EXPLICIT CONGRUENCES FOR EULER POLYNOMIALS

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Abstract In this paper we establish some explicit congruences for Euler polynomials modulo a general positive integer. As a consequence, if $a, m \in \mathbb{Z}$ and $2 \nmid m$ then

$$\frac{m^{k+1}}{2} E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2} E_k(x) \in \mathbb{Z}[x] \quad \text{for every } k = 0, 1, 2, \ldots,$$

which may be regarded as a refinement of a multiplication formula.

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1. Introduction

Congruences for Bernoulli numbers have been a very intriguing objective of research since the time of L. Euler, and they recently got revived in connection with p-adic interpolation of L-functions. Congruences for Euler numbers, being cognates of Bernoulli numbers, have also received much attention from the same point of view of p-adic interpolation. In [S4] the author determined Euler numbers modulo powers of two, while Euler numbers modulo any odd integer are essentially trivial.

As a natural further step, we are led to consider congruences among Bernoulli and Euler polynomials, the latter of which will be our main concern in this paper. We prove the integrity of coefficients of $f_k(x; a, m)$ (defined by (1.7)), which are related to the summands in the multiplica-

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tion formula (1.6) for Euler polynomials, and establish number-theoretic generalizations thereof (Theorems 1.2 and 2.1).

Hereafter, the labelled formulae with star indicate those known ones which have their counterparts for Bernoulli or Euler polynomials. Hopefully, these will serve also as a basic table for these polynomials (for more information, the reader is referred to [AS], [E], [S1]). In referring to them, we omit the star symbol.

Euler numbers $E_0, E_1, E_2, \ldots$ are defined by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{N} = \{1, 2, 3, \cdots\}. \tag{1.1*}$$

It is well known that they are integers and odd-numbered ones $E_1, E_3, E_5, \cdots$ are all zero.

For each $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the Euler polynomial $E_n(x)$ of degree n is given by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}. \tag{1.2*}$$

Note that

$$E_n = 2^n E_n(1/2). \tag{1.3*}$$

Here are basic properties of Euler polynomials:

$$E_n(x) + E_n(x+1) = 2x^n, \tag{1.4*}$$

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}, \tag{1.5*}$$

and

$$m^k \sum_{a=0}^{m-1} (-1)^a E_k\left(\frac{x+a}{m}\right) = E_k(x). \tag{1.6*}$$

From now on we always assume that q is `a fixed integer greater than one`, and let $\mathbb{Z}_q$ denote the ring of q-adic integers (see [M]). For $\alpha, \beta \in \mathbb{Z}_q$, by $\alpha \equiv \beta \pmod q$ we mean that $\alpha - \beta = q\gamma$ for some $\gamma \in \mathbb{Z}_q$. A rational number in $\mathbb{Z}_q$ is usually called a *q-integer*.

In this paper we aim at establishing some explicit congruences for Euler polynomials modulo a general positive integer.

We adopt some standard notations. For example, for a real number α, $\lfloor \alpha \rfloor$ stands for the greatest integer not exceeding α, and $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ the fractional part of α, (a, b) the greatest common divisor of $a, b \in \mathbb{Z}$, and $\Delta(P(x))$ the difference $P(x+1) - P(x)$ of a polynomial $P(x)$. For a prime p, $n \in \mathbb{N}_0$ and $a \in \mathbb{Z}$, we write $p^n \| a$ if $p^n \mid a$ and $p^{n+1} \nmid a$. For $a, b \in \mathbb{Z} \setminus \{0\}$, by $a \sim_2 b$ we mean that both $2^n \| a$ and $2^n \| b$ for a common $n \in \mathbb{N}_0$.

For convenience we also use the logical notations $\wedge$ (*and*), $\vee$ (*or*), $\Leftrightarrow$ (*if and only if*), and the special notation:

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

By (1.1) and (1.2) it is easy to get

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2} \quad \text{and} \quad E_2(x) = x^2 - x,$$

and verify that the polynomial

$$f_k(x) = f_k(x; a, m) = \frac{m^{k+1}}{2} E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2} E_k(x) \tag{1.7}$$

has integral coefficients for $k = 0, 1, 2$.

This phenomenon is not contingent but universal as asserted by the following general theorem.

