Acta Arith. 124(2006), no. 1, 41–57.

ON q-EULER NUMBERS, q-SALIÉ NUMBERS AND q-CARLITZ NUMBERS

HAO PAN AND ZHI-WEI SUN (NANJING)

ABSTRACT. Let $(a;q)_n = \prod_{0\leqslant k<n}(1-aq^k)$ for $n = 0,1,2,\ldots$. Define q-Euler numbers $E_n(q)$, q-Salié numbers $S_n(q)$ and q-Carlitz numbers $C_n(q)$ as follows:

$$\sum_{n=0}^{\infty} E_n(q)\frac{x^n}{(q,q)_n} = \left(\sum_{n=0}^{\infty}\frac{q^{n(2n-1)}x^{2n}}{(q;q)_{2n}}\right)^{-1},$$

$$\sum_{n=0}^{\infty} S_n(q)\frac{x^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^{2n}}{(q;q)_{2n}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(2n-1)}x^{2n}}{(q;q)_{2n}}$$

and

$$\sum_{n=0}^{\infty} C_n(q)\frac{x^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^{2n+1}}{(q;q)_{2n+1}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(2n+1)}x^{2n+1}}{(q;q)_{2n+1}}.$$

We show that

$$E_{2n}(q) - E_{2n+2^s t}(q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}}$$

for any nonnegative integers n, s, t with $2\nmid t$, where $[k]_q = (1-q^k)/(1-q)$; this is a q-analogue of Stern's congruence $E_{2n+2^s} \equiv E_{2n} + 2^s \pmod{2^{s+1}}$. We also prove that $(-q;q)_n = \prod_{0<k\leqslant n}(1+q^k)$ divides $S_{2n}(q)$ and the numerator of $C_{2n}(q)$; this extends Carlitz's result that 2^n divides the Salié number S_{2n} and the numerator of the Carlitz number C_{2n}. Our result on q-Salié numbers implies a conjecture of Guo and Zeng.

1. INTRODUCTION

The Euler numbers $E_0, E_1, E_2, \ldots$ are defined by

$$\sum_{n=0}^{\infty} E_n\frac{x^n}{n!} = \frac{2e^x}{e^{2x}+1} = \left(\frac{e^x+e^{-x}}{2}\right)^{-1} = \left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\right)^{-1};$$

2000 *Mathematics Subject Classifications*: Primary 11B65; Secondary 05A30, 11A07, 11B68.

The second author is responsible for communications, and supported by the National Science Fund for Distinguished Young Scholars (no. 10425103) and a Key Program of NSF in P. R. China.

they are all integers because there holds the recursion

$$\sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=\delta_{n,0}\quad (n\in\mathbb{N}=\{0,1,2,\ldots\}),$$

where the Kronecker symbol $\delta_{n,m}$ is 1 or 0 according as $n=m$ or not. It is easy to see that $E_{2k+1}=0$ for every $k=0,1,2,\ldots$. In 1871 Stern [St] obtained an interesting arithmetic property of the Euler numbers:

$$E_{2n+2^s}\equiv E_{2n}+2^s \pmod{2^{s+1}}\ \text{ for any } n,s\in\mathbb{N}; \tag{1.1}$$

equivalently we have

$$E_{2m}\equiv E_{2n} \pmod{2^{s+1}} \iff m\equiv n \pmod{2^s}\ \text{ for any } m,n,s\in\mathbb{N}. \tag{1.1$'$}$$

Later Frobenius amplified Stern's proof in 1910, and several different proofs of (1.1) or (1.1′) were given by Ernvall [E], Wagstaff [W] and Sun [Su]. Our first goal is to provide a complete q-analogue of the Stern congruence.

