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On the Function $w(x) = |\{1 \le s \le k\colon x \equiv a_s \pmod{n_s}\}|$

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Abstract

For a finite system $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ of arithmetic sequences the covering function is $w(x) = |\{1 \leqslant s \leqslant k : x \equiv a_s \pmod{n_s}\}|$. Using equalities involving roots of unity we characterize those systems with a fixed covering function $w(x)$. From the characterization we reveal some connections between a period n_0 of $w(x)$ and the moduli $n_1, \cdots, n_k$ in such a system A. Here are three central results: (a) For each $r = 0, 1, \cdots, n_k/(n_0, n_k)-1$ there exists a $J \subseteq \{1, \cdots, k-1\}$ such that $\sum_{s\in J} 1/n_s = r/n_k$. (b) If $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$ $(0 < l < k)$, then for any positive integer $r < n_k/n_{k-l}$ with $r \not\equiv 0 \pmod{n_k/(n_0, n_k)}$, the binomial coefficient $\binom{l}{r}$ can be written as the sum of some (not necessarily distinct) prime divisors of n_k. (c) $\max_{x\in\mathbb{Z}} w(x)$ can be written in the form $\sum_{s=1}^k m_s/n_s$ where $m_1, \cdots, m_k$ are positive integers.

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \cdots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we put

$$a(n) = a + n\mathbb{Z} = \{\cdots, a - 2n, a - n, a, a + n, a + 2n, \cdots\}$$

and call it an arithmetic sequence with modulus n. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1}$$

of such sequences we define its *covering function* $w_A : \mathbb{Z} \to \mathbb{N}$ by

$$w_A(x) = |\{1 \leqslant s \leqslant k : \ x \in a_s(n_s)\}|. \tag{2}$$

(Cf. [7].) The function $w_A(x)$ is periodic modulo the least common multiple $[n_1, \cdots, n_k]$ of $n_1, \cdots, n_k$, also the smallest positive period of $w_A(x)$ divides any period (including $[n_1, \cdots, n_k]$) of $w_A(x)$. For a positive integer m, if $w_A(x) \geqslant m$ (resp. $w_A(x) = m$) for all $x \in \mathbb{Z}$ then we say that (1) forms an *m-cover* (resp. *exact m-cover*) (of $\mathbb{Z}$). It is known that $\sum_{s=1}^k 1/n_s \geqslant m$ if (1) is an m-cover, and $\sum_{s=1}^k 1/n_s = m$ if (1) is an exact m-cover. (See, e.g. [13].) Usually one uses *cover* instead of 1-cover, and *disjoint covering system* instead of exact 1-cover. By [18], for each $m = 2, 3, 4, \cdots$ there are exact m-covers no subcover of which is an exact n-cover with $0 < n < m$. When the function $w_A(x)$ has a period n_0 (relatively) prime to some modulus n_t $(1 \leqslant t \leqslant k)$, for each $x \in \mathbb{Z}$ there is a $q \in \mathbb{Z}$ such that $a_t + n_t q \equiv x \pmod{n_0}$ and hence $w_A(x) = w_A(a_t + n_t q) > 0$, thus (1) forms a cover of $\mathbb{Z}$.

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Since P. Erdös ([3]) introduced the notion of cover in 1930's, covers and disjoint covering systems have been widely investigated by various mathematicians. (Cf. R. K. Guy [5], and Š. Porubský and J. Schönheim [9].) Two recent applications can be found in [4] and [17]. The most difficult problem in this area is to describe those moduli $n_1,\cdots,n_k$ in a general m-cover (or exact m-cover) (1).

Now we introduce some notations. For $n\in\mathbb{Z}^+$ we set

$$R(n)=\{x\in\mathbb{Z}:\ 0\leqslant x<n\},\tag{3}$$

and put

$$D(n)=\Big\{\sum_{p\mid n}px_p:\ x_p\in\mathbb{N}\text{ for any prime divisor }p\text{ of }n\Big\},\tag{4}$$

in other words $D(n)$ is the additive monoid generated by all the prime factors of n. For $a,b\in\mathbb{Z}$ by (a,b) we mean the greatest common divisor of a and b. For a complex number z, as usual we let

$$\binom{z}{n}=\frac{1}{n!}\prod_{0\leqslant j<n}(z-j)\quad\text{for }n=0,1,2,\cdots.$$

(An empty product is taken to be the multiplicative identity 1.) For any α in the field $\mathbb{R}$ of real numbers, we use $[\alpha]$ to denote the greatest integer not exceeding α, also we let $\alpha\mathbb{Z}=\{\alpha x\colon x\in\mathbb{Z}\}$ and $e(\alpha)=e^{2\pi i\alpha}$.

