GENERALIZATIONS OF KNOPP'S IDENTITY

Beifang Chen and Zhi-Wei Sun

(Communicated by D. Goss)

ABSTRACT. For integers a, b and $n > 0$ we define

$$S_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\!\left(\frac{ar}{n}\right)\!\right) \ln \Gamma\left(\left\{\frac{br}{n}\right\}\right)$$

and

$$T_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\!\left(\frac{ar}{n}\right)\!\right) \frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})},$$

which are similar to the homogeneous Dedekind sum $S(a, b, n)$. In this paper we establish functional equations for S_Γ and T_Γ. Moreover, by means of uniform function (introduced by Sun in 1989) we are able to extend Knopp's identity on Dedekind sums vastly.

1. Introduction

For a real number x, we use $\{x\}$ to denote its fractional part, and define

$$(1.1) \qquad ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Given $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, in 1892 R. Dedekind introduced the classical *Dedekind sum*

$$(1.2) \qquad S(m, n) = \sum_{r=0}^{n-1} \left(\!\left(\frac{r}{n}\right)\!\right) \left(\!\left(\frac{mr}{n}\right)\!\right)$$

in his study of the functional equation of Dedekind's eta function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}) \quad (\tau \text{ is in the upper half plane}).$$

When $m, n \in \mathbb{Z}^+$ are relatively prime, Dedekind determined $S(m,n) + S(n,m)$ explicitly. The result is now known as the reciprocity law for Dedekind sums. (See, e.g. W. S. Anglin [A], Z. W. Sun [S5] and D. Zagier [Z].)

By means of η's functional equation and Hecke operators, in 1980 M. I. Knopp [K] established the following arithmetic identity:

$$(1.3) \qquad \sum_{cd=m} \sum_{r=0}^{d-1} S(ac + rn, dn) = \sigma(m)S(a,n)$$

where $a \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$ and $\sigma(m)$ denotes the sum $\sum_{d \mid m} d$ of (positive) divisors of m. We can view this identity as a functional equation of Dedekind sums. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, the sum

$$(1.4) \qquad S(a, b, n) = \sum_{r=0}^{n-1} \left(\left(\frac{ar}{n} \right) \right) \left(\left(\frac{br}{n} \right) \right)$$

is called a *homogeneous Dedekind sum.* In 1996 Z. Y. Zheng [Zh] proved the following extension of Knopp's identity:

$$(1.5) \qquad \sum_{cd=m} \sum_{r_1, r_2 \in d_*} S(ac + r_1 n, bc + r_2 n, dn) = m\sigma(m)S(a, b, n)$$

where $a, b \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$ and

$$d_* = \{r \in \mathbb{Z} \colon 0 \leqslant r < d\}.$$

In this paper we will make a further generalization.

Definition 1.1. For a function F of two complex variables into the complex field $\mathbb{C}$, if for any ordered pair $\langle x, y \rangle$ in the domain $\mathrm{Dom}(F)$ of F we have

$$(1.6) \qquad \left\{ \left\langle \frac{x+r}{n}, ny \right\rangle \colon r \in n_* \right\} \subseteq \mathrm{Dom}(F) \text{ and } \sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny \right) = F(x, y)$$

for every $n = 1, 2, 3, \ldots$, then we call F a *uniform function* (into $\mathbb{C}$).

The concept of uniform function was first introduced by the second author in [S1] where he showed that, among functions F of two complex variables into $\mathbb{C}$ with $\{\langle \frac{x+r}{n}, ny \rangle \colon r \in n_*\} \subseteq \mathrm{Dom}(F)$ for all $\langle x, y \rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$, uniform functions are those F such that whenever

$$\sum_{\substack{1 \leqslant s \leqslant k \\ x \equiv a_s \pmod{n_s}}} \lambda_s = \sum_{\substack{1 \leqslant t \leqslant l \\ x \equiv b_t \pmod{m_t}}} \mu_t \qquad \text{for all } x \in \mathbb{Z}$$

(where $0 \leqslant a_s < n_s$, $0 \leqslant b_t < m_t$ and $\lambda_s, \mu_t \in \mathbb{C}$) we have

$$(1.7) \qquad \sum_{s=1}^{k} \lambda_s F\left(\frac{x + a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x + b_t}{m_t}, m_t y\right) \quad \text{for all } \langle x, y \rangle \in \mathrm{Dom}(F).$$

See also Theorem 2.1 of [S3] and Corollary 2 of [S4].