As usual we let $(a;q)_n=\prod_{0\leqslant k<n}(1-aq^k)$ for every $n\in\mathbb{N}$, where an empty product is regarded to have value 1 and hence $(a;q)_0=1$. For $n\in\mathbb{N}$ we set

$$[n]_q=\frac{1-q^n}{1-q}=\sum_{0\leqslant k<n}q^k,$$

this is the usual q-analogue of n. For any $n,k\in\mathbb{N}$, if $k\leqslant n$ then we call

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{\prod_{0<r\leqslant n}[r]_q}{(\prod_{0<s\leqslant k}[s]_q)(\prod_{0<t\leqslant n-k}[t]_q)}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

a *q-binomial coefficient*; if $k>n$ then we let $\begin{bmatrix}n\\k\end{bmatrix}_q=0$. Obviously we have $\lim_{q\to1}\begin{bmatrix}n\\k\end{bmatrix}_q=\binom{n}{k}$. It is easy to see that

$$\begin{bmatrix}n\\k\end{bmatrix}_q=q^k\begin{bmatrix}n-1\\k\end{bmatrix}_q+\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\qquad\text{for all } k,n=1,2,3,\ldots.$$

By this recursion, each q-binomial coefficient is a polynomial in q with integer coefficients.

We define q-Euler numbers $E_n(q)$ $(n\in\mathbb{N})$ by

$$\sum_{n=0}^{\infty}E_n(q)\frac{x^n}{(q;q)_n}=\left(\sum_{n=0}^{\infty}\frac{q^{\binom{2n}{2}}x^{2n}}{(q;q)_{2n}}\right)^{-1}. \tag{1.2}$$

Multiplying both sides by $\sum_{n=0}^{\infty} q^{\binom{2n}{2}} x^{2n}/(q;q)_{2n}$, we obtain the recursion

$$\sum_{\substack{k=0\\2|k}}^{n} \begin{bmatrix} n\\k \end{bmatrix}_q q^{\binom{k}{2}} E_{n-k}(q) = \delta_{n,0} \quad (n \in \mathbb{N})$$

which implies that $E_n(q) \in \mathbb{Z}[q]$. Observe that

$$\sum_{n=0}^{\infty} E_n(q) \frac{x^n}{\prod_{0<k\leqslant n}[k]_q} = \sum_{n=0}^{\infty} E_n(q) \frac{((1-q)x)^n}{(q;q)_n}$$
$$= \left(\sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}}((1-q)x)^{2n}}{(q;q)_{2n}}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}} x^{2n}}{\prod_{0<k\leqslant 2n}[k]_q}\right)^{-1}$$

and hence $\lim_{q\to 1} E_n(q) = E_n$.

The usual way to define a q-analogue of Euler numbers is as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_n(q) \frac{x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(q;q)_{2n}}\right)^{-1}.$$

(See, e.g., [GZ].) We assert that $\tilde{E}_n(q) = q^{\binom{n}{2}} E_n(1/q)$. In fact,

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} E_n(q^{-1}) \frac{x^n}{\prod_{0<k\leqslant n}(1-q^k)} = \sum_{n=0}^{\infty} E_n(q^{-1}) \frac{(-q^{-1}x)^n}{\prod_{0<k\leqslant n}(1-q^{-k})}$$
$$= \left(\sum_{n=0}^{\infty} \frac{q^{-\binom{2n}{2}}(-q^{-1}x)^{2n}}{\prod_{0<k\leqslant 2n}(1-q^{-k})}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{\prod_{0<k\leqslant 2n}(1-q^k)}\right)^{-1}.$$

Recently, with the help of cyclotomic polynomials, Guo and Zeng [GZ] proved that if $m, n, s, t \in \mathbb{N}$, $m - n = 2^s t$ and $2 \nmid t$ then

$$\tilde{E}_{2m}(q) \equiv q^{m-n} \tilde{E}_{2n}(q) \left(\bmod \prod_{r=0}^{s} (1 + q^{2^r t})\right).$$

This is a partial q-analogue of Stern's result.

Using our q-analogue of Euler numbers, we are able to give below a complete q-analogue of the classical result of Stern.