In this paper we fix system (1) and set $[1,k]=\{1,\cdots,k\}$. For $I\subseteq[1,k]$ we put $\bar{I}=[1,k]\setminus I$, and let $[n_s]_{s\in I}$ denote the least common multiple of those n_s with $s\in I$, which is regarded as 1 if $I=\emptyset$.

Let

$$S(A)=\Big\{\sum_{s\in J}\frac{1}{n_s}:J\subseteq[1,k]\Big\}.\tag{5}$$

If system A forms an exact m-cover or an m-cover, what can we say about the set $S(A)$? This question is very challenging and somewhat mysterious, though the author has made a series of investigations (see [12], [13], [14], [15] and [16]). In this paper we aim to tell something about $S(A)$ and the moduli $n_1,\cdots,n_k$ according to a period of the function $w_A(x)$.

Now we introduce our main theorems whose proofs will be given later.

Theorem 1.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$.*

(i) *For $I\subseteq\{1\leqslant s\leqslant k:\ n_s=n\}$ where $n\in\mathbb{Z}^+$, we have*

$$\Big\{\sum_{s\in J}\frac{1}{n_s}:\ J\subseteq\bar{I}\Big\}\supseteq\Big\{\frac{r}{n}:\ r\in R\Big(\frac{n}{(n_0,n)}\Big)\ \&\ \binom{|I|+r-1}{r}\notin D(n)\Big\}.\tag{6}$$

(ii) *If $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$, then for each positive integer $r<n_k/n_{k-l}$ either $r\equiv 0\pmod{n_k/(n_0,n_k)}$ or $\binom{l}{r}\in D(n_k)$.*

In the case $|I|=1$, Theorem 1.1 (i) yields result (a) described in the abstract.

Remark 1.1. Let (1) be an exact m-cover with $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$. In the case $m=1$, a result due to H. Davenport, L. Mirsky, D. Newman and R. Radó asserts that $l>1$, in 1971 M. Newman [6] confirmed a conjecture of Š. Znám by proving that $l \geqslant p$ where p is the smallest prime divisor of n_k. These can also be extended to exact m-covers. (See Porubský [8].) In 1991 the author [11] obtained a general result which implies that $l \geqslant \min_{1\leqslant s\leqslant k-l} n_k/(n_s,n_k)$, in 1995 Y. G. Chen and Porubský [2] showed that

$$l=\sum_{s=1}^{k-l}\frac{n_k}{(n_k,n_s)}x_s \qquad \text{for some } x_1,\cdots,x_{k-l}\in\mathbb{N}.$$

(By the way, it should be mentioned that the key equality (3) in [2] first appeared in [11].) It follows that $l \in D(n_k)$. By Theorem 1.1 (ii), actually $\{\binom{l}{r}\colon r\in\mathbb{Z}^+ \ \&\ r<n_k/n_{k-l}\}\subseteq D(n_k)$.

Theorem 1.2. *Let I be a subset of $[1,k]$ with $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$. Then*

$$\left\{\frac{a}{(n_0,[n_s]_{s\in I})}+\sum_{s\in J}\frac{1}{n_s}:\ a\in\mathbb{Z}\ \&\ J\subseteq\bar{I}\right\}\supseteq\frac{1}{[n_s]_{s\in I}}\mathbb{Z} \tag{7}$$

where $n_0\in\mathbb{Z}^+$ is an arbitrary period of $w_A(x)$.

Remark 1.2. Theorem 1.2 in the case $w_A(x)\equiv m$ was obtained by the author [15] in 1997.

Theorem 1.3. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function with period $n_0\in\mathbb{Z}^+$, and $\{a_s(n_s)\}_{s\in I}$ a (finite) nonempty system of arithmetic sequences with $|\{s\in I: x\in a_s(n_s)\}|\leqslant w(x)$ for all $x\in\mathbb{Z}$. Then there are $a_0\in R(n_0)$ and $m_s\in\mathbb{Z}^+$ for $s\in I$ such that*

$$a_0+n_0\sum_{s\in I}\frac{m_s}{n_s}=w(1)+\cdots+w(n_0), \tag{8}$$

thus

$$\sum_{s\in I}\frac{m_s}{n_s}=\left[\frac{w(1)+\cdots+w(n_0)}{n_0}\right] \tag{9}$$

providing all the n_s with $s\in I$ are prime to n_0.

