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ON ODD COVERING SYSTEMS WITH DISTINCT MODULI

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Abstract. A famous unsolved conjecture of P. Erdős and J. L. Selfridge states that there does not exist a covering system $\{a_s(\text{mod } n_s)\}_{s=1}^k$ with the moduli $n_1, \ldots, n_k$ odd, distinct and greater than one. In this paper we show that if such a covering system $\{a_s(\text{mod } n_s)\}_{s=1}^k$ exists with $n_1, \ldots, n_k$ all square-free, then the least common multiple of $n_1, \ldots, n_k$ has at least 22 prime divisors.

1. Introduction

For $a \in \mathbb{Z}$ and $n \in \{1, 2, 3, \ldots\}$, we simply let $a(n)$ denote the residue class

$$a(\text{mod } n) = \{a + nx : x \in \mathbb{Z}\}.$$

In the early 1930s P. Erdős called a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \qquad (*)$$

of residue classes a *covering system* if $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$. Clearly $(*)$ is a covering system if and only if it covers $0, 1, \ldots, N_A - 1$ where $N_A = [n_1, \ldots, n_k]$ is the least common multiple of the moduli $n_1, \ldots, n_k$.

Here are two covering systems with distinct moduli constructed by Erdős:

$$\{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\},$$

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$$\{0(2),\ 0(3),\ 0(5),\ 1(6),\ 0(7),\ 1(10),\ 1(14),\ 2(15),\\ 2(21),\ 23(30),\ 4(35),\ 5(42),\ 59(70),\ 104(105)\}.$$

Covering systems have been investigated by various number theorists and combinatorists, and many surprising applications have been found. (See [3], [4] and [7].)

A covering system with odd moduli is said to be an *odd covering system*. Here is a well-known open problem in the field (cf. [3]).

Erdős-Selfridge Conjecture. *There does not exist an odd covering system with the moduli distinct and greater than one.*

In 1986-1987, by a lattice-geometric method, M. A. Berger, A. Felzenbaum and A. S. Fraenkel ([1] and [2]) obtained some necessary conditions for system $(*)$ to be an odd covering system with $1 < n_1 < \cdots < n_k$, one of which is the inequality

$$\prod_{t=1}^{r} \frac{p_t - 1}{p_t - 2} - \sum_{t=1}^{r} \frac{1}{p_t - 2} > 2,$$

where $p_1, \ldots, p_r$ are the distinct prime divisors of N_A. They also showed that if $(*)$ is an odd covering system with $n_1, \ldots, n_k$ square-free, distinct and greater than one, then the above inequality can be improved as follows:

$$\prod_{t=1}^{r} \frac{p_t}{p_t - 1} - \sum_{t=1}^{r} \frac{1}{p_t - 1} \geqslant 2$$

and consequently $r \geqslant 11$. This was also deduced by the second author [6] in a simple way.

In 1991, by a complicated sieve method, R. J. Simpson and D. Zeilberger [5] proved that if $(*)$ is an odd covering system with $n_1, \ldots, n_k$ square-free, distinct and greater than one, then N_A has at least 18 prime divisors.

In this paper we obtain further improvement in this direction by a direct argument.

Theorem 1. *Suppose that* $(*)$ *is an odd covering system with* $1 < n_1 < \cdots < n_k$. *If* $N_A = [n_1, \ldots, n_k]$ *is square-free, then it has at least* 22 *prime divisors.*

In contrast with the Erdős-Selfridge conjecture, recently the second author [8] showed that if $(*)$ is a covering system with $1 < n_1 < \cdots < n_k$ then it cannot cover every integer an odd number of times.

2. Proof of Theorem 1

For convenience we let $[a,b]=\{x\in\mathbb{Z}:\ a\leqslant x\leqslant b\}$ for any $a,b\in\mathbb{Z}$.

Assume that $N=N_A=p_1\cdots p_r$ where $p_1<\cdots<p_r$ are distinct odd primes. For each $t\in[1,r]$, we set

$$d_t=\left\lfloor\frac{3}{5}(t-1)\right\rfloor$$

(where $\lfloor\cdot\rfloor$ is the greatest integer function), and define

$$M_t=\begin{cases}\{p_ip_t:\ 1\leqslant i\leqslant d_t\} & \text{if } t\leqslant 8,\\ \{p_ip_t:\ 1\leqslant i\leqslant d_t\}\cup\{p_1p_2p_t,p_1p_3p_t\} & \text{if } t\geqslant 9.\end{cases}$$

Note that $d_1=d_2=0$ and hence $M_1=M_2=\emptyset$.