Theorem 1.1. *For each $k \in \mathbb{N}_0$ and $a, m \in \mathbb{Z}$ with $2 \nmid m$, we have*

$$f_k(x) = f_k(x; a, m) = \frac{m^{k+1}}{2} E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2} E_k(x) \in \mathbb{Z}[x].$$

This result is remarkable in that (1.6) can be expressed as the vanishing arithmetic mean

$$\frac{1}{m} \sum_{a=0}^{m-1} (-1)^a f_k(x; a, m) = 0.$$

Theorem 1.1 follows from the following more general result whose proof will be given in Section 2.

Theorem 1.2. *Let $k \in \mathbb{N}_0$, $d, m \in \mathbb{N}$ and $d \mid m$. Let c be a real number, and let $P(x)$ denote the polynomial*

$$\frac{[2 \nmid q] + d\,[2 \mid q]}{2} m^k E_k\left(\frac{x}{m}\right) \\ - (-1)^{\lfloor \frac{c}{(d,q)} \rfloor} \frac{(d,q)^{k+1}}{2} \left(\frac{m}{d}\right)^k E_k\left(\frac{d}{(d,q)} \cdot \frac{x}{m} - \left\lfloor \frac{c}{(d,q)} \right\rfloor\right).$$

Then $P(x) \in \mathbb{Z}_q[x]$. Furthermore, if q is odd, then

$$P(x) \equiv \sum_{\substack{j=0 \\ 2\nmid (d-1)j+\lfloor \frac{c+jd}{q} \rfloor}}^{q-1} (-1)^j (x+jm)^k \pmod q; \tag{1.8}$$

if q is even, then

$$\begin{aligned} P(x) \equiv & \sum_{j=0}^{q-1} (-1)^{j-1}(x+jm)^k \left(\left\lfloor \frac{c+jd}{q} \right\rfloor + [2 \mid d \ \wedge \ 2 \mid j](-1)^{\lfloor \frac{c+jd}{q} \rfloor} \right) \\ & + \begin{cases} \frac{q}{2}[2 \nmid m \ \wedge \ 4 \mid d+1]\Delta(x^k) \pmod q & \text{if } 2 \mid k, \\ \frac{q}{2}[4 \mid d+1](\Delta(x^k) + [4 \mid q]\Delta(x^{k-1})) \pmod q & \text{if } 2 \nmid km, \\ \frac{q}{2}[d \sim_2 m]([2 \| m] + [2 \| q]) x^{k-1} \pmod q & \text{if } 2 \nmid k(m-1). \end{cases} \end{aligned} \tag{1.9}$$

Now we derive various consequences of Theorem 1.2.

Corollary 1.1. *Let $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, and let x be a q-integer. If q is odd, then*

$$\begin{aligned} & \frac{m^k}{2} E_k\left(\frac{x}{m}\right) - (-1)^{\lfloor \frac{x}{(m,q)} \rfloor} \frac{(m,q)^{k+1}}{2} E_k\left(\left\{\frac{x}{(m,q)}\right\}\right) \\ & \equiv \sum_{\substack{j=0 \\ 2\nmid (m-1)j+\lfloor \frac{x+jm}{q} \rfloor}}^{q-1} (-1)^j (x+jm)^k \pmod q. \end{aligned}$$

If q is even, then

$$\begin{aligned} & \frac{m^{k+1}}{2} E_k\left(\frac{x}{m}\right) - (-1)^{\lfloor \frac{x}{(m,q)} \rfloor} \frac{(m,q)^{k+1}}{2} E_k\left(\left\{\frac{x}{(m,q)}\right\}\right) \\ & \equiv \sum_{j=0}^{q-1} (-1)^{j-1}(x+jm)^k \left(\left\lfloor \frac{x+jm}{q} \right\rfloor + [2 \mid m \ \wedge \ 2 \mid j](-1)^{\lfloor \frac{x+jm}{q} \rfloor} \right) \\ & \quad + \begin{cases} \frac{q}{2}([2 \| m] + [2 \| q]) x^{k-1} \pmod q & \text{if } 2 \nmid k \text{ and } 2 \mid m, \\ 0 \pmod q & \text{if } k = 0, \\ \frac{q}{2}[4 \mid m+1 \ \wedge \ (k = 1 \ \vee \ 2 \mid k \ \vee \ 2 \| q)] \pmod q & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Just apply Theorem 1.2 with $c = x$ and $d = m$, and note that $\frac{q}{2}x^k \equiv \frac{q}{2}x \pmod{q}$ if $k > 0$ (cf. [S3, Lemma 2.1]). □