A uniform function F is said to be *periodic* if

$$(1.8) \qquad \langle x, y \rangle \in \mathrm{Dom}(F) \;\Rightarrow\; \langle x \pm 1, y \rangle \in \mathrm{Dom}(F) \;\&\; F(x \pm 1, y) = F(x, y).$$

We use PUF to denote the class of all periodic uniform functions. It is easy to see that the function D on $\mathbb{R} \times \mathbb{R}$ given by

$$(1.9) \qquad D(x, y) = ((x))$$

belongs to PUF where $\mathbb{R}$ is the field of real numbers. Generalizing the homogeneous Dedekind sums, we introduce the following definition.

Definition 1.2. Let $F, G \in \mathrm{PUF}$, $\langle x, y \rangle \in \mathrm{Dom}(F)$ and $\langle u, v \rangle \in \mathrm{Dom}(G)$. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we set

$$(1.10) \qquad \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix}(a, b, n) = \sum_{r=0}^{n-1} F\left(\frac{x + ar}{n}, ny\right) G\left(\frac{u + br}{n}, nv\right).$$

Now we give our extension of Knopp's identity.

Theorem 1.1. *Let $a, b \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$, $F, G \in \mathrm{PUF}$, $\langle x, y \rangle \in \mathrm{Dom}(F)$ and $\langle u, v \rangle \in \mathrm{Dom}(G)$. Then we have the identity*

$$(1.11) \qquad \sum_{cd=m} \sum_{r_1, r_2 \in d_*} \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix}(ac + r_1 n, bc + r_2 n, dn) = m \sum_{d \mid m} d \begin{bmatrix} F; x/d, dy \\ G; u/d, dv \end{bmatrix}(a, b, n).$$

Definition 1.3. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we define

$$(1.12) \qquad S_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \ln \Gamma\left(\left\{\frac{br}{n}\right\}\right)$$

and

$$(1.13) \qquad T_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})}$$

where $\Gamma(x)$ is the well-known gamma function.

By applying Theorem 1.1 to certain periodic uniform functions involving $\Gamma(x)$, we can deduce the following result.

Theorem 1.2. *Let $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$.*
  (i) *For the function T_Γ we have the following functional equation:*

$$\text{(1.14)} \qquad \sum_{cd=m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} T_\Gamma(ac + r_1 n, bc + r_2 n, dn) = md(m) T_\Gamma(a, b, n)$$

where $d(m)$ is the number of positive divisors of m.
  (ii) *For S_Γ we have*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} S_\Gamma(ac + r_1 n, bc + r_2 n, dn) - m\sigma(m) S_\Gamma(a, b, n)$$

$$\text{(1.15)}$$

$$= m \sum_{d|m} \Lambda(d) \sigma\left(\frac{m}{d}\right) \left(\frac{S(a, b, n/(d, n))}{d/(d, n)} - S(a, b, n)\right),$$

where (d, n) is the greatest common divisor of d and n, and the Mangoldt function Λ is given by

$$\text{(1.16)} \qquad \Lambda(d) = \begin{cases} \ln p & \text{if } d = p^\alpha \text{ for some prime } p \text{ and } \alpha \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Actually there are lots of examples of periodic uniform functions (see [S2,S3,S4]), so we can apply Theorem 1.1 to obtain many other results.