For $s\in[1,k]$ let

$$n_s'=\begin{cases}p_t & \text{if } n_s\in M_t \text{ for some } t,\\ n_s & \text{otherwise.}\end{cases}$$

Since $n_s'\mid n_s$, we have $a_s(n_s)\subseteq a_s(n_s')$. Thus $A'=\{a_s(n_s')\}_{s=1}^k$ is also an odd covering system. Let

$$\bar{I}=[1,k]\setminus\bigcup_{t=1}^r I_t\qquad\text{where } I_t=\{1\leqslant s\leqslant k:\ n_s'=p_t\}.$$

Then

$$\bigcup_{s\in\bar{I}}a_s(n_s')\supseteq[0,N-1]\setminus\bigcup_{t=1}^r\bigcup_{s\in I_t}a_s(n_s')=\bigcap_{t=1}^r\left([0,N-1]\setminus\bigcup_{s\in I_t}a_s(n_s')\right).$$

For each $t\in[1,r]$, clearly $|I_t|\leqslant d_t+1<p_t$ if $t\leqslant 8$, and $|I_t|\leqslant d_t+3$ otherwise. Observe that $d_t\leqslant 3(p_t-1)/5<p_t-3$ if $t\geqslant 9$. So there is a subset R_t of $[0,p_t-1]$ satisfying the following conditions:

(a) $|R_t|=p_t-1-d_t$ if $t\leqslant 8$, and $|R_t|=p_t-3-d_t$ if $t\geqslant 9$;

(b) $x\not\equiv a_s(\bmod\ p_t)$ for any $x\in R_t$ and $s\in I_t$.

Define

$$X=\{x\in[0,N-1]:\ \text{the remainder of } x \bmod p_t \text{ lies in } R_t \text{ for } t\in[1,r]\}.$$

Then $|X|=\prod_{t=1}^r|R_t|$ by the Chinese Remainder Theorem, also

$$X\subseteq\bigcap_{t=1}^r\left([0,N-1]\setminus\bigcup_{s\in I_t}a_s(n_s')\right)\subseteq\bigcup_{s\in\bar{I}}a_s(n_s')=\bigcup_{s\in\bar{I}}a_s(n_s)$$

and hence $X=\bigcup_{s\in J}X_s$, where

$$X_s = X\cap a_s(n_s) \ \text{ and } \ J=\{s\in \bar{I}:\ X_s\neq\emptyset\}.$$

For each $s\in J$, the set X_s consists of those $x\in[0,N-1]$ for which $x\equiv a_s \pmod{p_t}$ if $p_t\mid n_s$, and $x\equiv r_t \pmod{p_t}$ for some $r_t\in R_t$ if $p_t\nmid n_s$. Thus, by the Chinese Remainder Theorem,

$$|X_s| = \prod_{\substack{1\leqslant t\leqslant r\\ p_t\nmid n_s}}|R_t| = |X|\prod_{\substack{1\leqslant t\leqslant r\\ p_t\mid n_s}}|R_t|^{-1} \ \text{ for all } s\in J.$$

Let $a_0\in X$, $n_0=p_1p_2$ and $X_0=X\cap a_0(n_0)$. Again by the Chinese Remainder Theorem,

$$|X_0| = \prod_{2<t\leqslant r}|R_t| = |X|\prod_{\substack{1\leqslant t\leqslant r\\ p_t\mid n_0}}|R_t|^{-1}.$$

Let $j=0$ if $n_0\notin\{n_s:\ s\in J\}$, and let j be the unique element of J with $n_j=n_0$ if $n_0\in\{n_s:\ s\in J\}$. Set $J_0=\{s\in J:\ (n_s,n_0)=1\}$. Then

$$\begin{aligned}|X| = \Big|\bigcup_{s\in J\cup\{j\}}X_s\Big| &\leqslant \sum_{s\in J\setminus(J_0\cup\{j\})}|X_s| + \Big|X_j\cup\bigcup_{s\in J_0}X_s\Big|\\ &\leqslant \sum_{s\in J\setminus(J_0\cup\{j\})}|X_s| + |X_j| + \sum_{s\in J_0}|X_s\setminus X_j|\\ &= \sum_{s\in J\setminus(J_0\cup\{j\})}|X_s| + |X_j| + \sum_{s\in J_0}(|X_s|-|X_s\cap X_j|)\end{aligned}$$

and so

$$|X| \leqslant \sum_{s\in J\cup\{j\}}|X_s| - \sum_{s\in J_0}|X_s\cap X_j|.$$

If $s\in J_0$, then $X_s\cap X_j$ consists of those $x\in[0,N-1]$ for which $x\equiv a_j \pmod{n_j}$, $x\equiv a_s \pmod{n_s}$, and $x\equiv r_t \pmod{p_t}$ for some $r_t\in R_t$ if $p_t\nmid n_jn_s$, therefore

$$|X_s\cap X_j| = \prod_{\substack{1\leqslant t\leqslant r\\ p_t\nmid n_jn_s}}|R_t| = |X|\prod_{\substack{1\leqslant t\leqslant r\\ p_t\mid n_0n_s}}|R_t|^{-1}.$$

Set

$$D_1 = \{d>1:\ d\mid N\}\setminus\Big(\{p_1,\ldots,p_r\}\cup\bigcup_{t=1}^{r}M_t\Big),$$

and

$$D_2 = \{n_s : s \in J \cup \{j\}\} \text{ and } D_3 = \{d \in D_1 : (d, n_0) = 1\}.$$

If $s \in J$, then $n'_s \neq p_t$ for any $t \in [1, r]$, and thus $n_s = n'_s \in D_1$. Since $d_2 = 0$, we also have $n_j = p_1 p_2 \in D_1$. Therefore $D_2 \subseteq D_1$, and so $D_2 \cap D_3$ coincides with $D_4 = \{n_s : s \in J_0\}$.