Corollary 1.2. *Let $a \in \mathbb{Z}$, $k \in \mathbb{N}_0$, $m \in \mathbb{Z}^+$, $2 \mid q$ and $(m, q) = 1$. Then*

$$\begin{aligned} &f_k(x; a, m) \\ &\equiv \sum_{j=0}^{q-1} (-1)^{j-1} \left\lfloor \frac{a + jm}{q} \right\rfloor (x + a + jm)^k \\ &\quad + \frac{q}{2}[4 \mid m+1] \Big(\Delta(x^k) + [2 \nmid k \ \wedge \ 4 \mid q] \Delta(x^{k-1}) \Big) \pmod{q}. \end{aligned}$$

Proof. Clearly $2 \nmid m$ and $\Delta((x+a)^k) - \Delta(x^k) \in 2\mathbb{Z}[x]$. Applying Theorem 1.2 with $c = a$, $d = m$, and x replaced by $x + a$, we obtain the desired congruence. □

Proof of Theorem 1.1. Suppose first that $m > 0$. Then by Corollary 1.2,

$$f_k(x; a, m) \in \mathbb{Z}_2[x].$$

If p is an odd prime, then $E_k(x)/2 \in \mathbb{Z}_p[x]$, and also by (1.5)

$$\frac{m^{k+1}}{2} E_k\left(\frac{x+a}{m}\right) = \frac{m}{2} \sum_{l=0}^{k} \binom{k}{l} E_l(0) m^k \left(\frac{x+a}{m}\right)^{k-l} \in \mathbb{Z}_p[x].$$

Thus every coefficient of the polynomial $f_k(x) = f_k(x; a, m)$ is a p-integer for any prime p, which amounts to $f_k(x) \in \mathbb{Z}[x]$.

For the negative modulus case ($-m > 0$), using

$$E_k(1 - x) = (-1)^k E_k(x) \tag{1.10*}$$

and $2 \nmid m$, we may express $f_k(x; a, -m)$ as

$$-\frac{m^{k+1}}{2} E_k\left(\frac{x + (a+m)}{m}\right) + \frac{(-1)^{a+m}}{2} E_k(x).$$

Thus the positive modulus case applies and the proof is complete. □

In the spirit of Sun [S3], Theorem 1.2 can also be used to deduce some general congruences of Kummer's type for Euler polynomials. However, in order not to make this paper too long, we will not go into details.

2. Proof of Theorem 1.2

We introduce the Bernoulli polynomials $B_n(x)$ ($n \in \mathbb{N}_0$) by the generating power series

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}. \qquad (|z| < 2\pi) \tag{2.1*}$$

Their values $B_n(0)$ at $x = 0$ are rational numbers, called Bernoulli numbers and denoted by B_n; it is well known that $B_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$. Raabe's multiplication formula (counterpart of (1.6)) reads

$$m^{n-1} \sum_{r=0}^{m-1} B_n \left(\frac{x+r}{m} \right) = B_n(x) \text{ for any } m \in \mathbb{N}. \tag{2.2*}$$

Other properties include

$$\Delta(B_n(x)) = nx^{n-1} \tag{2.3*}$$

and

$$E_n(x) = \frac{2}{n+1} \left(B_{n+1}(x) - 2^{n+1} B_{n+1} \left(\frac{x}{2} \right) \right), \tag{2.4*}$$

the last one links the Euler and the Bernoulli polynomials.

Lemma 2.1. *Let k be a positive integer and y be a real number. Then*

$$kE_{k-1}(x + \{y\}) = 2(-1)^{\lfloor y \rfloor} \left(B_k(x + \{y\}) - 2^k B_k \left(\frac{x}{2} + \left\{ \frac{y}{2} \right\} \right) \right). \tag{2.5}$$

Proof. Observe that

$$\left\{ \frac{y}{2} \right\} < \frac{1}{2} \iff \{y\} = 2 \left\{ \frac{y}{2} \right\} \iff \lfloor y \rfloor = 2 \left\lfloor \frac{y}{2} \right\rfloor \iff 2 \mid \lfloor y \rfloor. \tag{2.6}$$