2. Proof of Theorem 1.1

Lemma 2.1. *Let $a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, $F \in \mathrm{PUF}$ and $\langle x, y \rangle \in \mathrm{Dom}(F)$. Then*

$$\text{(2.1)} \qquad \sum_{r=0}^{n-1} F\left(\frac{x + ar}{n}, ny\right) = (a, n) F\left(\frac{x}{(a, n)}, (a, n)y\right).$$

Proof. Let $d = (a, n)$. Then $a' = a/d$ is relatively prime to $n' = n/d$. Each $r \in n_*$ can be written uniquely in the form $sn' + t$ where $s \in d_*$ and $t \in n'_*$. Thus

$$\sum_{r=0}^{n-1} F\left(\frac{x + ar}{n}, ny\right) = \sum_{s=0}^{d-1} \sum_{t=0}^{n'-1} F\left(\frac{x}{n} + \frac{a'}{n'}(sn' + t), ny\right)$$

$$= d \sum_{t=0}^{n'-1} F\left(\frac{x}{n} + \left\{\frac{a't}{n'}\right\}, ny\right) = d \sum_{r=0}^{n'-1} F\left(\frac{x}{dn'} + \frac{r}{n'}, n'(dy)\right) = dF\left(\frac{x}{d}, dy\right).$$

We are done. $\square$

Proof of Theorem 1.1. Suppose that $c, d \in \mathbb{Z}^+$ and $cd = m$. Then

$$W(c, d) := \sum_{r_1, r_2 \in d_*} \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix} (ac + r_1 n, bc + r_2 n, dn)$$

$$= \sum_{r=0}^{dn-1} \sum_{r_1, r_2 \in d_*} F\left(\frac{x + (ac + r_1 n)r}{dn}, dny\right) G\left(\frac{u + (bc + r_2 n)r}{dn}, dnv\right)$$

$$= \sum_{r=0}^{dn-1} \sum_{r_1=0}^{d-1} F\left(\frac{(x + acr)/n + rr_1}{d}, dny\right) \sum_{r_2=0}^{d-1} F\left(\frac{(u + bcr)/n + rr_2}{d}, dnv\right)$$

$$= \sum_{r=0}^{dn-1} (d, r)^2 F\left(\frac{(x + acr)/n}{(d, r)}, (d, r)ny\right) G\left(\frac{(u + bcr)/n}{(d, r)}, (d, r)nv\right)$$

where we apply Lemma 2.1 in the last step. For the Möbius function μ, it is well known that $\sum_{r|k} \mu(r)$ equals 1 for $k = 1$, and 0 for $k = 2, 3, \ldots$. Thus

$$W(c, d) = \sum_{t|d} t^2 \sum_{\substack{0 \leqslant s < nd/t \\ (d, st) = t}} F\left(\frac{x + acst}{nt}, nty\right) G\left(\frac{u + bcst}{nt}, ntv\right)$$

$$= \sum_{t|d} t^2 \sum_{0 \leqslant s < nd/t} \sum_{r|(s, d/t)} \mu(r) F\left(\frac{x + acst}{nt}, nty\right) G\left(\frac{u + bcst}{nt}, ntv\right)$$

$$= \sum_{t|d} t^2 \sum_{r|\frac{d}{t}} \mu(r) \sum_{0 \leqslant s' < nd/(rt)} F\left(\frac{x + acrs't}{nt}, nty\right) G\left(\frac{u + bcrs't}{nt}, ntv\right)$$

and hence

$$W(c, d) = \sum_{rt|d} \mu(r) t^2 \sum_{k=0}^{\frac{d}{rt}-1} \sum_{l=0}^{n-1} F\left(\frac{x + acrt(kn + l)}{nt}, nty\right) G\left(\frac{u + bcrt(kn + l)}{nt}, ntv\right)$$

$$= \sum_{rt|d} \mu(r) t^2 \frac{d}{rt} \sum_{l=0}^{n-1} F\left(\frac{x/t + acrl}{n}, n(ty)\right) G\left(\frac{u/t + bcrl}{n}, n(tv)\right)$$

$$= \sum_{rst=d} \mu(r) st^2 \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} (acr, bcr, n).$$