Let

$$x_t = |R_t|^{-1} \leqslant 1 \quad \text{for } t = 1, \ldots, r,$$

and

$$I(d) = \{1 \leqslant t \leqslant r : p_t \mid d\}$$

for any positive divisor d of N. Observe that

$$\begin{aligned} &\sum_{d \in D_1 \setminus D_2} \prod_{t \in I(d)} x_t - x_1 x_2 \sum_{d \in D_3 \setminus D_4} \prod_{t \in I(d)} x_t \\ =& \sum_{d \in D_1 \setminus (D_2 \cup D_3)} \prod_{t \in I(d)} x_t + (1 - x_1 x_2) \sum_{d \in D_3 \setminus D_4} \prod_{t \in I(d)} x_t \geqslant 0. \end{aligned}$$

Thus

$$\begin{aligned} |X| \leqslant& \sum_{s \in J \cup \{j\}} |X_s| - \sum_{s \in J_0} |X_s \cap X_j| \\ =& \sum_{d \in D_2} |X| \prod_{t \in I(d)} x_t - \sum_{d \in D_4} |X| x_1 x_2 \prod_{t \in I(d)} x_t \\ \leqslant& |X| \Bigg(\sum_{d \in D_1} \prod_{t \in I(d)} x_t - x_1 x_2 \sum_{d \in D_3} \prod_{t \in I(d)} x_t \Bigg). \end{aligned}$$

Since $d_1 = d_2 = 0$ and $d_t < 3$ for $t < 6$, by the above we have

$$\begin{aligned} 1 \leqslant& \sum_{\substack{I \subseteq [1,r] \\ |I| > 1}} \prod_{t \in I} x_t - \sum_{t=1}^{r} \sum_{1 \leqslant i \leqslant d_t} x_i x_t - \sum_{9 \leqslant t \leqslant r} (x_1 x_2 x_t + x_1 x_3 x_t) \\ &- x_1 x_2 \Bigg(\sum_{\substack{I \subseteq [3,r] \\ |I| > 1}} \prod_{t \in I} x_t - \sum_{3 \leqslant t \leqslant r} \sum_{3 \leqslant i \leqslant d_t} x_i x_t \Bigg) \\ =& \prod_{t=1}^{r} (1 + x_t) - 1 - \sum_{t=1}^{r} x_t - \sum_{t=3}^{r} \sum_{i=1}^{d_t} x_i x_t - \sum_{9 \leqslant t \leqslant r} (x_1 x_2 x_t + x_1 x_3 x_t) \\ &- x_1 x_2 \Bigg(\prod_{t=3}^{r} (1 + x_t) - 1 - \sum_{t=3}^{r} x_t - \sum_{t=6}^{r} \sum_{i=3}^{d_t} x_i x_t \Bigg). \end{aligned}$$

It follows that $f(x_1,\dots,x_r)\geqslant 2$, where

$$f(x_1,...,x_r)=(1+x_1+x_2)\prod_{t=3}^{r}(1+x_t)-\sum_{t=1}^{r}x_t+x_1x_2-\sum_{t=3}^{r}\sum_{i=1}^{d_t}x_ix_t$$
$$+x_1x_2\sum_{t=3}^{8}x_t-x_1x_3\sum_{9\leqslant t\leqslant r}x_t+x_1x_2\sum_{t=6}^{r}\sum_{i=3}^{d_t}x_ix_t$$

can be written in the form $\sum_{i_1,\dots,i_r}c_{i_1,\dots,i_r}x_1^{i_1}\cdots x_r^{i_r}$ with $c_{i_1,\dots,i_r}\geqslant 0$.

Let $q_1=3<\cdots<q_r$ be the first r odd primes. For each $t\in[1,r]$, as $p_t\geqslant q_t$ we have $x_t\leqslant x'_t$, where

$$x'_t=\begin{cases}(q_t-d_t-1)^{-1} & \text{if } 1\leqslant t\leqslant 8,\\ (q_t-d_t-3)^{-1} & \text{if } 9\leqslant t\leqslant r.\end{cases}$$

Thus

$$f(x'_1,\dots,x'_r)\geqslant f(x_1,\dots,x_r)\geqslant 2.$$

By computation through computer we find that

$$f(x'_1,\dots,x'_{21})=1.995\cdots<2,$$

therefore $r\neq 21$. (This is why we define d_t and M_t in a somewhat curious way.)

In the case $r<21$, we let $p_{r+1}<\cdots<p_{21}$ be distinct primes greater than p_r, and then

$$\mathcal{A}=\{a_1(n_1),\dots,a_k(n_k),0(p_{r+1}),\dots,0(p_{21})\}$$

forms an odd covering system with $N_{\mathcal{A}}$ square-free and having exactly 21 distinct prime divisors. This is impossible by the above.

Now we can conclude that $r\geqslant 22$ and this completes the proof.

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