Hence, if $2 \mid \lfloor y \rfloor$, the right hand side of (2.5) is

$$2 \left(B_k(x + \{y\}) - 2^k B_k \left(\frac{x + \{y\}}{2} \right) \right),$$

which coincides with the left hand side of (2.5) in view of (2.4). Now, if $2 \nmid \lfloor y \rfloor$, then the right hand side of (2.5) is

$$-2 \left(B_k(x + \{y\}) - 2^k B_k \left(\frac{x + \{y\} + 1}{2} \right) \right).$$

We may express $B_k\left(\frac{x+\{y\}+1}{2}\right)$ as $2^{1-k} B_k\left(x + \{y\}\right) - B_k\left(\frac{x+\{y\}}{2}\right)$ by (2.2). Then what remains is $-2\left(2^k B_k\left(\frac{x+\{y\}}{2}\right) - B_k(x + \{y\})\right)$ which equals the left hand side of (2.5). This completes the proof. $\square$

Lemma 2.2. *Let $k \in \mathbb{N}_0$ and $m \in \mathbb{Z} \setminus \{0\}$. Then $(k+1)m^k E_k(x/m) \in \mathbb{Z}[x]$. Furthermore,*

$$(k+1)m^k E_k\left(\frac{x}{m}\right) - x^k - (x-km)\frac{(x+m)^k - x^k}{m} \in 2\mathbb{Z}[x].$$

Proof. First note that by (2.4),

$$(l+1)E_l(0) = 2(1-2^{l+1})B_{l+1}$$

for any $l \in \mathbb{N}_0$. Hence, if l is even then

$$(l+1)E_l(0) = [l=0];$$

while for odd l, we see that in the expression

$$\begin{aligned}(l+1)E_l(0) = 2^{l+1} - 1 - 2(2^{l+1}-1)\Bigg(B_{l+1} + \sum_{p-1\mid l+1}\frac{1}{p}\Bigg)\\ + 2\sum_{\substack{p\neq 2\\ p-1\mid l+1}}\frac{2^{l+1}-1}{p},\end{aligned}$$

the third and fourth terms are integers by the von Staudt–Clausen theorem (cf. [IR, pp. 233-236]) and Fermat's little theorem, respectively, so that $(l+1)E_l(0)$ is an odd integer.

Using (1.5) and writing $(k+1)\binom{k}{l}$ as $(l+1)\binom{k+1}{l+1}$, we find that

$$(k+1)m^k E_k\left(\frac{x}{m}\right) = \sum_{l=0}^{k}\binom{k+1}{l+1}(l+1)E_l(0)m^l x^{k-l}$$

lies in $\mathbb{Z}[x]$ and that

$$(k+1)m^k E_k\left(\frac{x}{m}\right) - \binom{k+1}{1}E_0(0)x^k - \sum_{\substack{l=1\\ 2\nmid l}}^{k}\binom{k+1}{l+1}m^l x^{k-l} \in 2\mathbb{Z}[x].$$

Now, since

$$\begin{aligned}\sum_{\substack{l=1\\ 2\nmid l}}^{k}\binom{k+1}{l+1}m^l x^{k-l} &= \frac{(x+m)^{k+1} + (x-m)^{k+1}}{2m} - \frac{x^{k+1}}{m}\\ &= \frac{(x+m)^{k+1} - x^{k+1}}{m} - \binom{k+1}{1}(x+m)^k\end{aligned}$$

$$-\sum_{j=2}^{k+1}\binom{k+1}{j}(-2m)^j(x+m)^{k+1-j},$$

the third term lying in $2\mathbb{Z}[x]$, we conclude that what we should subtract from $(k+1)m^kE_k\left(\frac{x}{m}\right)$ is

$$\begin{aligned}&\binom{k+1}{1}E_0(0)x^k+\frac{(x+m)^{k+1}-x^{k+1}}{m}-\binom{k+1}{1}(x+m)^k\\&=x^k+(x-km)\frac{(x+m)^k-x^k}{m},\end{aligned}$$

as asserted, and the proof is complete. □

Lemma 2.3 ([S3, Theorem 4.1])**.** *Let $k\in\mathbb{N}_0$, $d,m,n\in\mathbb{N}$, $d\mid n$, $m\mid qn$, and $2\nmid d$ or $2\nmid q$ or $2\mid\frac{qn}{m}$. Put $\bar{d}=(d,qn/m)$ and $\bar{m}=(m,qn/d)$. Then, for any real number y, the polynomial*