In view of the above, we have

$$\sum_{cd=m} W(c, d) = \sum_{rst|m} \mu(r) st^2 \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} \left(ar\frac{m}{rst}, br\frac{m}{rst}, n\right)$$

$$= \sum_{d|m} d \sum_{st=d} t \sum_{r|\frac{m}{d}} \mu(r) \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} \left(a\frac{m}{d}, b\frac{m}{d}, n\right)$$

$$= m \sum_{st=m} t \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} (a, b, n) = m \sum_{d|m} d \begin{bmatrix} F; x/d, dy \\ G; u/d, dv \end{bmatrix} (a, b, n).$$

This concludes the proof.   □

3. Proof of Theorem 1.2

For a uniform function F with $\mathrm{Dom}(F) \subseteq \mathbb{R} \times \mathbb{R}$, the function $\tilde{F}(x, y) = F(\{x\}, y)$ obviously lies in PUF.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Define $\Gamma^* : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$(3.1) \qquad \Gamma^*(x, y) = \begin{cases} \Gamma(x)y^x / \sqrt{2\pi y} & \text{if } x > 0, \\ \sqrt{2\pi y} & \text{if } x = 0. \end{cases}$$

By Example 2.2 of Sun [S2], the function $\ln \Gamma^*(x, y)$ is a uniform function. The reader can verify this by means of the following famous multiplication formula of Gauss (cf. [E]):

$$\prod_{r=0}^{n-1} \Gamma\left(z + \frac{r}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2 - nz} \Gamma(nz) \quad (n \in \mathbb{Z}^+ \text{ and } nz \neq 0, -1, \dots).$$

So the function $\Gamma_*(x, y) = \ln \Gamma^*(\{x\}, y)$ belongs to PUF.

Let $f(0) = -\gamma$ (where γ is the Euler constant $0.577\dots$), and $f(x) = \Gamma'(x)/\Gamma(x)$ for $x > 0$. By sections 1.7 and 1.7.1 of [E], we have

$$f(x) = \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{x+r}{n}\right) + \ln n = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)}$$

for any $x \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. Observe that

$$f(x) - f(0) - \frac{1}{n}\left(f\left(\frac{x}{n}\right) - f\left(\frac{0}{n}\right)\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x}\right) - \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x/n}\right)$$

$$= 1 - \frac{1}{n} + \sum_{k=1}^{\infty}\left(\left(1 - \frac{1}{n}\right)\frac{1}{k+1} + \frac{1}{kn+x} - \frac{1}{k+x}\right)$$

tends to zero as $x \to 0$. So $f(x) = \frac{1}{n}\sum_{r=0}^{n-1} f(\frac{x+r}{n}) + \ln n$ for all $x \geqslant 0$. For $x \geqslant 0$ and $y > 0$ let

$$(3.2) \qquad \Psi(x, y) = \begin{cases} (\Gamma'(x)/\Gamma(x) + \ln y)/y & \text{if } x > 0, \\ (-\gamma + \ln y)/y & \text{if } x = 0. \end{cases}$$

Then Ψ is a uniform function. (In fact, $\Psi(x, y) + \gamma/y$ is just the uniform function $G(x, y)$ given in Example 2.3 of [S2].) Thus the function $\psi(x, y) = \tilde{\Psi}(x, y)$ also lies in PUF.