Remark 1.3. By Theorem 1.3, $M(A)=\max_{x\in\mathbb{Z}}w_A(x)$ can be written in the form $\sum_{s=1}^{k}m_s/n_s$ with $m_1,\cdots,m_k\in\mathbb{Z}^+$.

In the next section we will characterize those systems with a fixed covering function, in Section 3 we mainly show Theorems 1.1–1.3.

Let us end this section with the following conjecture.

Conjecture. *Suppose that (1) forms a cover but none of its proper subsystems does. Then we have $|S(A)|\leqslant n_1+\cdots+n_k$, also $S(A)\supseteq\{r/d: r\in R(d)\}$ whenever $1/d\in S(A)$.*

2. Systems with a given covering function

In this section we employ roots of unity for our purposes.

Let $I\subseteq[1,k]$. For any c in the rational field $\mathbb{Q}$, we set

$$I^*[c]=\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I}x_s/n_s=c}} e\left(\sum_{s\in I}\frac{a_sx_s}{n_s}\right) \tag{10}$$

and

$$\bar{I}_*[c]=\sum_{\substack{J\subseteq\bar{I}\\ \sum_{s\in J}1/n_s=c}}(-1)^{|J|}e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big). \tag{11}$$

(For convenience we think that the equation $\sum_{s\in\emptyset}x_s/n_s=0$ over $\mathbb{N}$ only has the zero solution and so $\emptyset^*[0]=\emptyset_*[0]=1$.)

Lemma 2.1. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function with period $n_0\in\mathbb{Z}^+$. Put $N=[n_0,n_1,\cdots,n_k]$. Then (1) has covering function w if and only if we have the identity*

$$\prod_{s=1}^{k}\Big(1-z^{N/n_s}e^{2\pi i a_s/n_s}\Big)=\prod_{t=1}^{n_0}\Big(1-z^{N/n_0}e^{2\pi i t/n_0}\Big)^{w(t)} \tag{12}$$

Proof. Notice that all zeroes of either side are Nth roots of unity. For each $x\in\mathbb{Z}$, $e^{-2\pi i x/N}$ is a zero of multiplicity $w_A(x)$ of the left hand side, and a zero of multiplicity $w(x)$ of the right hand side. So, by Viète's theorem, identity (12) holds if and only if $w_A=w$. □

Theorem 2.1. *Let $w:\mathbb{Z}\to\mathbb{N}$ be a function having period $n_0\in\mathbb{Z}^+$. Let $I\subseteq[1,k]$ and $v\in\mathbb{Z}$. Then w is the covering function of (1) if and only if*

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*[c-n]=\sum_{a\geqslant 0}(-1)^a I_-[a]I^*\Big[c-\frac{a}{n_0}\Big] \tag{13}$$

holds for all $c\in\mathbb{Q}$, where

$$I_-[a]=\sum_{\substack{v_1,\cdots,v_{n_0}\in\mathbb{N}\\ v_1+\cdots+v_{n_0}=a}}\prod_{t=1}^{n_0}\binom{v+w(t)-|I|}{v_t}e\Big(\frac{\sum_{0<t<n_0}tv_t}{n_0}\Big). \tag{14}$$

Remark 2.1. Actually only finite sums are involved in (13).

Proof. Let $N=[n_0,n_1,\cdots,n_k]$. For $r\in\mathbb{R}$ with $|r|<1$, let

$$f_1(r)=(1-r^N)^v\prod_{s\in\bar{I}}\Big(1-r^{N/n_s}e^{2\pi i a_s/n_s}\Big)$$

and

$$f_2(r)=(1-r^N)^{v-|I|}\prod_{s\in I}\frac{1-r^N}{1-r^{N/n_s}e^{2\pi i a_s/n_s}}\times\prod_{t=1}^{n_0}\Big(1-r^{N/n_0}e^{2\pi i t/n_0}\Big)^{w(t)}.$$

By Lemma 2.1, w is the covering function of (1) if and only if for all $-1<r<1$ we have

$$\prod_{s=1}^{k}\Big(1-r^{N/n_s}e^{2\pi i a_s/n_s}\Big)=\prod_{t=1}^{n_0}\Big(1-r^{N/n_0}e^{2\pi i t/n_0}\Big)^{w(t)},\ \text{i.e. } f_1(r)=f_2(r).$$

