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POLYNOMIAL EXTENSION OF FLECK'S CONGRUENCE

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Abstract. Let p be a prime, and let $f(x)$ be an integer-valued polynomial. By a combinatorial approach, we obtain a nontrivial lower bound of the p-adic order of the sum

$$\sum_{k\equiv r\ (\mathrm{mod}\ p^{\beta})}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^{\alpha}}\right\rfloor\right),$$

where $\alpha\geqslant\beta\geqslant 0$, $n\geqslant p^{\alpha-1}$ and $r\in\mathbb{Z}$. This polynomial extension of Fleck's congruence has various backgrounds and several consequences such as

$$\sum_{k\equiv r\ (\mathrm{mod}\ p^{\alpha})}\binom{n}{k}a^k\equiv 0\ \left(\mathrm{mod}\ p^{\left\lfloor\frac{n-p^{\alpha-1}}{\varphi(p^{\alpha})}\right\rfloor}\right)$$

provided that $\alpha>1$ and $a\equiv -1\ (\mathrm{mod}\ p)$.

1. Introduction

As usual, we let $\binom{x}{0}=1$ and

$$\binom{x}{k}=\frac{x(x-1)\cdots(x-k+1)}{k!}\quad\text{for every } k=1,2,3,\ldots.$$

For convenience, we also set $\binom{x}{k}=0$ for any negative integer k.

Let p be a prime and r be an integer. In 1913 A. Fleck (cf. Dickson [D, p. 274]) discovered that

$$\sum_{k\equiv r\ (\mathrm{mod}\ p)}\binom{n}{k}(-1)^k\equiv 0\ \left(\mathrm{mod}\ p^{\left\lfloor\frac{n-1}{p-1}\right\rfloor}\right)\tag{1.1}$$

for all $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, where $\lfloor\cdot\rfloor$ is the well-known floor function. Sums of the form $\sum_{k\equiv r\ (\mathrm{mod}\ m)}\binom{n}{k}$ or $\sum_{k\equiv r\ (\mathrm{mod}\ m)}\binom{n}{k}(-1)^k$ (with $m\in$

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$\mathbb{Z}^+$) have various applications in number theory and combinatorics (see, e.g., [SS], [H] and [S02]).

In 1977, by a very complicated method, C. S. Weisman [W] extended Fleck's congruence to prime power moduli in the following way:

$$\sum_{k\equiv r\ (\mathrm{mod}\ p^{\alpha})}\binom{n}{k}(-1)^k\equiv 0\ \left(\mathrm{mod}\ p^{\lfloor\frac{n-p^{\alpha-1}}{\varphi(p^{\alpha})}\rfloor}\right),\tag{1.2}$$

where $\alpha,n\in\mathbb{N}=\{0,1,2,\dots\}$ and $n\geqslant p^{\alpha-1}$, and φ denotes Euler's totient function. Unaware of Fleck's previous work, Weisman was motivated by studying the relation between two different ways (Mahler's and van der Put's) to express a p-adically continuous function.

Quite recently, in his lecture notes on Fontaine's rings and p-adic L-functions given at Irvine (Spring, 2005), D. Wan got the following new extension of Fleck's congruence:

$$\sum_{k\equiv r\ (\mathrm{mod}\ p)}\binom{n}{k}(-1)^k\binom{(k-r)/p}{l}\equiv 0\ \left(\mathrm{mod}\ p^{\lfloor\frac{n-lp-1}{p-1}\rfloor}\right),\tag{1.3}$$

where $l,n\in\mathbb{N}$ and $n>lp$. Wan was led to this when trying to understand a sharp estimate for the ψ-operator in Fontaine's theory of (ϕ,Γ)-modules.

For a prime p, we let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the field of p-adic numbers and the ring of p-adic integers respectively; the p-adic order of $\omega\in\mathbb{Q}_p$ is defined by $\mathrm{ord}_p(\omega)=\sup\{a\in\mathbb{Z}:\ \omega/p^a\in\mathbb{Z}_p\}$ (whence $\mathrm{ord}_p(0)=+\infty$). Throughout this paper, the Kronecker symbol $\delta_{m,n}$ with $m,n\in\mathbb{N}$ takes 1 or 0 according as $m=n$ or not.