Lemma 3.1. *Let $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$. Then*

(3.3)
$$\begin{bmatrix} D; 0, m \\ \Gamma_*; 0, m \end{bmatrix}(a, b, n) = S_\Gamma(a, b, n) + \ln(mn)S(a, b, n)$$

and

(3.4)
$$\begin{bmatrix} D; 0, m \\ \psi; 0, m \end{bmatrix}(a, b, n) = \frac{T_\Gamma(a, b, n)}{mn}.$$

Proof. We first claim that both

$$R_1 = \sum_{\substack{r \in n_* \\ n | br}} \left(\!\left(\frac{ar}{n}\right)\!\right) \quad \text{and} \quad R_2 = \sum_{\substack{r \in n_* \\ n \nmid br}} \left(\!\left(\frac{ar}{n}\right)\!\right)$$

vanish. In fact, by Lemma 2.1 we have

$$\sum_{r=0}^{k-1} \left(\!\left(\frac{ar}{k}\right)\!\right) = (a, k)((0)) = 0 \quad \text{for every } k = 1, 2, 3, \ldots.$$

So $R_1 + R_2 = 0$. Let $d = (b, n)$, $b' = b/d$ and $n' = n/d$. Then

$$R_1 = \sum_{\substack{0 \leqslant r < n \\ n' | b'r}} \left(\!\left(\frac{ar}{n}\right)\!\right) = \sum_{r'=0}^{d-1} \left(\!\left(\frac{ar'n'}{n}\right)\!\right) = \sum_{r'=0}^{d-1} \left(\!\left(\frac{ar'}{d}\right)\!\right) = 0.$$

In view of the above,

$$\begin{bmatrix} D; 0, m \\ \Gamma_*; 0, m \end{bmatrix}(a, b, n) = \sum_{r=0}^{n-1} \left(\!\left(\frac{ar}{n}\right)\!\right) \Gamma_* \left(\frac{br}{n}, nm\right)$$

$$= \sum_{\substack{r \in n_* \\ n \nmid br}} \left(\!\left(\frac{ar}{n}\right)\!\right) \left(\ln \Gamma \left(\left\{\frac{br}{n}\right\}\right) + \left(\left\{\frac{br}{n}\right\} - \frac{1}{2}\right) \ln(mn) - \frac{\ln(2\pi)}{2} \right)$$

$$+ \sum_{\substack{r \in n_* \\ n | br}} \left(\!\left(\frac{ar}{n}\right)\!\right) \frac{\ln(2\pi mn)}{2}$$

$$= S_\Gamma(a, b, n) + \ln(mn)S(a, b, n)$$

and

$$\begin{bmatrix} D;0,m \\ \psi;0,m \end{bmatrix}(a,b,n) = \sum_{r=0}^{n-1} \left(\!\left(\frac{ar}{n}\right)\!\right) \psi\left(\frac{br}{n},nm\right)$$

$$= \sum_{\substack{r\in n_* \\ n\nmid br}} \left(\!\left(\frac{ar}{n}\right)\!\right) \frac{1}{mn} \left(\frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})} + \ln(mn)\right)$$

$$+ \sum_{\substack{r\in n_* \\ n\mid br}} \left(\!\left(\frac{ar}{n}\right)\!\right) \frac{1}{mn}(-\gamma + \ln(mn))$$

$$= \frac{T_\Gamma(a,b,n)}{mn}.$$

We are done.   □

Proof of Theorem 1.2. i) By Theorem 1.1,

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*} \begin{bmatrix} D;0,1 \\ \psi;0,1 \end{bmatrix}(ac+r_1n, bc+r_2n, dn) = m\sum_{d\mid m} d\begin{bmatrix} D;0,d \\ \psi;0,d \end{bmatrix}(a,b,n).$$

In light of Lemma 3.1, this says that

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*} \frac{1}{dn}T_\Gamma(ac+r_1n, bc+r_2n, dn) = m\sum_{d\mid m} d\frac{T_\Gamma(a,b,n)}{dn},$$

which is equivalent to (1.14).

