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# A RECIPROCITY LAW FOR UNIFORM FUNCTIONS

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Abstract. Let $F$ be a uniform function of two complex variables to an additive abelian group, i.e., $F$ satisfies the functional equation

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y) \quad (n=1,2,3,\ldots)$$

introduced by Z. W. Sun in the 1980s. Suppose that $\langle x+1, y\rangle \in \mathrm{Dom}(F)$ for all $\langle x,y\rangle \in \mathrm{Dom}(F)$. In this paper we establish the following reciprocity law:

$$\sum_{r=0}^{m-1} F\left(\frac{x+nr}{m}, my\right) = \sum_{r=0}^{n-1} F\left(\frac{x+mr}{n}, ny\right)$$

for any $\langle x,y\rangle \in \mathrm{Dom}(F)$ and $m,n=1,2,3,\cdots$. Several applications are also given.

## 1. Introduction

In 1989 Z. W. Sun [S1] introduced the following original concept in his study of covering equivalence.

*Definition* 1. For a function $F$ of two complex variables into an additive abelian group $M$, if for any ordered pair $\langle x,y\rangle$ in the domain of $F$ we have

$$\left\{\left\langle \frac{x+r}{n}, ny\right\rangle : r=0,1,\cdots,n-1\right\} \subseteq \mathrm{Dom}(F) \tag{1}$$

and

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y) \tag{2}$$

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for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$, then we call $F$ a *uniform function* (into $M$).

There are many examples of uniform functions, the reader may consult [S1], [S2], [S3], [S4] and [CS].

The main result of this paper is the following observation.

**Theorem 1** (Reciprocity law for uniform functions)**.** *Let $F$ be a uniform function into an additive abelian group. Suppose that $\langle x+1, y\rangle \in \mathrm{Dom}(F)$ for all $\langle x, y\rangle \in \mathrm{Dom}(F)$. Then, for any positive integers $m$ and $n$, we have*

$$\sum_{r=0}^{m-1} F\left(\frac{x+nr}{m}, my\right) = \sum_{r=0}^{n-1} F\left(\frac{x+mr}{n}, ny\right) \tag{3}$$

*for all $\langle x, y\rangle \in \mathrm{Dom}(F)$.*

*Remark* 1. If $F$ be a uniform function with $F(x+1, y) = F(x, y)$ for all $\langle x, y\rangle \in \mathrm{Dom}(F)$, then Lemma 2.1 of [CS] indicates that

$$\sum_{r=0}^{n-1} F\left(\frac{x+mr}{n}, ny\right) = (m, n)F\left(\frac{x}{(m, n)}, (m, n)y\right)$$

for all $m, n \in \mathbb{Z}^+$ and $\langle x, y\rangle \in \mathrm{Dom}(F)$, where $(m, n)$ denotes the greatest common divisor of $m$ and $n$.

Although Theorem 1 seems simple, it is very useful.

**Corollary 1** (Graham, Knuth and Patashnik [GKP, p.94])**.** *Let $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$ where $\mathbb{R}$ is the field of real numbers. Then we have*

$$\sum_{r=0}^{m-1} \left[\frac{x+nr}{m}\right] = \sum_{r=0}^{n-1} \left[\frac{x+mr}{n}\right], \tag{4}$$

*where $[x]$ denotes the greatest integer not exceeding the real number $x$.*

*Proof.* This is because the function $[\ ] : \mathbb{R} \times \mathbb{Z}^+ \to \mathbb{R}$ given by $[\ ](x, y) = [x]$ is uniform. The identity

$$\sum_{r=0}^{n-1} \left[\frac{x+r}{n}\right] = [x] \qquad (n \in \mathbb{Z}^+)$$

is due to Hermite. $\square$

*Remark* 2. In fact, if $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$ then both sides of (4) coincide with $(m, n)[x/(m, n)] + ((m, n) - 1)/2 + (m-1)(n-1)/2$.

**Corollary 2.** *Let $m, n$ be positive integers, and let $z$ be any complex number with $z/(m,n) \notin \mathbb{Z}$. Then we have the identity*

$$\sum_{r=0}^{m-1} \cot \pi \frac{z+nr}{m} = \frac{m}{n} \sum_{r=0}^{n-1} \cot \pi \frac{z+mr}{n}. \tag{5}$$

*Proof.* Let $\mathbb{C}$ denote the field of complex numbers. By Example 2.4 of [S4], the function $\cot_0(x,y) = y^{-1}\cot(\pi x)$ with $x \in \mathbb{C}\setminus\mathbb{Z}$ and $y > 0$ is uniform. In view of Theorem 1, for any $x \in \mathbb{C}\setminus\mathbb{Z}$ and $y > 0$ we have

$$\sum_{r=0}^{m-1} \frac{1}{my} \cot \pi \frac{x+nr}{m} = \sum_{r=0}^{n-1} \frac{1}{ny} \cot \pi \frac{x+mr}{n}.$$

Letting $x$ tend to $z$ we then obtain the desired identity. (Note that $z \notin m\mathbb{Z}+n\mathbb{Z} = (m,n)\mathbb{Z}$.) $\square$