$$L(x,y)=\frac{1}{k+1}\left(dm^kB_{k+1}\left(\frac{x}{m}\right)-\bar{d}\bar{m}^kB_{k+1}\left(\frac{x}{\bar{m}}-\left\lfloor\frac{y}{\bar{m}}\right\rfloor\right)\right)\tag{2.7}$$

is in $\mathbb{Z}_q[x]$ and is congruent to

$$\begin{aligned}R(x,y)=\sum_{j=0}^{qn/m-1}(x+jm)^k\left(\left\lfloor\frac{y+jm}{qn/d}\right\rfloor+\frac{1-d}{2}\right)-\frac{q}{3}\,[3\mid d]\,\frac{n}{d}\cdot\frac{qn}{m}kx^{k-1}\\+\frac{q}{2}k[d\sim_2 n]\Big(\left[2\mid n\ \wedge\ 2\|\frac{qn}{m}\right]x^{k-1}\\+[2\nmid m\ \wedge\ 2\|n\ \wedge\ 2\|q]\,\Delta(x^{k-1})\Big)\end{aligned}\tag{2.8}$$

modulo q.

Now we establish a result more general than Theorem 1.2 ((2.9) and (2.10) below are generalizations of (1.8) and (1.9), respectively).

Theorem 2.1. *Let $k\in\mathbb{N}_0$, $d,m,n\in\mathbb{N}$, $d\mid n$, $m\mid qn$, and $2\nmid d$ or $2\nmid q$ or $2\mid\frac{qn}{m}$. Put $\bar{d}=(d,qn/m)$ and $\bar{m}=(m,qn/d)$. Then for any real number y we have*

$$\begin{aligned}P_k(x,y):=&\frac{[2m\nmid qn]+d[2m\mid qn]}{2}m^kE_k\left(\frac{x}{m}\right)\\&-\frac{\bar{d}\bar{m}^k}{2}(-1)^{\lfloor y/\bar{m}\rfloor}E_k\left(\frac{x}{\bar{m}}-\left\lfloor\frac{y}{\bar{m}}\right\rfloor\right)\in\mathbb{Z}_q[x].\end{aligned}$$

Moreover, if qn/m is odd then

$$P_k(x,y) \equiv \sum_{\substack{j=0\\ 2\nmid (d-1)j+\lfloor \frac{y+jm}{qn/d}\rfloor}}^{qn/m-1} (-1)^j (x+jm)^k + \frac{q}{2}[d \sim_2 n \ \wedge \ d \not\equiv 0,1 \pmod 4] k x^{k-1} \pmod q; \quad (2.9)$$

if qn/m is even then

$$\begin{aligned} &P_k(x,y) \\ &\equiv \sum_{j=0}^{qn/m-1} (-1)^{j-1}(x+jm)^k \left(\left\lfloor \frac{y+jm}{qn/d} \right\rfloor + [2 \mid d \ \wedge \ 2 \mid j](-1)^{\lfloor \frac{y+jm}{qn/d} \rfloor} \right) \\ &\quad + R_k(x) \pmod q, \end{aligned} \quad (2.10)$$

where

$$R_k(x) = \begin{cases} \frac{q}{2}[d \sim_2 n \ \wedge \ d \not\equiv 0,1 \pmod 4 \ \wedge \ 2 \nmid m]\Delta(x^k) & \text{if } 2 \mid k, \\ \frac{q}{2}[d \sim_2 n \ \wedge \ d \not\equiv 0,1 \pmod 4](\Delta(x^k) + [4 \mid q]\Delta(x^{k-1})) & \text{if } 2 \nmid km, \\ \frac{q}{2}[d \sim_2 n]([d \not\equiv 0,1 \pmod 4] + [2 \mid n \ \wedge \ 2 \| (qn/m)])x^{k-1} & \\ & \text{if } 2 \nmid k(m-1). \end{cases}$$