ii) If $c,d\in\mathbb{Z}^+$ and $cd=m$, then by Lemma 3.1 we have

$$\sum_{r_1,r_2\in d_*} \begin{bmatrix} D;0,1 \\ \Gamma_*;0,1 \end{bmatrix}(ac+r_1n, bc+r_2n, dn)$$

$$= \sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn)$$

$$+ \ln(dn)\sum_{r_1,r_2\in d_*} S(ac+r_1n, bc+r_2n, dn).$$

This, together with (1.5), yields that

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*} \begin{bmatrix} D;0,1 \\ \Gamma_*;0,1 \end{bmatrix}(ac+r_1n, bc+r_2n, dn)$$

$$- \sum_{cd=m}\sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn)$$

$$= \sum_{cd=m}(\ln d + \ln n)\sum_{r_1,r_2\in d_*} S(ac+r_1n, bc+r_2n, dn)$$

$$= (\ln n)m\sigma(m)S(a,b,n) + \Sigma$$

where

$$\Sigma = \sum_{cd=m} \ln d \sum_{r_1,r_2 \in d_*} \begin{bmatrix} D; 0, 1 \\ D; 0, 1 \end{bmatrix} (ac + r_1 n, bc + r_2 n, dn).$$

It is well known that $\sum_{d|k} \Lambda(d) = \ln k$ for $k \in \mathbb{Z}^+$. (See, e.g. [A].) By Möbius' theorem, this implies that

$$\Lambda(k) = \sum_{d|k} \mu(d) \ln \frac{k}{d} = -\sum_{d|k} \mu(d) \ln d \quad \text{for any } k \in \mathbb{Z}^+.$$

By the proof of Theorem 1.1,

$$\Sigma = \sum_{rst|m} \ln(rst)\mu(r)st^2 \begin{bmatrix} D; 0, t \\ D; 0, t \end{bmatrix} \left(ar\frac{m}{rst}, br\frac{m}{rst}, n \right)$$

$$= \sum_{d|m} d \sum_{st=d} t \sum_{r|\frac{m}{d}} \mu(r) \ln(rd) \begin{bmatrix} D; 0, t \\ D; 0, t \end{bmatrix} \left(a\frac{m}{d}, b\frac{m}{d}, n \right)$$

and hence

$$\Sigma = \sum_{d|m} d \left(\sum_{r|\frac{m}{d}} \mu(r) \ln d + \sum_{r|\frac{m}{d}} \mu(r) \ln r \right) \sum_{t|d} tS \left(a\frac{m}{d}, b\frac{m}{d}, n \right)$$

$$= \sum_{d=m} d(\ln d)\sigma(d)S \left(a\frac{m}{d}, b\frac{m}{d}, n \right) - \sum_{d|m} d\Lambda \left(\frac{m}{d} \right) \sigma(d)S \left(a\frac{m}{d}, b\frac{m}{b}, n \right)$$

$$= m\sigma(m)(\ln m)S(a,b,n) - \sum_{d|m} d\sigma(d)\Lambda \left(\frac{m}{d} \right) S \left(a\frac{m}{d}, b\frac{m}{d}, n \right).$$

In view of Lemma 3.1,

$$\sum_{d|m} d \begin{bmatrix} D; 0/d, d \\ \Gamma_*; 0/d, d \end{bmatrix} (a,b,n) = \sum_{d|m} d \left(S_\Gamma(a,b,n) + \ln(dn)S(a,b,n) \right)$$

$$= \sigma(m)S_\Gamma(a,b,n) + S(a,b,n) \left(\sigma(m) \ln n + \sum_{d|m} d \ln d \right).$$

Thus, by Theorem 1.1 and the above,

$$m\sigma(m)S_\Gamma(a,b,n) + mS(a,b,n) \left(\sigma(m) \ln n + \sum_{d|m} d \ln d \right)$$

$$- \sum_{cd=m} \sum_{r_1,r_2 \in d_*} S_\Gamma(ac + r_1 n, bc + r_2 n, dn)$$

$$= m\sigma(m)(\ln n + \ln m)S(a,b,n) - \sum_{d|m} d\sigma(d)\Lambda \left(\frac{m}{d} \right) S \left(a\frac{m}{d}, b\frac{m}{d}, n \right).$$