**Corollary 3.** *Let $\Gamma(z)$ be the well known $\Gamma$-function. Then for any $m, n \in \mathbb{Z}^+$ and $z \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$ we have*

$$\prod_{r=0}^{m-1} \Gamma\left(\frac{z+nr}{m}\right) = (2\pi)^{(m-n)/2}\left(\frac{n}{m}\right)^{z+(mn-m-n)/2} \prod_{r=0}^{n-1} \Gamma\left(\frac{z+mr}{n}\right). \tag{6}$$

*Proof.* By Example 2.2 of [S4], the function $\gamma(x,y) = \Gamma(x)y^{x-1/2}/\sqrt{2\pi}$ with $x \neq 0,-1,-2,\ldots$ and $y > 0$ is a uniform function into the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers. Applying Theorem 1 we find that

$$\prod_{r=0}^{m-1} \gamma\left(\frac{z+nr}{m}, m\times 1\right) = \prod_{r=0}^{n-1} \gamma\left(\frac{z+mr}{n}, n\times 1\right).$$

This implies the desired identity (6). $\square$

For $k \in \mathbb{N} = \{0,1,2,\cdots\}$ let $B_k(x)$ denote the Bernoulli polynomial of degree $k$ and set

$$S_k(n) = 0^k + 1^k + \ldots + (n-1)^k \quad \text{for } n = 1,2,\cdots.$$

Theorem 1 also implies the following result.

**Theorem 2.** *Let $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}^+$. Then*

$$\sum_{j=0}^{k} \binom{k}{j} m^{k-j} n^{j-1} B_j(mx) S_{k-j}(n) = \sum_{j=0}^{k} \binom{k}{j} m^{j-1} n^{k-j} B_j(nx) S_{k-j}(m). \tag{7}$$

*Remark* 3. Theorem 2 in the case $x = 0$ yields the main result of Hans J. H. Tuenter [T]. When $x = 0$ and $n = 1$, (7) turns out to be a recursion for Bernoulli numbers which was proved by E. Y. Deeba and D. M. Rodriguez [DR] and used by F. T. Howard [H] to deduce those classical congruences concerning Bernoulli numbers.

Theorems 1 and 2 will be proved in the next section.

## 2. Proof of Theorems 1 and 2

*Proof of Theorem 1.* Fix $\langle x,y\rangle \in \mathrm{Dom}(F)$. If $q,r\in\mathbb{N}$ and $0\leqslant r<m$, then

$$\left\langle \frac{x+(mq+r)}{m}, my\right\rangle = \left\langle \frac{x+r}{m}+q, my\right\rangle \in \mathrm{Dom}(F).$$

Thus $\langle (x+l)/m, my\rangle \in \mathrm{Dom}(F)$ for every $l\in\mathbb{N}$. Similarly, $\langle (x+l)/n, ny\rangle \in \mathrm{Dom}(F)$ for all $l\in\mathbb{N}$.

Observe that

$$\begin{aligned}\sum_{r=0}^{m-1} F\left(\frac{x+nr}{m}, my\right) &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} F\left(\frac{(x+nr)/m+s}{n}, n(my)\right)\\ &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} F\left(\frac{x+nr+ms}{mn}, mny\right)\\ &= \sum_{s=0}^{n-1}\sum_{r=0}^{m-1} F\left(\frac{(x+ms)/n+r}{m}, m(ny)\right)\\ &= \sum_{s=0}^{n-1} F\left(\frac{x+ms}{n}, ny\right).\end{aligned}$$

This concludes the proof. □

*Proof of Theorem 2.* By Raabe's theorem, the function $b_k:\mathbb{C}\times\mathbb{Z}^+\to\mathbb{C}$ given by $b_k(x,y)=y^{m-1}B_k(x)$ is a uniform function as observed by Sun [S2].

Since

$$\sum_{l=0}^{\infty} B_l(x+y)\frac{z^l}{l!} = \frac{ze^{xz}}{e^z-1}e^{yz} = \sum_{j=0}^{\infty} B_j(x)\frac{z^j}{j!}\sum_{i=0}^{\infty} y^i\frac{z^i}{i!},$$

we have the identity

$$B_k(x+y)=\sum_{j=0}^{k}\binom{k}{j}B_j(x)y^{k-j}.$$

Observe that

$$\begin{aligned}\sum_{r=0}^{n-1} b_k\left(\frac{mnx+mr}{n},\ n\cdot 1\right) &= \sum_{r=0}^{n-1} n^{k-1}B_k\left(mx+\frac{mr}{n}\right)\\ &= \sum_{r=0}^{n-1} n^{k-1}\sum_{j=0}^{k}\binom{k}{j}B_j(mx)\left(\frac{mr}{n}\right)^{k-j}\\ &= \sum_{j=0}^{k}\binom{k}{j}B_j(mx)m^{k-j}n^{j-1}S_{k-j}(n)\end{aligned}$$

Similarly,

$$\sum_{r=0}^{m-1} b_k\left(\frac{mnx+nr}{m},\ m\cdot 1\right)=\sum_{j=0}^{k}\binom{k}{j}B_j(nx)m^{j-1}n^{k-j}S_{k-j}(m).$$

So the desired (7) follows from Theorem 1. □

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