Proof. We observe that the $(k+1)$-degree terms in (2.7) cancel each other in view of $d/m = \bar{d}/\bar{m}$. Hence $L(x,y)$ is of degree at most k. Writing

$$L(x,y) = \sum_{i=0}^{k} a_i(y) x^i,$$

we see that $2^k L(\frac{x}{2}, \frac{y}{2}) \in \mathbb{Z}_q[x]$, and similarly $2^k R(\frac{x}{2}, \frac{y}{2}) \in \mathbb{Z}_q[x]$, and *a fortiori* that

$$L(x,y) - 2^{k+1} L\left(\frac{x}{2}, \frac{y}{2}\right) \in \mathbb{Z}_q[x]$$

and

$$L(x,y) - 2^{k+1} L\left(\frac{x}{2}, \frac{y}{2}\right) \equiv R(x,y) - 2^{k+1} R\left(\frac{x}{2}, \frac{y}{2}\right) \pmod q. \quad (2.11)$$

By (2.7) we can express the left hand side of (2.11) in such a way that we may apply Lemma 2.1 to deduce that

$$\begin{aligned}&L(x,y)-2^{k+1}L\left(\frac{x}{2},\frac{y}{2}\right)\\&=\frac{dm^k}{2}E_k\left(\frac{x}{m}\right)-(-1)^{\lfloor y/\bar{m}\rfloor}\frac{\bar{d}\bar{m}^k}{2}E_k\left(\frac{x}{\bar{m}}-\left\lfloor\frac{y}{\bar{m}}\right\rfloor\right).\end{aligned}\tag{2.12}$$

On the other hand, the right hand side of (2.11) is congruent to

$$\Sigma-\frac{q}{3}[3|d]\,\frac{n}{d}\cdot\frac{qn}{m}k\left(x^{k-1}-2^{k+1}\left(\frac{x}{2}\right)^{k-1}\right)+r(x)$$

modulo q, where

$$\begin{aligned}r(x)=\frac{q}{2}k[d\sim_2 n]\Big(&\left[2\mid n\ \wedge\ 2\|\frac{qn}{m}\right]x^{k-1}\\&+[2\nmid m\ \wedge\ 2\|n\ \wedge\ 2\|q]\Delta(x^{k-1})\Big)\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\Sigma=\sum_{j=0}^{qn/m-1}\Bigg(&(x+jm)^k\left(\left\lfloor\frac{y+jm}{qn/d}\right\rfloor+\frac{1-d}{2}\right)\\&-(x+2jm)^k\left(2\left\lfloor\frac{y+2jm}{2qn/d}\right\rfloor+1-d\right)\Bigg).\end{aligned}\tag{2.14}$$

Hence

$$R(x,y)-2^{k+1}R\left(\frac{x}{2},\frac{y}{2}\right)\equiv\Sigma+r(x)\pmod q.\tag{2.15}$$

By the counterpart of (2.6), the second term on the right hand side of (2.14) becomes

$$-\sum_{j=0}^{qn/m-1}(x+2jm)^k\left(\left\lfloor\frac{y+2jm}{qn/d}\right\rfloor-d+\left[2\,\Big|\,\left\lfloor\frac{y+2jm}{qn/d}\right\rfloor\right]\right)$$

in which we shall divide the sum into two parts via midpoint. Then

$$\begin{aligned}\Sigma=&\sum_{j=0}^{qn/m-1}(x+jm)^k\left(\left\lfloor\frac{y+jm}{qn/d}\right\rfloor+\frac{1-d}{2}\right)\\&-\sum_{\substack{i=0\\2|i}}^{qn/m-1}(x+im)^k\left(\left\lfloor\frac{y+im}{qn/d}\right\rfloor-d+\left[2\,\Big|\,\left\lfloor\frac{y+im}{qn/d}\right\rfloor\right]\right)\end{aligned}$$

$$- \sum_{\substack{i=0 \\ 2 \mid i+qn/m}}^{qn/m-1} \Big(x + \Big(i + \frac{qn}{m}\Big)\, m\Big)^k \Bigg(\left\lfloor \frac{y + (i + qn/m)m}{qn/d} \right\rfloor - d + \left[2 \mid \left\lfloor \frac{y+im}{qn/d} + d \right\rfloor \right] \Bigg).$$