Clearly both Weisman's and Wan's extensions of Fleck's congruence follow from the special case $\alpha=\beta$ of the following theorem, which we will establish by a combinatorial approach.

Theorem 1.1. *Let p be a prime, and let $f(x)\in\mathbb{Q}_p[x]$, $\deg f\leqslant l\in\mathbb{N}$ and $f(a)\in\mathbb{Z}_p$ for all $a\in\mathbb{Z}$. Provided that $\alpha,\beta\in\mathbb{N}$ and $\alpha\geqslant\beta$, we have*

$$\sum_{k\equiv r\ (\mathrm{mod}\ p^{\beta})}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^{\alpha}}\right\rfloor\right)\in p^{\lfloor\frac{n-p^{\alpha-1}-l}{\varphi(p^{\alpha})}\rfloor-(l-1)\alpha-\beta}\mathbb{Z}_p\tag{1.4}$$

for all integers $n\geqslant p^{\alpha-1}$ and r; moreover, we can substitute $\delta_{\beta,0}$ for the first l in (1.4) *if α is greater than one.*

By Theorem 1.1 in the case $\alpha=\beta=r=0$, if $f(x)\in\mathbb{Z}[x]$ and $f(x)\neq 0$, then for any integer $n>\deg f+1$ we have $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)=0$ since the sum is divisible by all primes. In fact, a known identity due to L. Euler (cf. [LW, pp. 90-91]) states that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^l=\begin{cases}(-1)^n\,n! & \text{if } l=n\in\mathbb{N},\\ 0 & \text{if } 0\leqslant l<n.\end{cases}$$

Now we derive more consequences of Theorem 1.1.

Corollary 1.1. *Let* p *be a prime,* $m \in \mathbb{Z}^+$ *and* $\alpha = \operatorname{ord}_p(m)$*. Let* $l, n \in \mathbb{N}$ *and* $r \in \mathbb{Z}$*. Then*

$$\begin{aligned} \operatorname{ord}_p\Bigg(\sum_{k\equiv r \,(\mathrm{mod}\ p^\alpha)} \binom{n}{k}(-1)^k B_l\left(\frac{k-r}{m}\right)\Bigg) \\ \geqslant \left\lfloor \frac{n-p^{\alpha-1}-l(\delta_{\alpha,0}+\delta_{\alpha,1})}{\varphi(p^\alpha)} \right\rfloor - l\alpha, \end{aligned} \tag{1.5}$$

where $B_l(x)$ *is the Bernoulli polynomial of degree* l*.*

Proof. (1.5) holds trivially if $n < p^{\alpha-1}$. Below we suppose $n \geqslant p^{\alpha-1}$.

When $l = 0$, (1.5) reduces to Weisman's congruence (1.2). In the case $\alpha = 0$, if the lower bound in (1.5) is nonnegative (i.e., $l < n$) then the summation in (1.5) vanishes by Euler's identity.

Now we assume $l\alpha \neq 0$, and let $B_l = B_l(0)$ be the lth Bernoulli number. Note that $m_0 = m/p^\alpha$ is relatively prime to p. For any $a \in \mathbb{Z}$ we have $B_l(a/m_0) - B_l \in \mathbb{Z}_p$, because

$$m_0^l\left(B_l\left(\frac{a}{m_0}\right) - B_l\right) = \left(m_0^l B_l\left(\frac{a}{m_0}\right) - B_l\right) - \left(m_0^l B_l(0) - B_l\right) \in \mathbb{Z}_p$$

by [S03, Corollary 1.3]. Applying Theorem 1.1 with $f(x) = B_l(x/m_0) - B_l$ and $\beta = \alpha$, we get that

$$\begin{aligned} \operatorname{ord}_p\Bigg(\sum_{k\equiv r \,(\mathrm{mod}\ p^\alpha)} \binom{n}{k}(-1)^k B_l\left(\frac{k-r}{m}\right) - B_l\Sigma\Bigg) \\ \geqslant \left\lfloor \frac{n-p^{\alpha-1}-l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - l\alpha, \end{aligned}$$

where $\Sigma = \sum_{k\equiv r \,(\mathrm{mod}\ p^\alpha)} \binom{n}{k}(-1)^k$. Recall that $pB_l \in \mathbb{Z}_p$ by the von Staudt–Clausen theorem (cf. [IR, pp. 233-236]). This, together with (1.2), shows that

$$\operatorname{ord}_p(B_l\Sigma) \geqslant \operatorname{ord}_p(\Sigma) - 1 \geqslant \left\lfloor \frac{n-p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor - 1 \geqslant \left\lfloor \frac{n-p^{\alpha-1}-l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - l\alpha.$$