Observe that

$$\begin{aligned}f_1(r)&=\sum_{n=0}^{\infty}(-1)^n\binom{v}{n}r^{nN}\sum_{J\subseteq\bar{I}}(-1)^{|J|}e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big)r^{\sum_{s\in J}N/n_s}\\&=\sum_{h=0}^{\infty}r^h\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\sum_{\substack{J\subseteq\bar{I}\\ n+\sum_{s\in J}1/n_s=h/N}}(-1)^{|J|}e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big)\\&=\sum_{h=0}^{\infty}r^h\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*\Big[\frac{h}{N}-n\Big].\end{aligned}$$

Since $1-r^N=\prod_{t=1}^{n_0}(1-r^{N/n_0}e^{2\pi it/n_0})$, we also have

$$\begin{aligned}f_2(r)&=\prod_{s\in I}\Big(\sum_{x_s=0}^{n_s-1}r^{Nx_s/n_s}e^{2\pi ia_sx_s/n_s}\Big)\times\prod_{t=1}^{n_0}\Big(1-r^{N/n_0}e^{2\pi it/n_0}\Big)^{w(t)+v-|I|}\\&=\sum_{x_s\in R(n_s)\text{ for }s\in I}e\Big(\sum_{s\in I}\frac{a_sx_s}{n_s}\Big)r^{N\sum_{s\in I}x_s/n_s}\\&\quad\times\prod_{t=1}^{n_0}\sum_{v_t\geqslant 0}(-1)^{v_t}\binom{w(t)+v-|I|}{v_t}e\Big(\frac{tv_t}{n_0}\Big)r^{Nv_t/n_0},\end{aligned}$$

that is,

$$\begin{aligned}f_2(r)&=\sum_{h=0}^{\infty}r^h\sum_{a\geqslant 0}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \sum_{s\in I}x_s/n_s=h/N-a/n_0}}e\Big(\sum_{s\in I}\frac{a_sx_s}{n_s}\Big)\\&\quad\times\sum_{\substack{v_1,\cdots,v_{n_0}\in\mathbb{N}\\ v_1+\cdots+v_{n_0}=a}}(-1)^{\sum_{t=1}^{n_0}v_t}\prod_{t=1}^{n_0}\binom{v+w(t)-|I|}{v_t}e\Big(\frac{\sum_{t=1}^{n_0}tv_t}{n_0}\Big)\\&=\sum_{h=0}^{\infty}r^h\sum_{a\geqslant 0}(-1)^aI_-[a]I^*\left[\frac{h}{N}-\frac{a}{n_0}\right].\end{aligned}$$

Thus, for (1) to have covering function w, it is necessary and sufficient that

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*\left[\frac{h}{N}-n\right]=\sum_{a\geqslant 0}(-1)^aI_-[a]I^*\left[\frac{h}{N}-\frac{a}{n_0}\right]$$

for every $h\in\mathbb{N}$. Clearly both sides of (13) vanish if $cN\notin\mathbb{N}$, so the desired result follows. □

Corollary 2.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$. Let I be a subset of $[1,k]$ and c be a rational with denominator d. Suppose that $I^*[c-n/d_0]=0$ for every positive integer n where $d_0=(n_0,[d,[n_s]_{s\in I}])$. Then $\bar{I}_*[c]=I^*[c]$.*

Proof. Assume that a is a positive integer with $I^*[c-a/n_0]\neq 0$. Then $\sum_{s\in I}x_s/n_s=c-a/n_0$ for some $x_s\in R(n_s)$ $(s\in I)$, and hence $\frac{a}{n_0}[d,[n_s]_{s\in I}]\in\mathbb{Z}$. Since d_0 can be written as $n_0x+[d,[n_s]_{s\in I}]y$ with $x,y\in\mathbb{Z}$, we have $a/n_0=n/d_0$ for some $n\in\mathbb{Z}^+$. This contradicts our supposition.

