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ON THE RANGE OF A COVERING FUNCTION

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ABSTRACT. Let $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k>1)$ be a finite system of residue classes with the moduli $n_1,\ldots,n_k$ distinct. By means of algebraic integers we show that the range of the covering function $w(x)=|\{1\leqslant s\leqslant k: x\equiv a_s \pmod{n_s}\}|$ is not contained in any residue class with modulus greater than one. In particular, the values of $w(x)$ cannot have the same parity.

1. INTRODUCTION

For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, let $a(n)$ stand for the residue class $\{x\in\mathbb{Z}: x\equiv a \pmod n\}$. A finite system

$$\{a_s(n_s)\}_{s=1}^k \quad (k>1) \tag{1.1}$$

of residue classes is said to be a *cover* of $\mathbb{Z}$ if $\bigcup_{s=1}^k a_s(n_s)=\mathbb{Z}$.

The concept of cover of $\mathbb{Z}$ was introduced by P. Erdős ([E50]) in the early 1930s, who was particularly interested in those covers (1.1) with the moduli $n_1,\ldots,n_k$ distinct. By Example 3 of the author [S96], if $n>1$ is odd then

$$\{1(2),2(2^2),\ldots,2^{n-2}(2^{n-1}),2^{n-1}(n),2^{n-1}2(2n),\ldots,2^{n-1}n(2^{n-1}n)\}$$

forms a cover of $\mathbb{Z}$ with distinct moduli. Covers of $\mathbb{Z}$ have been studied by various researchers (cf. [G04] and [PS]) and many surprising applications have been found (see, e.g. [F02], [S00], [S01] and [S03b]).

Here are two major open problems concerning covers of $\mathbb{Z}$ (see sections E23, F13 and F14 of [G04] for references to these and other conjectures).

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Erdős–Selfridge Conjecture. *Let* (1.1) *be a cover of* $\mathbb{Z}$ *with distinct moduli. Then* $n_1, \ldots, n_k$ *cannot be all odd and greater than one.*

Schinzel's Conjecture. *If* (1.1) *is a cover of* $\mathbb{Z}$*, then there is a modulus* n_t *dividing another modulus* n_s*.*

For system (1.1), the function $w : \mathbb{Z} \to \mathbb{Z}$ given by

$$w(x) = |\{1 \leqslant s \leqslant k : \ x \in a_s(n_s)\}| \tag{1.2}$$

is called its *covering function*. Obviously $w(x)$ is periodic modulo the least common multiple $N = [n_1, \ldots, n_k]$ of the moduli $n_1, \ldots, n_k$.

Now we list some known results concerning the covering function $w(x)$.

(i) The arithmetic mean of $w(x)$ with x in a period equals $\sum_{s=1}^{k} 1/n_s$.

(ii) (Z. W. Sun [S95, S96]) The covering function $w(x)$ takes its minimum on every set of

$$\left|\left\{0 \leqslant \theta < 1 : \sum_{s\in I} \frac{m_s}{n_s} - \theta \in \mathbb{Z} \text{ for some } I \subseteq \{1, \ldots, k\}\right\}\right|$$

consecutive integers, where $m_1, \ldots, m_k$ are given integers relatively prime to $n_1, \ldots, n_k$ respectively.

(iii) (Z. W. Sun [S03a]) The maximal value of $w(x)$ can be written in the form $\sum_{s=1}^{k} m_s/n_s$ with $m_1, \ldots, m_k \in \mathbb{Z}^+$.

(iv) (Š. Porubský [P75]) If $n_1, \ldots, n_k$ are distinct, then $[n_1, \ldots, n_k]$ is the smallest positive period of the function $w(x)$.

(v) (Z. W. Sun [S03a]) If $n_0 \in \mathbb{Z}^+$ is a period of the function $w(x)$, then for any $t = 1, \ldots, k$ we have

$$\left\{\sum_{s\in I} \frac{1}{n_s} : I \subseteq \{1, \ldots, k\} \setminus \{t\}\right\} \supseteq \left\{\frac{r}{n_t} : r \in \mathbb{Z} \text{ and } 0 \leqslant r < \frac{n_t}{(n_0, n_t)}\right\},$$

where (n_0, n_t) denotes the greatest common divisor of n_0 and n_t.