So the desired (1.5) follows. □

Corollary 1.2. *Let* p *be a prime, and let* $f(x) \in \mathbb{Q}_p[x]$, $\deg f = l \geqslant 0$ *and* $f(a) \in \mathbb{Z}_p$ *for all* $a \in \mathbb{Z}$*. Let* $\alpha \in \mathbb{N}$ *and* $r \in \mathbb{Z}$*. Then, for any integer* $n \geqslant p^{\alpha-1}$*, we have*

$$\begin{aligned} \operatorname{ord}_p\Bigg(\sum_{k=0}^{n} \binom{n}{k}(-1)^k (k-r, p^\alpha) f\left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor\right)\Bigg) \\ \geqslant \left\lfloor \frac{n-p^{\alpha-1}-l(\delta_{\alpha,0}+\delta_{\alpha,1})}{\varphi(p^\alpha)} \right\rfloor - (l-1)\alpha - 1, \end{aligned} \tag{1.6}$$

where $(k-r,p^\alpha)$ *is the greatest common divisor of* $k-r$ *and* p^α*.*

Proof. Let $g(1)=p$ and $g(p^\beta)=p-1$ if $0<\beta\leqslant\alpha$. By Theorem 1.1, the p-adic order of

$$\begin{aligned}&\sum_{\beta=0}^{\alpha}g(p^\beta)p^\beta\sum_{k\equiv r\ (\mathrm{mod}\ p^\beta)}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^\alpha}\right\rfloor\right)\\&=\sum_{k=0}^{n}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^\alpha}\right\rfloor\right)\sum_{d\mid(k-r,p^\alpha)}g(d)d\end{aligned}$$

is at least

$$\nu=\left\lfloor\frac{n-p^{\alpha-1}-l(\delta_{\alpha,0}+\delta_{\alpha,1})}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha.$$

We note in passing that in the case $\alpha>1$,

$$\mathrm{ord}_p(g(p^0))+\left\lfloor\frac{n-p^{\alpha-1}-\delta_{0,0}}{\varphi(p^\alpha)}\right\rfloor\geqslant\left\lfloor\frac{n-p^{\alpha-1}}{\varphi(p^\alpha)}\right\rfloor.$$

Now, since

$$\sum_{d\mid(k-r,p^\alpha)}g(d)d=p+\sum_{1<d\mid(k-r,p^\alpha)}(p-1)d=\sum_{d\mid(k-r,p^\alpha)}\varphi(d)p=(k-r,p^\alpha)p,$$

by the above the sum in (1.6) has p-adic order at least $\nu-1$. □

Corollary 1.3. *Let* p *be a prime, and let* α,β,a,n,r *be integers for which*

$$\alpha>1,\ \alpha\geqslant\beta\geqslant0,\ a\equiv\ 1\ (\mathrm{mod}\ p^\alpha),\ n\geqslant p^{\alpha-1}\ \text{and}\ r<p^\beta.$$

Then we have the congruence

$$\sum_{k\equiv r\ (\mathrm{mod}\ p^\beta)}\binom{n}{k}(-1)^k a^{\lfloor\frac{k-r}{p^\alpha}\rfloor}\equiv0\ \left(\mathrm{mod}\ p^{\left\lfloor\frac{n-p^{\alpha-1}-\delta_{\beta,0}}{\varphi(p^\alpha)}\right\rfloor+\alpha-\beta}\right).\tag{1.7}$$

Proof. When $a=1$, (1.7) holds by Theorem 1.1 in the case $l=0$. So it suffices to show that

$$D:=\sum_{k\equiv r\ (\mathrm{mod}\ p^\beta)}\binom{n}{k}(-1)^k\left(a^{\lfloor\frac{k-r}{p^\alpha}\rfloor}-1\right)$$

is divisible by p^λ where

$$\lambda=\left\lfloor\frac{n-p^{\alpha-1}-\delta_{\beta,0}}{\varphi(p^\alpha)}\right\rfloor+\alpha-\beta.$$

