_i}{h_i}\leqslant 1\right\}\right|\\=&\frac{1}{H}\sum_{a=0}^{H-1}\prod_{i=1}^{n}\left(h_i-\left[\frac{h_ia}{H}\right]\right).\end{aligned}$$

Thus, by Lemma 2.2 we have

$$m(h_1,\cdots,h_n)=\sum_{I\subseteq[1,n]}(-1)^{|I|}A(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}(h_j-1)$$

and

$$H-M(h_1,\cdots,h_n)=\sum_{I\subseteq[1,n]}(-1)^{|I|}A(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}h_j=\sum_{I\subseteq[1,n]}m(\{h_i\}_{i\in I}).$$

(Note that $h_i-[h_ia/H]=1+(h_i-1-[h_ia/H])$.)

Clearly $\prod_{i\in\emptyset}h_i-M(\emptyset)=1=\sum_{I\subseteq\emptyset}m(\{h_i\}_{i\in I})$. Applying Lemma 2.1 we get that

$$\begin{aligned}(-1)^n m(h_1,\cdots,h_n)=&\sum_{I\subseteq[1,n]}(-1)^{|I|}\Big(\prod_{i\in I}h_i-M(\{h_i\}_{i\in I})\Big)\\=&\prod_{i=1}^{n}(1-h_i)-\sum_{I\subseteq[1,n]}(-1)^{|I|}M(\{h_i\}_{i\in I}).\end{aligned}$$

So (1.12) holds. Since $m(\{h_i\}_{i\in I})=A(\{h_i\}_{i\in I})$ for all $I\subseteq[1,n]$, the first equality in (1.13) follows from the above. As

$$1-\frac{M(h_1,\cdots,h_n)}{H}=\sum_{I\subseteq[1,n]}(-1)^{|I|}\frac{m(\{h_i\}_{i\in I})}{\prod_{i\in I}h_i},$$

by Lemma 2.1 we have

$$\begin{aligned}\frac{m(h_1,\cdots,h_n)}{H}=&\sum_{I\subseteq[1,n]}(-1)^{|I|}\Big(1-\frac{M(\{h_i\}_{i\in I})}{\prod_{i\in I}h_i}\Big)\\=&(1-1)^n-\sum_{I\subseteq[1,n]}(-1)^{|I|}\frac{M(\{h_i\}_{i\in I})}{\prod_{i\in I}h_i}.\end{aligned}$$

Thus (1.13) is valid.

Observe that

$$\begin{aligned}&M(h_1,\cdots,h_n)\\=&H-\sum_{I\subseteq[1,n]}(-1)^{|I|}m(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}h_j=\sum_{\emptyset\neq I\subseteq[1,n]}(-1)^{|I|-1}m(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}h_j\\=&\sum_{\emptyset\neq I\subseteq[1,n]}(-1)^{|I|-1}\Big(\prod_{i\in I}(h_i-1)-\sum_{J\subseteq I}(-1)^{|I|-|J|}M(\{h_j\}_{j\in J})\Big)\prod_{j\in\bar{I}}h_j\\=&-H\sum_{\emptyset\neq I\subseteq[1,n]}(-1)^{|I|}\prod_{i\in I}\frac{h_i-1}{h_i}+\sum_{J\subseteq[1,n]}(-1)^{|J|}M(\{h_j\}_{j\in J})\sum_{J\subseteq I\subseteq[1,n]}\prod_{j\in\bar{I}}h_j\\=&H-H\sum_{I\subseteq[1,n]}\prod_{i\in I}\Big(\frac{1}{h_i}-1\Big)+\sum_{J\subseteq[1,n]}(-1)^{|J|}M(\{h_j\}_{j\in J})\sum_{I'\subseteq\bar{J}}\prod_{j\in I'}h_j\\=&H-1+\sum_{I\subseteq[1,n]}(-1)^{|I|}M(\{h_i\}_{i\in I})\prod_{j\in\bar{I}}(h_j+1).\end{aligned}$$