(vi) (Z. W. Sun [S04]) The function $w(x)$ is constant if $w(x)$ equals a constant for $|S|$ consecutive integers x where

$$S = \left\{\frac{r}{n_s} : \ r = 0, \ldots, n_s - 1; \ s = 1, \ldots, k\right\}.$$

In this paper we study the range of a covering function (via algebraic integers) for the first time. Proofs of the theorems below will be given in the next section.

Theorem 1.1. *Suppose that the range of the covering function of* (1.1) *is contained in a residue class with modulus* m*. Then, for any* $t = 1, \ldots, k$ *with* $mn_t \nmid [n_1, \ldots, n_k]$*, we have* $n_t \mid n_s$ *for some* $1 \leqslant s \leqslant k$ *with* $s \neq t$*.*

Corollary 1.1. *If the covering function $w(x)$ of* (1.1) *is constant, then for any $t = 1, \dots, k$ there is an $s \neq t$ such that $n_t \mid n_s$, and in particular $n_k = n_{k-1}$ provided that $n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$.*

Proof. Suppose that $w(x) = c$ for all $x \in \mathbb{Z}$. Choose an integer $m > [n_1, \dots, n_k]$. As $c(m)$ contains the range of $w(x)$, the desired result follows from Theorem 1.1. □

Remark 1.1. When (1.1) is a disjoint cover of $\mathbb{Z}$, i.e., $w(x) = 1$ for all $x \in \mathbb{Z}$, the first part of Corollary 1.1 was given by B. Novák and Š. Znám [NZ] and the second part was originally obtained by H. Davenport, L. Mirsky, D. Newman and R. Radó independently. Corollary 1.1 appeared in Porubský [P75].

Corollary 1.2. *Suppose that those moduli in* (1.1) *which are maximal with respect to divisibility are distinct. Then $w(\mathbb{Z}) = \{w(x) : x \in \mathbb{Z}\}$ cannot be contained in a residue class other than $0(1) = \mathbb{Z}$, i.e., for any prime p there is an $x \in \mathbb{Z}$ with $w(x) \not\equiv w(0) \pmod p$. In particular, those $w(x)$ with $x \in \mathbb{Z}$ cannot have the same parity.*

Proof. Assume that $w(\mathbb{Z})$ is contained in a residue class with modulus $m \in \mathbb{Z}^+$. For each modulus n_t maximal with respect to divisibility, there is no $s \neq t$ such that $n_t \mid n_s$, thus mn_t divides $N = [n_1, \dots, n_k]$ by Theorem 1.1. Since N is also the least common multiple of those moduli n_t maximal with respect to divisibility, we must have $mN \mid N$ and hence $m = 1$. This ends the proof. □

Remark 1.2. In contrast with the Erdős–Selfridge conjecture, Corollary 1.2 indicates that *if* (1.1) *is a cover of $\mathbb{Z}$ with distinct moduli then not every integer is covered by* (1.1) *an odd number of times.*

Here is another related result.

Theorem 1.2. *Let $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ both have distinct moduli. Then A and B are identical provided that $w_A(x) \equiv w_B(x) \pmod m$ for all $x \in \mathbb{Z}$, where w_A and w_B are covering functions of A and B respectively, and m is an integer not dividing $N = [n_1, \dots, n_k, m_1, \dots, m_l]$.*

Remark 1.3. In 1975 Znám [Z75] extended a uniqueness theorem of S. K. Stein [St] as follows: Under the condition of Theorem 1.2, we have $A = B$ if $w_A = w_B$. This follows from Theorem 1.2 by taking $m > N$.

Theorem 1.1 can be refined as follows.

Theorem 1.3. *Let $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$ be weights assigned to the k residue classes in* (1.1) *respectively. Suppose that $n_0 \in \mathbb{Z}^+$ is the smallest positive period of $w(x) = \sum_{1 \leqslant s \leqslant k,\, n_s \mid x - a_s} \lambda_s$ modulo $m \in \mathbb{Z}$, and that $d \in \mathbb{Z}^+$ does* *not divide* n_0 *but* $I(d)=\{1\leqslant s\leqslant k:\ d\mid n_s\}\neq\emptyset$. *Then, either* m *divides* $[n_1,\ldots,n_k]\sum_{s\in I(d)}\lambda_s/n_s$, *or we have*

$$|I(d)|\geqslant|\{a_s \bmod d:\ s\in I(d)\}|\geqslant\min_{\substack{0\leqslant s\leqslant k\\ s\notin I(d)}}\frac{d}{(d,n_s)}\geqslant p(d) \tag{1.3}$$

where $p(d)$ *denotes the smallest prime divisor of* d.