In view of the above and Theorem 2.1,

$$\bar{I}_*[c]=\sum_{n\geqslant 0}(-1)^n\binom{0}{n}\bar{I}_*[c-n]=(-1)^0I^*\left[c-\frac{0}{n_0}\right]=I^*[c].$$

This ends the proof. □

Corollary 2.2. *Let $I\subseteq[1,k]$, $m\in\mathbb{Z}^+$ and $v\in\mathbb{Z}$. Then (1) forms an exact m-cover if and only if for all $c\in\mathbb{Q}$ we have*

$$\sum_{n\geqslant 0}(-1)^n\binom{v}{n}\bar{I}_*[c-n]=\sum_{n\geqslant 0}(-1)^n\binom{v+m-|I|}{n}I^*[c-n].\tag{15}$$

Proof. The constant function $w:\mathbb{Z}\to\mathbb{N}$ given by $w(x)\equiv m$ has period $n_0=1$. For $a\in\mathbb{N}$ clearly $I_-[a]=\binom{v+m-|I|}{a}$. So the desired result follows from Theorem 2.1. □

Remark 2.2. In the cases $v=0$ and $v=|I|-m$, (15) turns out to be

$$\sum_{n\geqslant 0}(-1)^n\binom{m-|I|}{n}I^*[c-n]=\bar{I}_*[c] \tag{16}$$

and

$$\sum_{n\geqslant 0}(-1)^n\binom{|I|-m}{n}\bar{I}_*[c-n]=I^*[c] \tag{17}$$

respectively.

3. Proof of Theorems 1.1–1.3

For $I\subseteq[1,k]$ and $c\in\mathbb{Q}$ we set

$$I^*(c)=\left|\left\{\text{vector } \langle x_s\rangle_{s\in I}:\ x_s\in R(n_s)\text{ for } s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c\right\}\right| \tag{18}$$

and

$$\bar{I}_*(c)=\left|\left\{J\subseteq\bar{I}:\ \sum_{s\in J}\frac{1}{n_s}=c\right\}\right|=\left|\left\{\langle\delta_s\rangle_{s\in\bar{I}}:\ \delta_s\in\{0,1\}\ \&\ \sum_{s\in\bar{I}}\frac{\delta_s}{n_s}=c\right\}\right|. \tag{19}$$

In this section we will get rid of roots of unity to obtain relations between $I^*(c)$ and $\bar{I}_*(c)$, and then give proofs of our theorems stated in the first section.

Lemma 3.1. *Let $\lambda_1,\cdots,\lambda_k$ be nth roots of unity and $c_1,\cdots,c_k$ nonnegative integers such that $c_1\lambda_1+\cdots+c_k\lambda_k=0$. Then $c_1+\cdots+c_k\in D(n)$.*

Proof. Let a be any integer divisible by none of the prime divisors of n. Then a is prime to n. Applying the automorphism σ_a of the cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$ with $\sigma_a(e^{2\pi i/n})=e^{2\pi ia/n}$ we obtain that $c_1\lambda_1^a+\cdots+c_k\lambda_k^a=0$. Thus $c_1+\cdots+c_k\in D(n)$ by Lemma 9 of [14] (thanks to Chen's ingenious idea in [2]). □

Theorem 3.1. *Let $n_0\in\mathbb{Z}^+$ be a period of $w_A(x)$ and I be a subset of $[1,k]$. Let $r\in\mathbb{Z}$ and*

$$c=\min\left\{\sum_{s\in I}\frac{x_s}{n_s}:\ x_s\in\mathbb{N}\text{ for } s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}-\frac{r}{[n_s]_{s\in I}}\in\frac{1}{(n_0,[n_s]_{s\in I})}\mathbb{Z}\right\}. \tag{20}$$

(i) *If $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, then $\bar{I}_*(c)\geqslant I^*(c)>0$.*

(ii) *If $\bar{I}_*(c)=0$ then $I^*(c)\in D([n_s]_{s\in I})$.*

Proof. The existence of c follows from Proposition 2.2 of [15]. Note that $I^*(c)>0$. For $n\in\mathbb{Z}^+$ clearly $I^*(c-n/(n_0,[n_s]_{s\in I}))=0$ and hence $I^*[c-n/(n_0,[n_s]_{s\in I})]=0$. By Corollary 2.1, $\bar{I}_*[c]=I^*[c]$.

i) When $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$, by the Chinese Remainder Theorem in general form (see Lemma 1 of [10]) there exists an integer a_I congruent to a_s mod n_s for all $s\in I$, therefore

$$I^*[c]=\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I}x_s/n_s=c}}e\bigg(\sum_{s\in I}a_I\frac{x_s}{n_s}\bigg)=e(a_Ic)I^*(c)$$

and hence $\bar{I}_*(c) \geqslant |\bar{I}_*[c]| = |I^*[c]| = I^*(c) > 0$.

ii) As $\bar{I}_*(c) = 0$ we have $I^*[c] = \bar{I}_*[c] = 0$, so $I^*(c) \in D([n_s]_{s\in I})$ by Lemma 3.1.