This proves (1.14). □

*Proof of Theorem 1.5.* Let $\vec{x}\in D=D_{h_1}\times\cdots\times D_{h_n}$ and $J(\vec{x})=\{1\leqslant j\leqslant n: x_j=\max\vec{x}\}$. If $\emptyset\ne I\subseteq J(\vec{x})$, then $\max\vec{x}\in\bigcap_{i\in I}D_{h_i}=D_{(h_i)_{i\in I}}$. Note that

$$\sum_{\emptyset\ne I\subseteq J(\vec{x})}(-1)^{|I|-1}=(-1)^{|\emptyset|}-\sum_{I\subseteq J(\vec{x})}(-1)^{|I|}=1-(1-1)^{|J(\vec{x})|}=1.$$

Let $D^*=\{\vec{x}\in D: x_1\cdots x_n\ne 0\}$. For any nonempty $I\subseteq[1,n]$ we have

$$\begin{aligned}\sum_{\substack{\vec{x}\in D^*\\ J(\vec{x})\supseteq I}}\max\vec{x}&=\sum_{x\in D_{(h_i)_{i\in I}}}\ \sum_{x_j\in D_{h_j}\cap(0,x]\ \text{for}\ j\in\bar{I}}x\\ &=\sum_{x\in D_{(h_i)_{i\in I}}}x\prod_{j\in\bar{I}}|\{r\in\mathbb{Z}\colon 0<r\leqslant h_jx\}|=\sum_{x\in D_{(h_i)_{i\in I}}}x\prod_{j\in\bar{I}}[h_jx].\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\vec{x}\in D^*}\max\vec{x}&=\sum_{\vec{x}\in D^*}\ \sum_{\emptyset\ne I\subseteq J(\vec{x})}(-1)^{|I|-1}\max\vec{x}\\ &=\sum_{\emptyset\ne I\subseteq[1,n]}(-1)^{|I|-1}\sum_{\substack{\vec{x}\in D^*\\ J(\vec{x})\supseteq I}}\max\vec{x}\\ &=\sum_{\emptyset\ne I\subseteq[1,n]}(-1)^{|I|-1}\sum_{r=0}^{(h_i)_{i\in I}-1}\frac{r}{(h_i)_{i\in I}}\prod_{j\in\bar{I}}\left[\frac{h_jr}{(h_i)_{i\in I}}\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{\vec{x}\in D^*}\max\vec{x}&=\sum_{0<s_1<h_1}\cdots\sum_{0<s_n<h_n}\max_{1\leqslant i\leqslant n}\frac{s_i}{h_i}\\ &=\sum_{0<r_1<h_1}\cdots\sum_{0<r_n<h_n}\max_{1\leqslant i\leqslant n}\left(1-\frac{r_i}{h_i}\right)\\ &=\prod_{i=1}^n(h_i-1)-\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\min_{1\leqslant i\leqslant n}\frac{r_i}{h_i}.\end{aligned}$$

So we have (1.16) with the help of (1.9). □

*Proof of Theorem 1.6.* Since $(h_i)_{i\in I}=1$ for any $I\subseteq[1,n]$ with $|I|>1$, (1.19) follows directly from (1.16). By (1.9) and (1.19), for any $I\subseteq[1,n]$ we have $B(\{h_i\}_{i\in I})=\prod_{i\in I}(h_i-1)-m(\{h_i\}_{i\in I})$. Using this we can easily deduce (1.20) and (1.21) from (1.13) and (1.12). □

For $\vec{x} = \langle x_1, \cdots, x_n\rangle \in \mathbb{R}^n$, if $x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_n}$ where $\{i_1, \cdots, i_n\} = [1, n]$, then we let

$$\min{}_t\ \vec{x} = x_{i_t} \quad \text{for } t = 1, \cdots, n. \tag{2.4}$$

Let $\{h_i\}_{i=1}^n$ be a finite sequence of positive integers. For $t \in [1, n]$ we define

$$m_t(h_1, \cdots, h_n) = m_t(\{h_i\}_{i=1}^n) = \sum_{\vec{x} \in D_{h_1} \times \cdots \times D_{h_n}} \min{}_t\ \vec{x}. \tag{2.5}$$