Theorem 1.1. *Let $n, s, t \in \mathbb{N}$ and $2 \nmid t$. Then*

$$E_{2n}(q) - E_{2n+2^s t}(q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}}. \tag{1.3}$$

The Salié numbers S_n $(n \in \mathbb{N})$ are given by

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \frac{\cosh x}{\cos x} = \frac{(e^x + e^{-x})/2}{(e^{ix} + e^{-ix})/2} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) \Big/ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ we get the recursion

$$\sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}(-1)^{k/2}S_{n-k}=\frac{1+(-1)^n}{2}\quad (n\in\mathbb{N})$$

which implies that all Salié numbers are integers and $S_{2k+1}=0$ for all $k\in\mathbb{N}$.

By a sophisticated use of some deep properties of Bernoulli numbers, in 1965 Carlitz [C2] proved that $2^n \mid S_{2n}$ for any $n\in\mathbb{N}$ (which was first conjectured by Gandhi [G]). Recently Guo and Zeng [GZ] defined a q-analogue of Salié numbers in the following way:

$$\sum_{n=0}^{\infty}\tilde{S}_n(q)\frac{x^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}x^{2n}}{(q;q)_{2n}}\Big/\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(q;q)_{2n}}$$

and hence

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^k\tilde{S}_{2n-2k}(q)=q^{n^2}\quad\text{for any } n\in\mathbb{N}.$$

They conjectured that $(-q;q)_n=\prod_{0<k\leqslant n}(1+q^k)$ divides $\tilde{S}_{2n}(q)$ (in $\mathbb{Z}[q]$).

We define q-Salié numbers by

$$\sum_{n=0}^{\infty}S_n(q)\frac{x^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^{2n}}{(q;q)_{2n}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{2n}{2}}x^{2n}}{(q;q)_{2n}}.\tag{1.4}$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^n q^{\binom{2n}{2}}x^{2n}/(q;q)_{2n}$ one finds the recursion

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^k q^{\binom{2k}{2}}S_{2n-2k}(q)=q^{n(n-1)}\quad (n\in\mathbb{N}).\tag{1.5}$$

In this paper we are able to prove the following q-analogue of Carlitz's result concerning Salié numbers.

Theorem 1.2. *Let $n\in\mathbb{N}$. Then $(-q;q)_n=\prod_{0<k\leqslant n}(1+q^k)$ divides $S_{2n}(q)$ in the ring $\mathbb{Z}[q]$.*

Corollary 1.1. *For any $n\in\mathbb{N}$ we have $(-q;q)_n\mid\tilde{S}_{2n}(q)$ in the ring $\mathbb{Z}[q]$ as conjectured by Guo and Zeng.*

Proof. By Theorem 1.2, $S_{2n}(q)=(-q;q)_nP_n(q)$ for some $P_n(q)\in\mathbb{Z}[q]$. Let m be a natural number not smaller than $\deg P$. Then $q^mP(q^{-1})\in\mathbb{Z}[q]$. Since

$$q^{\binom{n+1}{2}}\prod_{0<k\leqslant n}(1+q^{-k})=\prod_{0<k\leqslant n}(1+q^k),$$

$q^{m+\binom{n+1}{2}}S_{2n}(q^{-1})$ is in $\mathbb{Z}[q]$ and divisible by $(-q;q)_n$. If the equality

$$\tilde{S}_{2n}(q)=q^{\binom{2n}{2}}S_{2n}(q^{-1})$$

holds, then $q^m\tilde{S}_{2n}(q)$ is divisible by $(-q;q)_n$ and hence so is $\tilde{S}_{2n}(q)$ since q^m is relatively prime to $(-q;q)_n$.

