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RESTRICTED SUMSETS AND A CONJECTURE OF LEV

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Abstract. Let A, B, S be finite subsets of an abelian group G. Suppose that the restricted sumset

$$C = \{a + b\colon a \in A,\ b \in B,\ \text{and } a - b \notin S\}$$

is nonempty and some $c \in C$ can be written as $a + b$ with $a \in A$ and $b \in B$ in at most m ways. We show that if G is torsion-free or elementary abelian then $|C| \geqslant |A| + |B| - |S| - m$. We also prove that $|C| \geqslant |A| + |B| - 2|S| - m$ if the torsion subgroup of G is cyclic. In the case $S = \{0\}$ this provides an advance on a conjecture of Lev.

1. Introduction

Let A and B be finite nonempty subsets of an (additively written) abelian group G. The sumset of A and B is defined by

$$A + B = \{a + b\colon a \in A \text{ and } b \in B\}.$$

The Cauchy-Davenport theorem (cf. [N, pp. 43-48]), a basic result in additive combinatorial number theory, states that

$$|A + B| \geqslant \min\{p,\ |A| + |B| - 1\}$$

if $G = \mathbb{Z}/p\mathbb{Z}$ with p prime. Another theorem due to Kemperman and Scherk (cf. [Sc], [Ke] and [L2]) asserts that

$$|A + B| \geqslant |A| + |B| - \min_{c \in A+B} \nu_{A,B}(c), \tag{1.1}$$

where

$$\nu_{A,B}(c) = |\{(a, b) \in A \times B\colon a + b = c\}|; \tag{1.2}$$

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in particular, we have $|A+B| \geqslant |A|+|B|-1$ if some $c \in A+B$ can be uniquely written as $a+b$ with $a \in A$ and $b \in B$.

Now we define the restricted sumset

$$A \dotplus B = \{a+b\colon a \in A,\ b \in B,\ \text{and } a \neq b\}. \tag{1.3}$$

In 1964, Erdős and Heilbronn [EH] conjectured that if $G = \mathbb{Z}/p\mathbb{Z}$ with p prime then

$$|A \dotplus A| \geqslant \min\{p,\, 2|A|-3\}.$$

This is much more difficult than the Cauchy-Davenport theorem concerning unrestricted sumsets. It had been open for thirty years until Dias da Silva and Hamidoune [DH] confirmed it in 1994 using representations of symmetric groups. Later Alon, Nathanson and Ruzsa [ANR1, ANR2] developed a powerful polynomial method to give a simpler proof of the Erdős-Heilbronn conjecture (see also [A2]). They showed that if $G = \mathbb{Z}/p\mathbb{Z}$ with p prime then

$$|A \dotplus B| \geqslant \min\{p,\, |A|+|B|-2-\delta\},$$

where δ is 1 or 0 according to whether $|A| = |B|$ or not. The reader may consult [HS], [K1], [K2], [L1], [LS] and [SY] for various extensions of the Erdős-Heilbronn conjecture.

Motivated by the Kemperman-Scherk theorem and the Erdős-Heilbronn conjecture, Lev [L2] proposed the following interesting conjecture.

Conjecture 1.1 (Lev)**.** *Let G be an abelian group, and let A and B be finite nonempty subsets of G. Then we have*

$$|A \dotplus B| \geqslant |A|+|B|-2-\min_{c \in A+B} \nu_{A,B}(c). \tag{1.4}$$

This conjecture is known to be true for torsion-free abelian groups and elementary abelian 2-groups. It also holds when $|G|$ is prime, or G is cyclic and $|G| \leqslant 25$. (Cf. [L2].)

Now we state our main results.

Theorem 1.1. *Let A and B be finite nonempty subsets of a field F. Let $P(x,y) \in F[x,y]$ and*

$$C = \{a+b\colon a \in A,\ b \in B,\ \text{and } P(a,b) \neq 0\}. \tag{1.5}$$

If C is nonempty, then

$$|C| \geqslant |A|+|B|-\deg P-\min_{c \in C} \nu_{A,B}(c). \tag{1.6}$$

Remark 1.1. When $P(x,y)=1$, (1.6) becomes (1.1).

Notice the difference between the minima in (1.4) and (1.6): as $C \subseteq A+B$ we have $\min_{c \in A+B} \nu_{A,B}(c) \leqslant \min_{c \in C} \nu_{A,B}(c)$.