Clearly, $m(h_1, \cdots, h_n) = m_1(h_1, \cdots, h_n)$ and $M(h_1, \cdots, h_n) = m_n(h_1, \cdots, h_n)$. Note that

$$\sum_{t=1}^n m_t(h_1, \cdots, h_n) = \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \left(\frac{r_1}{h_1} + \cdots + \frac{r_n}{h_n}\right)$$
$$= \sum_{k=1}^n \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \frac{r_k}{h_k} = \sum_{k=1}^n \sum_{r_k=0}^{h_k-1} \frac{r_k}{h_k} \prod_{\substack{j=1 \\ j \neq k}}^n h_j = \sum_{k=1}^n \frac{h_k - 1}{2} \cdot \frac{h_1 \cdots h_n}{h_k}.$$

So we have

$$\sum_{t=1}^n m_t(h_1, \cdots, h_n) = \frac{h_1 \cdots h_n}{2} \left(n - \sum_{k=1}^n \frac{1}{h_k}\right). \tag{2.6}$$

If $h_1, \cdots, h_n$ are pairwise coprime, then we can show that

$$m_t(\{h_i\}_{i=1}^n) = \sum_{I \subseteq [1,n]} B(\{h_i\}_{i \in I}) \sum_{s=0}^{n-t} \binom{|I| - 1}{s} (-1)^s \sum_{\substack{J \subseteq \bar{I} \\ |J| = n-t-s}} \prod_{j \in J} (h_j - 1). \tag{2.7}$$

## 3. Computations of Some Explicit Values

Let $h$ be a positive integer. Clearly

$$m(h) = M(h) = A(h) = B(h) = \sum_{r=0}^{h-1} \frac{r}{h} = \frac{h-1}{2}. \tag{3.1}$$

**Theorem 3.1.** *Let $h$ and $k$ be positive integers. Then*

$$m(h, k) = A(h, k) = \frac{hk}{3} - \frac{h + k - 1}{4} - \frac{h^2 + k^2 - (h, k)^2}{12hk}, \tag{3.2}$$

$$M(h, k) = \frac{2}{3}hk - \frac{h + k + 1}{4} + \frac{h^2 + k^2 - (h, k)^2}{12hk} \tag{3.3}$$

*and*

$$B(h,k)=\frac{2}{3}hk-\frac{3(h+k)-1}{4}+\frac{(h,k)}{2}+\frac{h^2+k^2-(h,k)^2}{12hk}. \tag{3.4}$$

*Proof.* Observe that

$$\begin{aligned}F(h,k)=&\sum_{a=0}^{hk-1}\frac{a}{h}\left\{\frac{a}{k}\right\}=\sum_{q=0}^{h-1}\sum_{r=0}^{k-1}\frac{kq+r}{h}\left\{\frac{kq+r}{k}\right\}\\=&\sum_{r=0}^{k-1}\frac{r}{k}\sum_{q=0}^{h-1}\frac{kq+r}{h}=\sum_{r=0}^{k-1}\frac{r}{k}\left(k\frac{h-1}{2}+r\right)\\=&\frac{h-1}{2}\sum_{r=0}^{k-1}r+\frac{1}{k}\sum_{r=0}^{k-1}r^2=\frac{h-1}{2}\cdot\frac{k(k-1)}{2}+\frac{(k-1)(2k-1)}{6}.\end{aligned}$$

Similarly,

$$F(k,h)=\frac{h(h-1)(k-1)}{4}+\frac{(h-1)(2h-1)}{6}.$$

Let $d=(h,k)$, $h'=h/d$ and $k'=k/d$. Obviously $(h',k')=1$. When $v$ runs from 0 to $k'-1$, $h'v+r$ runs over a complete system of residues modulo $k'$ where $r$ is an arbitrary integer. Thus