Now let us explain why $\tilde{S}_n(q)=q^{\binom{n}{2}}S_n(q^{-1})$ for any $n\in\mathbb{N}$. In fact,

$$\sum_{n=0}^{\infty}q^{\binom{n}{2}}S_n(q^{-1})\frac{x^n}{\prod_{0<k\leqslant n}(1-q^k)}=\sum_{n=0}^{\infty}S_n(q^{-1})\frac{(-q^{-1}x)^n}{\prod_{0<k\leqslant n}(1-q^{-k})}$$

$$=\sum_{n=0}^{\infty}\frac{q^{-n(n-1)}(-q^{-1}x)^{2n}}{\prod_{0<k\leqslant 2n}(1-q^{-k})}\Big/\sum_{n=0}^{\infty}\frac{(-1)^nq^{-\binom{2n}{2}}(-q^{-1}x)^{2n}}{\prod_{0<k\leqslant 2n}(1-q^{-k})}$$

$$=\sum_{n=0}^{\infty}\frac{q^{n^2}x^{2n}}{\prod_{0<k\leqslant 2n}(1-q^k)}\Big/\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{\prod_{0<k\leqslant 2n}(1-q^k)}=\sum_{n=0}^{\infty}\tilde{S}_n(q)\frac{x^n}{(q;q)_n}.$$

This concludes our proof. □

In 1956 Carlitz [C1] investigated the coefficients of

$$\frac{\sinh x}{\sin x}=\sum_{n=0}^{\infty}C_n\frac{x^n}{n!},$$

where

$$\sinh x=\frac{e^x-e^{-x}}{2}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

We call those numbers C_n ($n\in\mathbb{N}$) *Carlitz numbers.* In 1965 Carlitz [C2] proved a conjecture of Gandhi [G] which states that 2^n divides the numerator of C_{2n}.

Now we define q-Carlitz numbers $C_n(q)$ ($n\in\mathbb{N}$) by

$$\sum_{n=0}^{\infty}C_n(q)\frac{x^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^{2n+1}}{(q;q)_{2n+1}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(2n+1)}x^{2n+1}}{(q;q)_{2n+1}}. \quad (1.6)$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^nq^{n(2n+1)}x^{2n+1}/(q;q)_{2n+1}$ we get the recursion

$$\sum_{k=0}^{n}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q(-1)^kq^{k(2k+1)}C_{2n-2k}(q)=q^{n(n-1)}\quad (n\in\mathbb{N}). \quad (1.7)$$

By (1.7) and induction,

$$[1]_q[3]_q\cdots[2n+1]_qC_{2n}(q)\in\mathbb{Z}[q];$$

in particular, $(2n+1)!!C_{2n}\in\mathbb{Z}$. If $j,k\in\mathbb{N}$ and $q^j=-1$, then $q^{j(2k+1)}=-1$ and hence $q^{2k+1}\neq 1$. Thus q^j+1 is relatively prime to $1-q^{2k+1}$ for any $j,k\in\mathbb{N}$, and hence $(-q;q)_n=\prod_{0<j\leqslant n}(1+q^j)$ is relatively prime to the denominator of $C_{2n}(q)$. This basic property will be used later.

Here is our q-analogue of Carlitz's divisibility result concerning Carlitz numbers.

Theorem 1.3. *For any $n \in \mathbb{N}$, $(-q;q)_n$ divides the numerator of $C_{2n}(q)$.*

Note that $E_{2k+1}(q) = S_{2k+1}(q) = C_{2k+1}(q) = 0$ for all $k \in \mathbb{N}$ because

$$\sum_{n=0}^{\infty} E_n(q)\frac{x^n}{(q;q)_n},\ \sum_{n=0}^{\infty} S_n(q)\frac{x^n}{(q;q)_n},\ \sum_{n=0}^{\infty} C_n(q)\frac{x^n}{(q;q)_n}$$

are even functions.

Our approach to q-Euler numbers, q-Salié numbers and q-Carlitz numbers is quite different from that of Guo and Zeng [GZ]. The proofs of Theorems 1.1–1.3 depend on new recursions for q-Euler numbers, q-Salié numbers and q-Carlitz numbers. In the next section we will prove Theorem 1.1. In Section 3 we establish an auxiliary theorem which essentially says that if $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ then

$$\sum_{\substack{k\in\mathbb{Z}\\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n}. \tag{1.8}$$

(We can also substitute $2n+1$ for $2n$ in (1.8).) Section 4 is devoted to the proofs of Theorems 1.2 and 1.3 on the basis of Section 3.