Write $a = 1 + p^\alpha b$ with $b \in \mathbb{Z}$. Then

$$\begin{aligned} D &= \sum_{k \equiv r \ (\mathrm{mod}\ p^\beta)} \binom{n}{k} (-1)^k \sum_{0 < l \leqslant \lfloor \frac{k-r}{p^\alpha} \rfloor} \binom{\lfloor (k-r)/p^\alpha \rfloor}{l} (p^\alpha b)^l \\ &= \sum_{0 < l \leqslant \lfloor \frac{n-r}{p^\alpha} \rfloor} p^{l\alpha} b^l \sum_{k \equiv r \ (\mathrm{mod}\ p^\beta)} \binom{n}{k} (-1)^k \binom{\lfloor (k-r)/p^\alpha \rfloor}{l}. \end{aligned}$$

For each $0 < l \leqslant \lfloor (n-r)/p^\alpha \rfloor$, applying Theorem 1.1 with $f(x) = \binom{x}{l}$ we find that

$$p^{l\alpha} \sum_{k \equiv r \ (\mathrm{mod}\ p^\beta)} \binom{n}{k} (-1)^k \binom{\lfloor (k-r)/p^\alpha \rfloor}{l} \equiv 0 \pmod{p^\lambda}.$$

Therefore $D \equiv 0 \pmod{p^\lambda}$. This concludes the proof. □

Let $a \in \mathbb{Z}$ be congruent to 1 modulo a prime p. By induction, $a^{p^\alpha} \equiv 1 \pmod{p^{\alpha+1}}$ for any $\alpha \in \mathbb{N}$. Let $n, r \in \mathbb{Z}$ and $n \geqslant p^{\alpha-1}$. If $\alpha \geqslant 2$, then by Corollary 1.3 in the case $\beta = \alpha$ we have

$$\sum_{k \equiv r \ (\mathrm{mod}\ p^\alpha)} \binom{n}{k} (-a)^k \equiv 0 \left(\mathrm{mod}\ p^{\lfloor \frac{n - p^{\alpha-1}}{\varphi(p^\alpha)} \rfloor} \right). \tag{1.8}$$

By the binomial theorem, (1.8) is also valid with $\alpha = 0$. We remark that (1.8) also holds when $\alpha = 1$, as pointed out by Fleck (cf. [D, p. 274]).

In the next section we will provide some lemmas. Section 3 is devoted to our proof of Theorem 1.1.

2. Some Lemmas

Let us recall the following well-known convolution identity of Chu and Vandermonde (see, e.g., [GKP, (5.27)]):

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n} \quad \text{for all } n = 0, 1, 2, \ldots.$$

This can be seen by comparing the power series expansions of $(1+t)^x(1+t)^y$ and $(1+t)^{x+y}$.

Lemma 2.1. *Let $f(x)$ be a function from $\mathbb{Z}$ to a field, and let $m, n \in \mathbb{Z}^+$. Then, for any $r \in \mathbb{Z}$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor \right) = \sum_{k \equiv \bar{r} \ (\mathrm{mod}\ m)} \binom{n-1}{k} (-1)^{k-1} \Delta f\left(\frac{k - \bar{r}}{m} \right),$$

where $\bar{r} = r + m - 1$ *and* $\Delta f(x) = f(x+1) - f(x)$.

Proof. By the Chu-Vandermonde identity, for any $h \in \mathbb{N}$ we have

$$\sum_{k=0}^{h}\binom{n}{k}(-1)^k = (-1)^h\sum_{k=0}^{h}\binom{n}{k}\binom{-1}{h-k} = (-1)^h\binom{n-1}{h}.$$

Therefore

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{m}\right\rfloor\right) = \sum_{j\in\mathbb{Z}} c_j f(j),$$

where

$$\begin{aligned}c_j = &\sum_{\substack{k\in\mathbb{Z}\\ \lfloor\frac{k-r}{m}\rfloor=j}}\binom{n}{k}(-1)^k\\ = &\sum_{0\leqslant k<(j+1)m+r}\binom{n}{k}(-1)^k - \sum_{0\leqslant k<jm+r}\binom{n}{k}(-1)^k\\ =&(-1)^{(j+1)m+r-1}\binom{n-1}{(j+1)m+r-1} - (-1)^{jm+r-1}\binom{n-1}{jm+r-1}.\end{aligned}$$