Theorem 1.2. *Let A and B be finite nonempty subsets of an abelian group G whose torsion subgroup*

$$\mathrm{Tor}(G) = \{g \in G\colon g \textit{ has a finite order}\}$$

is cyclic. For $i = 1, \ldots, l$ let m_i and n_i be nonnegative integers and let $d_i \in G$. Suppose that

$$C = \{a + b\colon a \in A,\ b \in B,\ \textit{and } m_i a - n_i b \neq d_i \textit{ for all } i = 1, \ldots, l\} \tag{1.7}$$

is nonempty. Then

$$|C| \geqslant |A| + |B| - \sum_{i=1}^{l}(m_i + n_i) - \min_{c \in C} \nu_{A,B}(c). \tag{1.8}$$

Remark 1.2. When A and B are finite subsets of $\mathbb{Z}$, the restricted sumset in (1.7) was first studied by Sun [Su1].

From Theorems 1.1 and 1.2 we deduce the following result on difference-restricted sumsets.

Theorem 1.3. *Let G be an abelian group, and let A, B, S be finite nonempty subsets of G with*

$$C = \{a + b\colon a \in A,\ b \in B,\ \textit{and } a - b \notin S\} \neq \emptyset. \tag{1.9}$$

(i) *If G is torsion-free or elementary abelian, then*

$$|C| \geqslant |A| + |B| - |S| - \min_{c \in C} \nu_{A,B}(c). \tag{1.10}$$

(ii) *If* $\mathrm{Tor}(G)$ *is cyclic, then*

$$|C| \geqslant |A| + |B| - 2|S| - \min_{c \in C} \nu_{A,B}(c). \tag{1.11}$$

Proof. Without loss of generality we can assume that G is generated by the finite set $A \cup B \cup S$.

If $G \cong \mathbb{Z}^n$, then we can simply view G as the ring of algebraic integers in an algebraic number field K with $[K : \mathbb{Q}] = n$. If $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ where p is a prime, then G is isomorphic to the additive group of the finite field with p^n elements. Thus part (i) follows from Theorem 1.1 in the case $P(x, y) = \prod_{s \in S}(x - y - s)$.

Let $d_1, \ldots, d_l$ be all the distinct elements of S. Applying Theorem 1.2 with $m_i = n_i = 1$ for all $i = 1, \ldots, l$ we immediately get the second part. $\square$

Remark 1.3. It is interesting to compare Theorem 1.3 in the case $S = \{0\}$ with Conjecture 1.1.

Concerning the set C given by (1.9), there are some known results of different types. When A, B, S are finite nonempty subsets of a field whose characteristic is an odd prime p, the authors [PS] proved that $|C| \geqslant \min\{p, |A|+|B|-|S|-q-1\}$, where q is the largest power of p not exceeding $|S|$. By modifying Károlyi's proof of [K1, Theorem 3], we can show that if $q > 1$ is a power of a prime p, and A, B, S are subsets of $\mathbb{Z}/q\mathbb{Z}$ with $\min\{|A|, |B|\} > |S|$, then $|C| \geqslant \min\{p, |A| + |B| - 2|S| - 1\}$.

We will give a key lemma in the next section and prove Theorems 1.1 and 1.2 in Section 3. Our proofs use a version of the polynomial method.

2. Some preparations

Our basic tool is as follows.

Combinatorial Nullstellensatz ([A1, Theorem 1.1])**.** *Let $A_1, \ldots, A_n$ be finite nonempty subsets of a field F, and set $g_i(x) = \prod_{a \in A_i}(x-a)$ for $i = 1, \ldots, n$. Then $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ vanishes over the Cartesian product $A_1 \times \cdots \times A_n$ if and only if it can be written in the form*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} g_i(x_i) h_i(x_1, \ldots, x_n)$$

where $h_i(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ and $\deg h_i \leqslant \deg f - \deg g_i$.

With help of the Combinatorial Nullstellensatz, we provide a lemma for our purposes.

Lemma 2.1. *Let A and B be finite nonempty subsets of a field F, and write*

$$\nu_i = |\{(a, b) \in A \times B \colon a + \lambda_i b = \mu_i\}| \tag{2.1}$$

for $i = 1, \ldots, k$ where $\lambda_i \in F \setminus \{0\}$ and $\mu_i \in F$. Let $P(x, y) \in F[x, y]$. Suppose that for any $i = 1, \ldots, k$ there are $a \in A$ and $b \in B$ with $P(a, b) \neq 0$ and $a + \lambda_i b = \mu_i$, and that for each $(a, b) \in A \times B$ with $P(a, b) \neq 0$ there is a unique $i \in \{1, \ldots, k\}$ with $a + \lambda_i b = \mu_i$. Then we have

$$k + \min\{\nu_1, \ldots, \nu_k\} \geqslant |A| + |B| - \deg P. \tag{2.2}$$

Proof. Clearly

$$f(x, y) := P(x, y) \prod_{j=1}^{k} (x + \lambda_j y - \mu_j)$$

vanishes over $A\times B$. Set $g_A(x)=\prod_{a\in A}(x-a)$ and $g_B(y)=\prod_{b\in B}(y-b)$. By the Combinatorial Nullstellensatz, there are $h_A(x,y),h_B(x,y)\in F[x,y]$ such that