Remark 1.4. Theorem 1.3 in the case $m=0$ was first obtained by the author [S91] in 1991, an extension of this was given in [S04].

Instead of (1.1) we can also consider a finite system of residue classes in $\mathbb{Z}^n$ (cf. [S04]) and deduce n-dimensional versions of Theorems 1.1–1.3.

2. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. Without any loss of generality we assume that $0\leqslant a_s<n_s$ for $s=1,\ldots,k$. Set $N=[n_1,\ldots,n_k]$. Then

$$\begin{aligned}\sum_{r=0}^{N-1}w(r)z^r&=\sum_{r=0}^{N-1}\sum_{\substack{1\leqslant s\leqslant k\\ n_s\mid a_s-r}}z^r=\sum_{s=1}^{k}\sum_{\substack{0\leqslant r<N\\ r\in a_s(n_s)}}z^r\\ &=\sum_{s=1}^{k}z^{a_s}\sum_{0\leqslant q<N/n_s}(z^{n_s})^q\\ &=\sum_{\substack{1\leqslant s\leqslant k\\ z^{n_s}=1}}\frac{N}{n_s}z^{a_s}+(1-z^N)\sum_{\substack{1\leqslant s\leqslant k\\ z^{n_s}\neq 1}}\frac{z^{a_s}}{1-z^{n_s}}.\end{aligned}$$

Suppose that $w(r)=a+mq_r$ for each $r\in\mathbb{Z}$ where $a,q_r\in\mathbb{Z}$. If $\alpha\notin\mathbb{Z}$ but $\alpha N\in\mathbb{Z}$, then

$$\sum_{r=0}^{N-1}w(r)e^{2\pi i\alpha r}=m\sum_{r=0}^{N-1}q_re^{2\pi i\alpha r}$$

and also

$$\sum_{r=0}^{N-1}w(r)e^{2\pi i\alpha r}=\sum_{\substack{s=1\\ \alpha n_s\in\mathbb{Z}}}^{k}\frac{N}{n_s}e^{2\pi i\alpha a_s},$$

therefore we have the following congruence

$$\sum_{\substack{s=1\\ \alpha n_s\in\mathbb{Z}}}^{k}\frac{N}{n_s}e^{2\pi i\alpha a_s}\equiv 0 \pmod m \tag{2.1}$$

in the ring of all algebraic integers.

If $1 \leqslant t \leqslant k$ and $n_t \mid n_s$ for no $s \in \{1, \dots, k\} \setminus \{t\}$, then by applying (2.1) with $\alpha = 1/n_t < 1$ we obtain that

$$\frac{N}{n_t} e^{2\pi i a_t/n_t} \equiv 0 \pmod{m}$$

and hence m divides N/n_t in $\mathbb{Z}$.

The proof of Theorem 1.1 is now complete. $\square$

Proof of Theorem 1.2. Without any loss of generality, we assume that $n_1 > \cdots > n_k$ and $m_1 > \cdots > m_l$. As $w_A(x) - w_B(x) \equiv 0 \pmod{m}$ for all $x \in \mathbb{Z}$, by modifying the proof of Theorem 1.1 slightly, we find that if $\alpha \notin \mathbb{Z}$ but $\alpha N \in \mathbb{Z}$ then

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^{k} \frac{N}{n_s} e^{2\pi i \alpha a_s} - \sum_{\substack{t=1 \\ \alpha m_t \in \mathbb{Z}}}^{l} \frac{N}{m_t} e^{2\pi i \alpha b_t} \equiv 0 \pmod{m}. \tag{2.2}$$

In the case $d = \max\{m_1, n_1\} > 1$, by applying (2.2) with $\alpha = 1/d$ and the hypothesis $m \nmid N$, we get that $m_1 = n_1$ and

$$\frac{N}{d}\left(e^{2\pi i a_1/d} - e^{2\pi i b_1/d}\right) \equiv 0 \pmod{m}.$$