(Note that $\binom{n-1}{i} \neq 0$ only for those $i \in \{0, \ldots, n-1\}$.) So we have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{m}\right\rfloor\right)\\ =&\sum_{j\in\mathbb{Z}}(-1)^{(j+1)m+r-1}\binom{n-1}{(j+1)m+r-1}f(j)\\ &-\sum_{j\in\mathbb{Z}}(-1)^{jm+r-1}\binom{n-1}{jm+r-1}f(j)\\ =&\sum_{k\equiv\bar{r}\ (\mathrm{mod}\ m)}\binom{n-1}{k}(-1)^k\left(f\left(\frac{k-\bar{r}}{m}\right) - f\left(\frac{k-\bar{r}}{m}+1\right)\right)\\ =&\sum_{k\equiv\bar{r}\ (\mathrm{mod}\ m)}\binom{n-1}{k}(-1)^{k-1}\Delta f\left(\frac{k-\bar{r}}{m}\right).\end{aligned}$$

This proves the desired identity. □

It is interesting to compare the identity in Lemma 2.1 with the following observation

$$\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv r\ (\mathrm{mod}\ m)}}\Delta f\left(\frac{k-r}{m}\right) = f\Big(\left\lfloor\frac{n-r}{m}\right\rfloor+1\Big) - f\Big(\left\lfloor\frac{-r-1}{m}\right\rfloor+1\Big),$$

which appeared in the author's proof of [S03, Lemma 3.1].

Lemma 2.2. *Let p be a prime and α be a positive integer. Then, for any $k = 0, 1, \dots, \varphi(p^\alpha)$, we have*

$$\binom{\varphi(p^\alpha)}{k} \equiv \begin{cases} (-1)^k \pmod{p} & \textit{if } p^{\alpha-1} \mid k, \\ 0 \pmod{p} & \textit{otherwise.} \end{cases}$$

Proof. Let $k = k_0 + k_1 p + \cdots + k_{\alpha-1} p^{\alpha-1}$ be the p-adic expansion of k, where $k_0, k_1, \dots, k_{\alpha-1} \in \{0, \dots, p-1\}$. By a well-known theorem of E. Lucas (see, e.g., [HS]),

$$\begin{aligned} \binom{\varphi(p^\alpha)}{k} &= \binom{\sum_{0 \leqslant j < \alpha-1} 0p^j + (p-1)p^{\alpha-1}}{\sum_{0 \leqslant j < \alpha-1} k_j p^j + k_{\alpha-1} p^{\alpha-1}} \\ &\equiv \binom{p-1}{k_{\alpha-1}} \prod_{0 \leqslant j < \alpha-1} \binom{0}{k_j} \pmod{p}. \end{aligned}$$

If $p^{\alpha-1} \nmid k$, then $k_j > 0$ for some $j < \alpha - 1$, and hence $\binom{\varphi(p^\alpha)}{k} \equiv 0 \pmod{p}$. When $p^{\alpha-1} \mid k$, we have $k_j = 0$ for all $j < \alpha - 1$, and thus

$$\begin{aligned} \binom{\varphi(p^\alpha}{k} \equiv \binom{p-1}{k_{\alpha-1}} &= \prod_{0 < s \leqslant k_{\alpha-1}} \frac{p-s}{s} \pmod{p} \\ &\equiv (-1)^{k_{\alpha-1}} \equiv (-1)^{p^{\alpha-1} k_{\alpha-1}} = (-1)^k \pmod{p}. \end{aligned}$$

This completes the proof. □

3. Proof of Theorem 1.1

We use induction on $w_l(\alpha, \beta) := l(\alpha + 1) + \beta$.

In the case $w_l(\alpha, \beta) = 0$ (i.e., $l = \beta = 0$), the desired result is trivial because $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0$ for all $n \in \mathbb{Z}^+$.

Let w be a positive integer, and assume that the desired result holds whenever $w_l(\alpha, \beta) < w$. Now we deal with the case $w_l(\alpha, \beta) = w$.

Case 1: $\beta = 0$.