Whence, writing

$$c_j(y) = \left\lfloor \frac{y + jm}{qn/d} \right\rfloor \tag{2.16}$$

for $j = 0, 1, \ldots$, we obtain

$$\begin{aligned} \Sigma \equiv & \sum_{j=0}^{qn/m-1} (x+jm)^k c_j(y)[2 \nmid j] + \frac{d-1}{2} \sum_{j=0}^{qn/m-1} (x+jm)^k (2\,[2 \mid j] - 1) \\ & + \sum_{j=0}^{qn/m-1} (x+jm)^k ([2 \mid j \ \wedge \ 2 \nmid c_j(y)]) \\ & - \sum_{j=0}^{qn/m-1} (x+jm)^k c_j(y) \left[2 \mid \frac{qn}{m} + j\right] \\ & - \sum_{j=0}^{qn/m-1} (x+jm)^k \left[2 \mid \frac{qn}{m} + j \ \wedge \ 2 \mid c_j(y) + d\right] \pmod{q}. \end{aligned}$$

Recombination of terms yields

$$\Sigma \equiv \sum_{j=0}^{qn/m-1} (x+jm)^k c_j(y) \left([2 \nmid j] - \left[2 \mid \frac{qn}{m} - j\right]\right) + \Sigma_1 + \Sigma_2 \pmod{q},$$

where

$$\begin{aligned} \Sigma_1 = & \frac{d-1}{2} \sum_{j=0}^{qn/m-1} (-1)^j (x+jm)^k, \\ \Sigma_2 = & \sum_{j=0}^{qn/m-1} (x+jm)^k \bigg([2 \mid j \ \wedge \ 2 \nmid c_j(y)] \\ & \qquad - \left[2 \mid \frac{qn}{m} - j \ \wedge \ 2 \mid c_j(y) + d\right] \bigg). \end{aligned} \tag{2.17}$$

Thus, in view of the equality $[2\nmid j]-[2\mid \frac{qn}{m}-j]=[2\mid \frac{qn}{m}](-1)^{j-1}$, we deduce that

$$\Sigma\equiv\left[2\mid\frac{qn}{m}\right]\sum_{j=0}^{qn/m-1}(-1)^{j-1}(x+jm)^k c_j(y)+\Sigma_1+\Sigma_2\pmod q\,. \tag{2.18}$$

Writing

$$\Sigma_1=\frac{d-1}{2}m^k\sum_{j=0}^{qn/m-1}(-1)^j\left(\frac{x}{m}+j\right)^k$$

and applying (1.4) and (1.5) successively, we obtain

$$\begin{aligned}\Sigma_1&=\frac{d-1}{2}\cdot\frac{m^k}{2}\sum_{j=0}^{qn/m-1}\left((-1)^jE_k\left(\frac{x}{m}+j\right)-(-1)^{j+1}E_k\left(\frac{x}{m}+j+1\right)\right)\\&=\frac{d-1}{2}\cdot\frac{m^k}{2}\left(E_k\left(\frac{x}{m}\right)-(-1)^{qn/m}E_k\left(\frac{x}{m}+\frac{qn}{m}\right)\right)\\&=\frac{d-1}{2}\cdot\frac{m^k}{2}\left(E_k\left(\frac{x}{m}\right)-(-1)^{qn/m}\sum_{l=0}^{k}\binom{k}{l}E_{k-l}\left(\frac{x}{m}\right)\left(\frac{qn}{m}\right)^l\right);\end{aligned}$$

whence separating the term with $l=0$,

$$\begin{aligned}\Sigma_1=&\frac{d-1}{2}\cdot\left[2\nmid\frac{qn}{m}\right]m^kE_k\left(\frac{x}{m}\right)\\&-(-1)^{qn/m}\sum_{0<l\le k}\binom{k}{l-1}(k-l+1)m^{k-l}E_{k-l}\left(\frac{x}{m}\right)\frac{d-1}{2}\cdot\frac{n^lq^l}{2l}.\end{aligned} \tag{2.19}$$