$$f(x,y)=g_A(x)h_A(x,y)+g_B(y)h_B(x,y)$$

and

$$\max\{\deg g_A+\deg h_A,\ \deg g_B+\deg h_B\}\leqslant \deg f.$$

Fix $1\leqslant i\leqslant k$. Write $h_B(x,y)=\sum_{s,t\geqslant 0}c_{st}x^sy^t$ where $c_{st}\in F$. Then

$$h_B(x,y)=\sum_{s,t\geqslant 0}c_{st}((x+\lambda_iy-\mu_i)+\mu_i-\lambda_iy)^sy^t=(x+\lambda_iy-\mu_i)q(x,y)+r(y),$$

where $q(x,y)\in F[x,y]$, and $r(y)=h_B(\mu_i-\lambda_iy,y)$ has degree not greater than $\deg h_B$.

Now assume that $k+\nu_i<|A|+|B|-\deg P$. We want to deduce a contradiction. Set

$$A_0=\{a\in A\colon (\mu_i-a)/\lambda_i\notin B\}.$$

Obviously $|A_0|=|A|-\nu_i$ and $g_B((\mu_i-a)/\lambda_i)\neq 0$ for any $a\in A_0$. If $a\in A_0$, then

$$g_B\left(\frac{\mu_i-a}{\lambda_i}\right)h_B\left(a,\frac{\mu_i-a}{\lambda_i}\right)=f\left(a,\frac{\mu_i-a}{\lambda_i}\right)-g_A(a)h_A\left(a,\frac{\mu_i-a}{\lambda_i}\right)=0$$

and hence

$$r\left(\frac{\mu_i-a}{\lambda_i}\right)=h_B\left(a,\frac{\mu_i-a}{\lambda_i}\right)=0.$$

Since $\deg r\leqslant\deg f-\deg g_B<|A|-\nu_i=|A_0|$, we must have $r(y)=0$, i.e., $h_B(x,y)$ is divisible by $x+\lambda_iy-\mu_i$. Recall that there are $a_0\in A$ and $b_0\in B$ such that $P(a_0,b_0)\neq 0$ and $a_0+\lambda_ib_0=\mu_i$. Since $h_B(a_0,b_0)=0$, the polynomial $P(a_0,y)\prod_{j=1}^k(a_0+\lambda_jy-\mu_j)=f(a_0,y)=g_B(y)h_B(a_0,y)$ is divisible by $(y-b_0)^2$. As $a_0+\lambda_jb_0\neq\mu_j$ for any $j\neq i$, we must have $y-b_0\mid P(a_0,y)$, which contradicts the fact that $P(a_0,b_0)\neq 0$. $\square$

3. Proofs of Theorems 1.1–1.2

Proof of Theorem 1.1. Let $\mu_1,\dots,\mu_k$ be all the distinct elements of C. Applying Lemma 2.1 with $\lambda_1=\cdots=\lambda_k=1$, we find that

$$|C|+\min_{c\in C}\nu_{A,B}(c)\geqslant|A|+|B|-\deg P$$

which is equivalent to (1.6). $\square$

Proof of Theorem 1.2. Without loss of generality, we can assume that G is finitely generated, and furthermore that G is a subgroup of the multiplicative group of the field of complex numbers (see the proof of Theorem 1.1 of [Su2]); thus, C is the set

$$\{ab\colon a \in A,\ b \in B,\ \text{and } a^{m_i}b^{-n_i} \neq d_i \text{ for all } i = 1, \ldots, l\}.$$

Let $-\lambda_1, \ldots, -\lambda_k$ be all the distinct elements of C, and set

$$P(x, y) = \prod_{i=1}^{l}(x^{m_i}y^{n_i} - d_i).$$

Then, for each $j \in \{1, \ldots, k\}$, there are $a \in A$ and $b \in B$ such that $a + \lambda_j b^{-1} = 0$ and $P(a, b^{-1}) \neq 0$. If $a \in A$, $b \in B$ and $P(a, b^{-1}) \neq 0$, then there is a unique $j \in \{1, \ldots, k\}$ such that $\lambda_j = -ab$ (i.e., $a + \lambda_j b^{-1} = 0$). Applying Lemma 2.1 to the sets A and $B^{-1} = \{b^{-1}\colon b \in B\}$ with $\mu_1 = \cdots = \mu_k = 0$, we obtain that

$$k + \min_{1 \leqslant j \leqslant k} |\{(a, b) \in A \times B\colon a + \lambda_j b^{-1} = 0\}| \geqslant |A| + |B^{-1}| - \deg P.$$

Therefore

$$|C| + \min_{c \in C} |\{(a, b) \in A \times B\colon ab = c\}| \geqslant |A| + |B| - \sum_{i=1}^{l}(m_i + n_i)$$

as desired. $\square$

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