The proof is now complete. □

Proof of Theorem 1.1. i) As $0 \in D(n)$ the case $I = \emptyset$ is trivial. Let $I \neq \emptyset$ and $r \in R(n/(n_0, n))$. It is well known in combinatorics that the equation $\sum_{s\in I} x_s = r$ has exactly $\binom{|I|+r-1}{r}$ solutions over $\mathbb{N}$. (Cf. p.38 of [1].) By Theorem 3.1 (ii), either $I^*(r/n) = \binom{|I|+r-1}{r} \in D(n)$ or $\sum_{s\in J} 1/n_s = r/n$ for some $J \subseteq \bar{I}$. This proves part (i).

ii) Let r be an integer with $1 \leqslant r < n_k/n_{k-l}$ and $r \not\equiv 0 \pmod{n_k/(n_0, n_k)}$. If $1 \leqslant j \leqslant k-l$ then $1/n_j \geqslant 1/n_{k-l} > r/n_k$. So, for any $J \subseteq [1, k]$ with $\sum_{s\in J} 1/n_s = r/n_k$, we have $J \subseteq \{k-l+1, \cdots, k\}$ and $|J| = r$. Since $\emptyset^*[r/n_k - n/(n_0, n_k)] = 0$ for all $n \in \mathbb{N}$, applying Corollary 2.1 we find that

$$\sum_{\substack{J\subseteq\{k-l+1,\cdots,k\}\\ |J|=r}} e\Big(\frac{\sum_{s\in J} a_s}{n_k}\Big) = \sum_{\substack{J\subseteq[1,k]\\ \sum_{s\in J} 1/n_s = r/n_k}} (-1)^{|J|-r} e\Big(\sum_{s\in J}\frac{a_s}{n_s}\Big) = 0.$$

Thus, with the help of Lemma 3.1,

$$\binom{l}{r} = |\{J \subseteq \{k-l+1, \cdots, k\} : |J| = r\}| \in D(n_k).$$

This concludes the proof. □

Proof of Theorem 1.2. Let r be any integer and c be as in (20). Then $c + a/(n_0, [n_s]_{s\in I}) = r/[n_s]_{s\in I}$ for some $a \in \mathbb{Z}$. By Theorem 3.1 (i), there is a $J \subseteq \bar{I}$ such that $\sum_{s\in J} 1/n_s = c$. So we have the desired result. □

Proof of Theorem 1.3. By adding some residue classes modulo $N = [n_0, [n_s]_{s\in I}]$ we can extend $\{a_s(n_s)\}_{s\in I}$ to a system having covering function $w(x)$. Without any loss of generality, we may suppose that $I \subseteq [1, k]$ and (1) has covering function $w(x)$. Let $c = \sum_{s\in\bar{I}} 1/n_s$. Then $\bar{I}_*[c] \neq 0$. Applying Theorem 2.1 with $v = 0$ we find that $I^*[c - a/n_0] \neq 0$ for some $a \in \mathbb{N}$. So there are $x_s \in R(n_s)$ for $s \in I$ such that $c - a/n_0 = \sum_{s\in I} x_s/n_s$. Note that

$$c + \sum_{s\in I}\frac{1}{n_s} = \sum_{s=1}^{k}\frac{1}{n_s} = \frac{1}{n_0}\sum_{t=1}^{n_0} w(t)$$

where in the last step we calculate the degrees of both sides of (12). Write $a = a_0 + n_0 q$ where $a_0 \in R(n_0)$ and $q \in \mathbb{N}$. Then

$$\frac{1}{n_0}\sum_{t=1}^{n_0} w(t) - \frac{a_0}{n_0} = q + \sum_{s\in I}\frac{x_s+1}{n_s} = \sum_{s\in I}\frac{m_s}{n_s}$$

where those m_s with $s \in I$ are suitable positive integers. If all the n_s with $s \in I$ are prime to n_0, then n_0 must divide $\sum_{t=1}^{n_0} w(t) - a_0$ and hence

$$\sum_{s\in I}\frac{m_s}{n_s} = \frac{w(1) + \cdots + w(n_0) - a_0}{n_0} = \left\lfloor\frac{w(1) + \cdots + w(n_0)}{n_0}\right\rfloor.$$

We are done. □

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