If $a_1 \not\equiv b_1 \pmod{d}$, then $z = 1 - e^{2\pi i(b_1 - a_1)/d}$ is a zero of the monic polynomial $(-1)^{d-1}P(1-x) \in \mathbb{Z}[x]$ where $P(x) = (1-x^d)/(1-x) = 1 + x + \cdots + x^{d-1}$, hence z divides the constant term $P(1) = d$ of $P(1-x)$ in the ring of algebraic integers. As m does not divide N, we must have $a_1 \equiv b_1 \pmod{d}$ and so $a_1(n_1) = b_1(m_1)$. Now that

$$|\{1 < s \leqslant k : x \in a_s(n_s)\}| \equiv |\{1 < t \leqslant l : x \in b_t(m_t)\}| \pmod{m}$$

for all $x \in \mathbb{Z}$, we can continue the above procedure to obtain that

$$a_2(n_2) = b_2(m_2), \ \dots, \ a_{\min\{k,l\}}(n_{\min\{k,l\}}) = b_{\min\{k,l\}}(m_{\min\{k,l\}}).$$

If $k \neq l$, say $k > l$, then $m\mathbb{Z}$ contains the range of the covering function of $\{a_s(n_s)\}_{s=l+1}^{k}$ and this contradicts Theorem 1.1 since $m \nmid [n_{l+1}, \dots, n_k]$ and $n_{l+1} > \cdots > n_k$. So $A = B$ and we are done. $\square$

Proof of Theorem 1.3. Let $N = [n_1, \dots, n_k]$. Clearly $(n_0, N) \in n_0\mathbb{Z} + N\mathbb{Z}$ is also a period of $w(x)$ mod m, so $(n_0, N) = n_0$ and hence $n_0 \mid N$. Observe that

$$\sum_{\substack{s=1 \\ x \in a_s(n_s)}}^{k} \lambda_s - \sum_{\substack{r=0 \\ x \in r(n_0)}}^{n_0 - 1} w(r) \equiv 0 \pmod{m}$$

for each $x \in \mathbb{Z}$. As in the proof of Theorem 1.1, if $c \in \mathbb{Z}$ and $d \nmid c$ then

$$\sum_{\substack{s=1\\(c/d)n_s\in\mathbb{Z}}}^{k} \lambda_s \frac{N}{n_s} e^{2\pi i \frac{c}{d} a_s} - \sum_{\substack{r=0\\(c/d)n_0\in\mathbb{Z}}}^{n_0-1} w(r)\frac{N}{n_0} e^{2\pi i \frac{c}{d} r} \equiv 0 \pmod{m}. \tag{2.3}$$

For any $c \in \mathbb{Z}^+$ divisible by none of those $d/(d, n_s)$ with $0 \leqslant s \leqslant k$ and $s \notin I(d)$, we have

$$d \mid cn_s \iff \frac{d}{(d,n_s)} \mid c \iff d \mid n_s \iff s \in I(d),$$

therefore (2.3) yields that

$$\sum_{s\in I(d)} \lambda_s \frac{N}{n_s} e^{2\pi i \frac{c}{d} a_s} \equiv 0 \pmod{m}.$$

Let

$$R = \{0 \leqslant r < d:\ a_s \equiv r \ (\mathrm{mod}\ d) \text{ for some } s \in I(d)\}$$

and suppose that $|R| < \min_{0\leqslant s\leqslant k,\, s\notin I(d)} d/(d,n_s)$. By the above,

$$u_n := \sum_{r\in R} c_r \left(e^{2\pi i \frac{r}{d}}\right)^n \equiv 0 \ (\mathrm{mod}\ m) \ \text{ for every } n = 1, \ldots, |R|,$$

where $c_r = N\sum_{s\in I(d),\, a_s\in r(d)} \lambda_s/n_s \in \mathbb{Z}$. As $\{u_n\}_{n\geqslant 0}$ is a linear recurrence of order $|R|$ with characteristic polynomial $\prod_{r\in R}(x - e^{2\pi i r/d})$ whose coefficients are algebraic integers, we have $u_n \equiv 0 \ (\mathrm{mod}\ m)$ for every $n = |R|+1, |R|+2, \ldots$. In particular, $\sum_{r\in R} c_r = u_d \equiv 0 \ (\mathrm{mod}\ m)$, i.e., m divides $N\sum_{s\in I(d)} \lambda_s/n_s$. We are done. $\square$

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