Therefore

$$m\sigma(m)S_\Gamma(a,b,n) - \sum_{cd=m}\sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn)$$

$$=m\left(\sum_{d|m} d\ln m - \sum_{d|m} d\ln d\right)S(a,b,n) - \sum_{d|m}\frac{m}{d}\sigma\left(\frac{m}{d}\right)\Lambda(d)S(ad,bd,n)$$

$$=m\sum_{d|m} d\ln\frac{m}{d}S(a,b,n) - m\sum_{d|m}\Lambda(d)\frac{S(ad,bd,n)}{d}\sigma\left(\frac{m}{d}\right).$$

$$=m\sum_{d|m}\Lambda(d)\sigma\left(\frac{m}{d}\right)\left(S(a,b,n) - \frac{S(ad,bd,n)}{d}\right)$$

where in the last step we note that

$$\sum_{d|m} d\ln\frac{m}{d} = \sum_{d|m} d\sum_{t|\frac{m}{d}}\Lambda(t) = \sum_{t|m}\Lambda(t)\sum_{d|\frac{m}{t}} d = \sum_{t|m}\Lambda(t)\sigma\left(\frac{m}{t}\right).$$

Fix $d\in\mathbb{Z}^+$ and let $d' = d/(d,n)$ and $n' = n/(d,n)$. Evidently,

$$\frac{S(ad,bd,n)}{(d,n)} = \frac{1}{(d,n)}\sum_{s=0}^{(d,n)-1}\sum_{t=0}^{n'-1}\left(\!\left(\frac{ad'(sn'+t)}{n'}\right)\!\right)\left(\!\left(\frac{bd'(sn'+t)}{n'}\right)\!\right)$$

$$=\sum_{t=0}^{n'-1}\left(\!\left(\frac{ad't}{n'}\right)\!\right)\left(\!\left(\frac{bd't}{n'}\right)\!\right) = \sum_{r=0}^{n'-1}\left(\!\left(\frac{ar}{n'}\right)\!\right)\left(\!\left(\frac{br}{n'}\right)\!\right) = S(a,b,n').$$

By the above we finally get (1.15). This concludes the proof. $\square$

Acknowledgment. The authors are grateful to the referee for his/her comments.

REFERENCES

[A]   W. S. Anglin, *The Queen of Mathematics—An Introduction to Number Theory*, Kluwer, Dordrecht, 1995, pp. 336–349.
[E]   Erdélyi et al., *Higher Transcendental Functions*, vol. I, McGraw-Hill, 1953, Ch. 1.
[K]   M. I. Knopp, *Hecke operators and an identity for Dedekind sums*, J. Number Theory **12** (1980), 2–9.
[S1] Z. W. Sun, *Systems of congruences with multipliers*, Nanjing Univ. J. Math. Biquarterly **6** (1989), no. 1, 124–133. MR 90m:11006; Zbl. M. 703.11002.
[S2] Z. W. Sun, *On covering equivalence*, in: "Analytic Number Theory" (Beijing/Kyoto, 1999), 277–302, Dev. Math., 6, Kluwer Acad. Publ., Dordrecht, 2002.
[S3] Z. W. Sun, *Products of binomial coefficients modulo p^2*, Acta Arith. **97** (2001), 87–98.
[S4] Z. W. Sun, *Algebraic approaches to periodic arithmetical maps*, J. Algebra **240** (2001), 723–743.
[S5] Z. W. Sun, *Sums of minima and maxima*, Discrete Math. **257** (2002), 143–159.
[Z]   D. Zagier, *Higher dimensional Dedekind sums*, Math. Ann. **202** (1973), 149–172.

[Zh] Z. Y. Zheng, *Th Petersson–Knopp identity for the homogeneous Dedekind sums*, J. Number Theory **57** (1996), 223–230.

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG. *E-mail:* `mabfchen@ust.hk`

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, THE PEOPLE'S REPUBLIC OF CHINA. *E-mail:* `zwsun@nju.edu.cn`