In this case, l is positive. Let $n \in \mathbb{N}$, $n \geqslant p^{\alpha-1}$, $r \in \mathbb{Z}$ and $\bar{r} = r + p^\alpha - 1$. By Lemma 2.1,

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor\right) \\ = &\sum_{k \equiv \bar{r} \pmod{p^\alpha}} \binom{n-1}{k} (-1)^{k-1} \Delta f\left(\frac{k-\bar{r}}{p^\alpha}\right). \end{aligned} \tag{3.1}$$

Clearly $\Delta f(x)$ is a polynomial of degree at most $l-1$, and $\Delta f(a) \in \mathbb{Z}_p$ for all $a \in \mathbb{Z}$. Also, $w_{l-1}(\alpha,\alpha) < w_l(\alpha,0) = w$. In view of (3.1) and the induction hypothesis,

$$\begin{aligned}&\operatorname{ord}_p\bigg(\sum_{k=0}^{n}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^\alpha}\right\rfloor\right)\bigg)\\ \geqslant&\left\lfloor\frac{(n-1)-p^{\alpha-1}-(l-1)}{\varphi(p^\alpha)}\right\rfloor-(l-2)\alpha-\alpha\\ =&\left\lfloor\frac{n-p^{\alpha-1}-l}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-0.\end{aligned}$$

(Note that this is trivial if $n-1 < p^{\alpha-1}$.) Similarly, when $\alpha > 1$, by (3.1) and the induction hypothesis we have

$$\begin{aligned}&\operatorname{ord}_p\bigg(\sum_{k=0}^{n}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^\alpha}\right\rfloor\right)\bigg)\\ \geqslant&\left\lfloor\frac{(n-1)-p^{\alpha-1}-\delta_{\alpha,0}}{\varphi(p^\alpha)}\right\rfloor-(l-2)\alpha-\alpha\\ =&\left\lfloor\frac{n-p^{\alpha-1}-\delta_{0,0}}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-0.\end{aligned}$$

Case 2: $0 < \beta \leqslant \alpha$.

If $l = 0$ (i.e., $f(x)$ is constant), then $w_l(\beta,\beta) = w_l(\alpha,\beta) = w$ and it suffices to handle the case $\alpha = \beta$. In fact, when $l = 0$, $n \geqslant p^{\alpha-1}$ and $r \in \mathbb{Z}$, provided that

$$\sum_{k\equiv r\,(\mathrm{mod}\ p^\beta)}\binom{n}{k}(-1)^k f\left(\frac{k-r}{p^\beta}\right) \in p^{\left\lfloor\frac{n-p^{\beta-1}}{\varphi(p^\beta)}\right\rfloor}\mathbb{Z}_p$$

we have

$$\sum_{k\equiv r\,(\mathrm{mod}\ p^\beta)}\binom{n}{k}(-1)^k f\left(\left\lfloor\frac{k-r}{p^\alpha}\right\rfloor\right) \in p^{\left\lfloor\frac{n-p^{\alpha-1}}{\varphi(p^\alpha)}\right\rfloor-(0-1)\alpha-\beta}\mathbb{Z}_p,$$

because

$$\frac{n-p^{\beta-1}}{\varphi(p^\beta)}-\frac{n-p^{\alpha-1}}{\varphi(p^\alpha)}=\frac{n}{p^{\alpha-1}}\sum_{0\leqslant s<\alpha-\beta}p^s \geqslant \alpha-\beta.$$

Below we simply let $(l-1)\alpha+\beta \geqslant 0$ (i.e., $\alpha = \beta$ if $l = 0$).

Let us use induction on $n \geqslant p^{\alpha-1}$. The desired result is trivial when $n-p^{\alpha-1} < \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$.

Below we let $n \geqslant p^\alpha$ and assume that the desired result holds for smaller values of n not less than $p^{\alpha-1}$. Note that $n' = n - \varphi(p^\beta) < n$ and also $n' \geqslant n - \varphi(p^\alpha) \geqslant p^{\alpha-1}$.