$$\begin{aligned}G(h,k)=&\sum_{a=0}^{hk-1}\left\{\frac{a}{h}\right\}\left\{\frac{a}{k}\right\}=\sum_{r=0}^{h-1}\frac{r}{h}\sum_{q=0}^{k-1}\left\{\frac{hq+r}{k}\right\}\\=&\sum_{r=0}^{h-1}\frac{r}{h}\sum_{u=0}^{d-1}\sum_{v=0}^{k'-1}\left\{\frac{h(k'u+v)+r}{k}\right\}=\sum_{r=0}^{h-1}\frac{r}{h'}\sum_{v=0}^{k'-1}\left\{\frac{h'v+[r/d]+\{r/d\}}{k'}\right\}\\=&\sum_{r=0}^{h-1}\frac{r}{h'}\sum_{s=0}^{k'-1}\frac{s+\{r/d\}}{k'}=\sum_{r=0}^{h-1}\frac{r}{h'}\left(\frac{k'-1}{2}+\left\{\frac{r}{d}\right\}\right)=\frac{h-1}{2}\cdot\frac{k-d}{2}+F(h',d)\\=&\frac{h-1}{2}\cdot\frac{k-d}{2}+\frac{h-d}{2}\cdot\frac{d-1}{2}+\frac{(d-1)(2d-1)}{6}=\frac{(h-1)(k-1)}{4}+\frac{d^2-1}{12}.\end{aligned}$$

Note that

$$\begin{aligned}hkA(h,k)=&\sum_{a=0}^{hk-1}\left(\frac{a}{h}-\left\{\frac{a}{h}\right\}\right)\left(\frac{a}{k}-\left\{\frac{a}{k}\right\}\right)\\=&\frac{1}{hk}\sum_{a=0}^{hk-1}a^2-F(h,k)-F(k,h)+G(h,k).\end{aligned}$$

In view of the above, we can compute $m(h,k)=A(h,k)$ and verify (3.2). As

$$M(h,k)+m(h,k)=\sum_{r=0}^{h-1}\sum_{s=0}^{k-1}\left(\frac{r}{h}+\frac{s}{k}\right)=hk-\frac{h+k}{2},$$

(3.3) follows from (3.2). Combining (3.2) with (1.17), we then obtain (3.4). □

*Remark* 3.1. Let $h,k$ be positive integers, and $\alpha,\beta\in[0,1)$. That $G(h,k)=(h-1)(k-1)/4+((h,k)^2-1)/12$ can be extended as follows:

$$\sum_{a=0}^{hk-1}\left\{\frac{\alpha+a}{h}\right\}\left\{\frac{\beta+a}{k}\right\}=\left(\alpha+\frac{h-1}{2}\right)\left(\beta+\frac{k-1}{2}\right)+\frac{(h,k)^2-1}{12}. \tag{3.5}$$

Dedekind's reciprocity law is the following consequence in the case $(h,k)=1$.

**Corollary 3.2.** *Let $h$ and $k$ be positive integers. Then*

$$s(h,k)+s(k,h)=\frac{h}{12k}+\frac{k}{12h}+\frac{(h,k)^2}{12hk}-\frac{1}{4}. \tag{3.6}$$

*Proof.* Write $d=(h,k)$ and $k'=k/d$. For $s'(h,k)=\sum_{r=0}^{k-1}\frac{r}{k}[\frac{hr}{k}]$, we have

$$s(h,k)+s'(h,k)=\sum_{r=0}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\frac{1}{2}\right)+\frac{1}{2}\sum_{\substack{r=0\\ k|hr}}^{k-1}\frac{r}{k}$$

$$=\frac{h}{k^2}\sum_{r=0}^{k-1}r^2-\frac{1}{2k}\sum_{r=0}^{k-1}r+\frac{1}{2k}\sum_{\substack{r=0\\ k'|r}}^{k-1}r=\frac{k-1}{12k}(4hk-2h-3k)+\frac{d-1}{4}.$$

Similarly,

$$s(k,h)+s'(k,h)=\frac{h-1}{12h}(4hk-2k-3h)+\frac{d-1}{4}.$$

Since

$$s'(h,k)+s'(k,h)=B(h,k)=\frac{2}{3}hk-\frac{3(h+k)-1}{4}+\frac{d}{2}+\frac{h^2+k^2-d^2}{12hk},$$

we can obtain (3.6) from the above. □

*Proof of Theorem 1.2.* By (1.13) we have

$$m(h,k,l)=m(\emptyset)(h-1)(k-1)(l-1)+d_1+d_2+d_3-m(h,k,l)$$

where

$$\begin{aligned} d_1 &= -m(h)(k-1)(l-1) + m(k,l)(h-1), \\ d_2 &= -m(k)(h-1)(l-1) + m(h,l)(k-1), \\ d_3 &= -m(l)(h-1)(k-1) + m(h,k)(l-1). \end{aligned}$$