Now, by [S3, Lemma 2.1], $q^{l-1}/l\equiv 0\pmod q$ for $l>2$, so that if $0<l\le k$, then

$$\begin{aligned}\frac{d-1}{2}\cdot\frac{n^lq^l}{2l}&=\binom{d}{2}\frac{n^l}{d}\cdot\frac{q}{2}\cdot\frac{q^{l-1}}{l}\\&\equiv\frac{q}{2}\left[d\sim_2 n\ \wedge\ d\not\equiv 0,1\ (\mathrm{mod}\ 4)\right]\\&\qquad\times\left([l=1]+[l=2\ \wedge\ 2\nmid n\ \wedge\ 2\|q]\right)\pmod q,\end{aligned} \tag{2.20}$$

and in particular, only two terms with $l=1,2$ appear on the right hand side of (2.19) modulo q. Hence the sum on the right hand side of (2.19) is congruent to

$$\frac{d-1}{2}\cdot\frac{qn}{2}[k>0]km^{k-1}E_{k-1}\left(\frac{x}{m}\right)$$
$$+\frac{d-1}{2}\cdot\frac{n^2q^2}{4}[k>1]k(k-1)m^{k-2}E_{k-2}\left(\frac{x}{m}\right)$$

modulo q. Thus, by Lemma 2.2, Σ_1 is congruent to

$$\frac{d-1}{2}\left[2\nmid\frac{qn}{m}\right]m^kE_k\left(\frac{x}{m}\right)-(-1)^{qn/m}\frac{d-1}{2}$$
$$\times\left(\frac{nq}{2}[k>0]\left(x^{k-1}+(x-(k-1)m)\frac{(x+m)^{k-1}-x^{k-1}}{m}\right)\right.$$
$$\left.+\frac{n^2q^2}{4}[k>1]k\left(x^{k-2}+(x-(k-2)m)\frac{(x+m)^{k-2}-x^{k-2}}{m}\right)\right)$$

modulo q. By (2.20) with $l=1,2$, the second term of the above expression is congruent to $-(-1)^{qn/m}\bar{r}(x)$ modulo q, where

$$\begin{aligned}\bar{r}(x)=&\frac{q}{2}\left[d\sim_2 n\ \wedge\ d\not\equiv 0,1\ (\mathrm{mod}\ 4)\ \wedge\ 2\mid k\right]\frac{(x+m)^k-x^k}{m}\\&+\frac{q}{2}[d\sim_2 n\ \wedge\ d\not\equiv 0,1\ (\mathrm{mod}\ 4)\ \wedge\ 2\nmid k]\\&\qquad\times\frac{x(x+m)^{k-1}-(x+m)x^{k-1}}{m}\\&+\frac{q}{2}[2\|q\ \wedge\ 2\nmid kn\ \wedge\ 4\mid d+1]\frac{(x+m)^{k-1}-x^{k-1}}{m}.\end{aligned}\tag{2.21}$$

Thus

$$\Sigma_1\equiv\frac{d-1}{2}[2\nmid\frac{qn}{m}]m^kE_k\left(\frac{x}{m}\right)+\bar{r}(x)\pmod{q}.\tag{2.22}$$

Now, from (2.11)–(2.13), (2.18) and (2.22), it follows that

$$\frac{dm^k}{2}E_k\left(\frac{x}{m}\right)-(-1)^{\lfloor y/\bar{m}\rfloor}\frac{\bar{d}\bar{m}^k}{2}E_k\left(\frac{x}{\bar{m}}-\left\lfloor\frac{y}{\bar{m}}\right\rfloor\right)\equiv\Sigma+r(x)$$
$$\equiv\left[2\mid\frac{qn}{m}\right]\sum_{j=0}^{qn/m-1}(-1)^{j-1}(x+jm)^kc_j(y)$$
$$+\frac{d-1}{2}\left[2\nmid\frac{qn}{m}\right]m^kE_k\left(\frac{x}{m}\right)+r(x)+\bar{r}(x)$$
$$+\sum_{j=0}^{qn/m-1}(x+jm)^k\Big([2\mid j\ \wedge\ 2\nmid c_j(y)]$$

$$-\left[2 \mid \frac{qn}{m} - j \ \wedge \ 2 \mid c_j(y) + d\right]\Big) \pmod q.$$

With the help of the binomial theorem, we can easily verify that $r(x) + \bar{r}(x) \equiv R_k(x) \pmod q$. If qn/m is odd, then either $2 \nmid q$ or $2 \mid m$, whence

$$R_k(x) \equiv \frac{q}{2}[d \sim_2 n \ \wedge \ d \not\equiv 0, 1 \pmod 4]kx^{k-1} \pmod q.$$

and the desired results follow. □

Proof of Theorem 1.2. Just apply Theorem 2.1 with $n = m$ and $y = cm/d$.

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