Let r be any integer, and set

$$S = \sum_{k \equiv r \,(\mathrm{mod}\ p^\beta)} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor\right). \tag{3.2}$$

By the Chu-Vandermonde identity,

$$\begin{aligned} S &= \sum_{k \equiv r \,(\mathrm{mod}\ p^\beta)} \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} \binom{n'}{k-j} (-1)^k f\left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor\right) \\ &= \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} \sum_{k \equiv r \,(\mathrm{mod}\ p^\beta)} \binom{n'}{k-j} (-1)^k f\left(\left\lfloor \frac{k-j-(r-j)}{p^\alpha} \right\rfloor\right) \\ &= \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} (-1)^j S_j, \end{aligned}$$

where

$$S_j = \sum_{k \equiv r-j \,(\mathrm{mod}\ p^\beta)} \binom{n'}{k} (-1)^k f\left(\left\lfloor \frac{k-(r-j)}{p^\alpha} \right\rfloor\right). \tag{3.3}$$

For any $j = 0, 1, \ldots, \varphi(p^\beta)$, by the induction hypothesis we have

$$\mathrm{ord}_p(S_j) \geqslant \gamma = \left\lfloor \frac{n' - p^{\alpha-1} - l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - (l-1)\alpha - \beta,$$

and Lemma 2.2 yields that

$$\binom{\varphi(p^\beta)}{j} \equiv \begin{cases} (-1)^j \pmod p & \text{if } p^{\beta-1} \mid j, \\ 0 \pmod p & \text{if } p^{\beta-1} \nmid j. \end{cases}$$

Thus, if $\gamma \geqslant 0$ then

$$S \equiv \sum_{j=0}^{p-1} \binom{\varphi(p^\beta)}{p^{\beta-1}j} (-1)^{p^{\beta-1}j} S_{p^{\beta-1}j} \equiv \sum_{j=0}^{p-1} S_{p^{\beta-1}j} \pmod{p^{\gamma+1}}.$$

Observe that

$$\sum_{j=0}^{p-1} S_{p^{\beta-1}j} = \sum_{k \equiv r \,(\mathrm{mod}\ p^{\beta-1})} \binom{n'}{k} (-1)^k f\left(\left\lfloor \frac{k-(r-p^{\beta-1}j_k)}{p^\alpha} \right\rfloor\right),$$

where j_k is the unique integer in $\{0,\ldots,p-1\}$ with $p^\beta \mid k-(r-p^{\beta-1}j_k)$. For $k\equiv r \pmod{p^{\beta-1}}$, clearly

$$\frac{k-r+p^{\beta-1}j_k}{p^\beta}=\frac{k-r'-p^{\beta-1}(p-1-j_k)}{p^\beta}=\left\lfloor\frac{k-r'}{p^\beta}\right\rfloor$$

where $r'=r-\varphi(p^\beta)$. Therefore $\sum_{j=0}^{p-1}S_{p^{\beta-1}j}=S'$, where

$$S'=\sum_{k\equiv r' \pmod{p^{\beta-1}}}\binom{n'}{k}(-1)^k f\left(\left\lfloor\frac{k-r'}{p^\alpha}\right\rfloor\right). \tag{3.4}$$

From the above it follows that

$$\mathrm{ord}_p(S-S')\geqslant\gamma+1\geqslant\left\lfloor\frac{n-p^{\alpha-1}-l\delta_{\alpha,1}}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-\beta.$$

Let $l_0=l$ if $\alpha=1$, and $l_0=\min\{l,\delta_{\beta-1,0}\}$ if $\alpha>1$. As $w_l(\alpha,\beta-1)<w_l(\alpha,\beta)=w$, by the induction hypothesis we have

$$\begin{aligned}\mathrm{ord}_p(S')&\geqslant\left\lfloor\frac{n'-p^{\alpha-1}-l_0}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-(\beta-1)\\&\geqslant\left\lfloor\frac{n-p^{\alpha-1}-l\delta_{\alpha,1}}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-\beta.\end{aligned}$$

(Note that if $\alpha>1=\delta_{\beta-1,0}$ then $\beta=1<\alpha$ and hence $n'-1+\varphi(p^\alpha)\geqslant n'+\varphi(p^\beta)=n$.)

Combining the above we finally obtain that

$$\mathrm{ord}_p(S)=\mathrm{ord}_p((S-S')+S')\geqslant\left\lfloor\frac{n-p^{\alpha-1}-l\delta_{\alpha,1}}{\varphi(p^\alpha)}\right\rfloor-(l-1)\alpha-\beta.$$

Since $\delta_{\beta,0}=0$, this concludes the induction step in Case 2.

The proof of Theorem 1.1 is now complete.

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