In view of Theorem 3.1, $d_1$ coincides with

$$\begin{aligned} &-\frac{(h-1)(k-1)(l-1)}{2} + (h-1)\left(\frac{(k-1)(l-1)}{4} + \frac{(k^2-1)(l^2-1)+(k,l)^2-1}{12kl}\right) \\ &= (h-1)(k-1)(l-1)\left(\frac{(k+1)(l+1)}{12kl} - \frac{1}{4}\right) + (h-1)\frac{(k,l)^2-1}{12kl} \\ &= (h-1)(k-1)(l-1)\left(\frac{k+l+1}{12kl} - \frac{1}{6}\right) + (h-1)\frac{(k,l)^2-1}{12kl}. \end{aligned}$$

Similarly,

$$d_2 = (h-1)(k-1)(l-1)\left(\frac{h+l+1}{12hl} - \frac{1}{6}\right) + (k-1)\frac{(h,l)^2-1}{12hl}$$

and

$$d_3 = (h-1)(k-1)(l-1)\left(\frac{h+k+1}{12hk} - \frac{1}{6}\right) + (l-1)\frac{(h,k)^2-1}{12hk}.$$

Therefore

$$\begin{aligned} 2m(h,k,l) =&(h-1)(k-1)(l-1) + d_1 + d_2 + d_3 \\ =&(h-1)(k-1)(l-1)\left(\frac{hk+hl+kl}{6hkl} + \frac{1}{2} + \frac{h+k+l}{12hkl}\right) \\ &+ (h-1)\frac{(k,l)^2-1}{12kl} + (k-1)\frac{(h,l)^2-1}{12hl} + (l-1)\frac{(h,k)^2-1}{12hk} \end{aligned}$$

and hence (1.10) follows.

By (1.12), (3.1) and (3.2),

$$\begin{aligned} &m(h,k,l) + M(h,k,l) \\ =&hkl - m(\emptyset) - m(h) - m(k) - m(l) - m(h,k) - m(h,l) - m(k,l) \\ =&hkl - 1 - \frac{h-1}{2} - \frac{k-1}{2} - \frac{l-1}{2} - \frac{hk+hl+kl}{3} + \frac{2h+2k+2l-3}{4} \\ &+ \frac{lh^2+lk^2-l(h,k)^2+kh^2+kl^2-k(h,l)^2+hk^2+hl^2-h(k,l)^2}{12hkl} \\ =&hkl - \frac{hk+hl+kl}{3} - \frac{1}{4} + \frac{h+k}{12l} + \frac{h+l}{12k} + \frac{k+l}{12h} - \frac{(h,k)^2}{12hk} - \frac{(h,l)^2}{12hl} - \frac{(k,l)^2}{12kl}. \end{aligned}$$

This together with (1.10) yields (1.11). □

*Remark* 3.2. Let $h, k, l$ be positive integers. Recall that

$$m_2(h,k,l) + m(h,k,l) + M(h,k,l) = \sum_{t=1}^{3} m_t(h,k,l) = \frac{hkl}{2}\left(3 - \frac{1}{h} - \frac{1}{k} - \frac{1}{l}\right).$$

Using the explicit formula for $m(h,k,l) + M(h,k,l)$, we find that $m_2(h,k,l)$ equals

$$\frac{hkl}{2} - \frac{hk+hl+kl}{6} + \frac{1}{4} - \frac{h+k}{12l} - \frac{h+l}{12k} - \frac{k+l}{12h} + \frac{(h,k)^2}{12hk} + \frac{(h,l)^2}{12hl} + \frac{(k,l)^2}{12kl}.$$

*Proof of Corollary 1.4.* By (1.12),

$$\begin{aligned} & hkln - m(h,k,l,n) - M(h,k,l,n) \\ =& m(\emptyset) + m(h) + m(k) + m(l) + m(n) \\ & + m(h,k) + m(h,l) + m(h,n) + m(k,l) + m(k,n) + m(l,n) \\ & + m(h,k,l) + m(h,k,n) + m(h,l,n) + m(k,l,n). \end{aligned}$$

Thus, we can calculate $m(h,k,l,n) + M(h,k,l,n)$ with helps of Theorems 1.2 and 3.1. The desired result then follows. □

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