2. Proof of Theorem 1.1

Lemma 2.1. *For any $n \in \mathbb{N}$ we have*

$$E_{2n}(q) = 1 - \sum_{0<k\leqslant n} (-q;q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2(n-k)}(q). \tag{2.1}$$

Proof. Let us recall the following three known identities (cf. Theorem 10.2.1 and Corollary 10.2.2 of [AAR]):

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q;q)_n} = (x;q)_\infty$$

where $(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)$,

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \quad\text{and}\quad \sum_{n=0}^{\infty} \frac{(-1;q)_n x^n}{(q;q)_n} = \frac{(-x;q)_\infty}{(x;q)_\infty}.$$

Observe that

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{\infty} E_n(q)\frac{x^n}{(q;q)_n} =&\left(\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n} + \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-x)^n}{(q;q)_n}\right)^{-1}\\ =&\frac{1}{(x;q)_\infty + (-x;q)_\infty}\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\left(\sum_{n=0}^{\infty}E_n(q)\frac{x^n}{(q;q)_n}\right)\left(1+\sum_{n=0}^{\infty}\frac{(-1;q)_nx^n}{(q;q)_n}\right)\\=&\frac{1}{(x;q)_\infty+(-x;q)_\infty}\left(1+\frac{(-x;q)_\infty}{(x;q)_\infty}\right)=\frac{1}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}.\end{aligned}$$

Comparing the coefficients of x^n we obtain that

$$\frac{1}{2}E_n(q)+\frac{1}{2}\sum_{k=0}^{n}(-1;q)_k\begin{bmatrix}n\\k\end{bmatrix}_q E_{n-k}(q)=1,$$

i.e.,

$$E_n(q)=1-\sum_{0<k\leqslant n}(-q;q)_{k-1}\begin{bmatrix}n\\k\end{bmatrix}_q E_{n-k}(q).$$

Substituting $2n$ for n in the last equality and recalling that $E_{2j+1}(q)=0$ for $j\in\mathbb{N}$, we immediately obtain the desired (2.1). $\square$

Corollary 2.1. *For any $n\in\mathbb{N}$ we have*

$$E_{2n}(q)\equiv 1 \pmod{1+q}. \tag{2.2}$$

Proof. This follows from (2.1) because $1+q$ divides $(-q;q)_m$ for all $m=1,2,3,\ldots$. $\square$

The following trick is simple but useful.

$$\prod_{k=0}^{n}(1+q^{2^k})=[2^{n+1}]_q \quad \text{for any } n\in\mathbb{N}. \tag{2.3}$$

In fact,

$$\begin{aligned}(1-q)\prod_{k=0}^{n}(1+q^{2^k})=&(1-q^2)\prod_{0<k\leqslant n}(1+q^{2^k})\\=&\cdots=(1-q^{2^n})(1+q^{2^n})=1-q^{2^{n+1}}.\end{aligned}$$

Lemma 2.2. *Let m,n,s,t be positive integers with $2m\geqslant n$ and $2\nmid t$. Then $(-q;q)_m\begin{bmatrix}2^st\\n\end{bmatrix}_q$ is divisible by $(1+q)^{\lfloor(m-1)/2\rfloor}[2^s]_{q^t}$, where we use $\lfloor\alpha\rfloor$ to denote the greatest integer not exceeding a real number α.*

Proof. Write $n=2^kl$ with $k,l\in\mathbb{N}$ and $2\nmid l$. Then

$$[n]_q=\frac{1-q^n}{1-q}=\frac{1-q^{2^kl}}{1-q^l}\cdot\frac{1-q^l}{1-q}=[2^k]_{q^l}[l]_q.$$

Obviously $[2^k]_{q^l}=\prod_{0\leqslant j<k}(1+q^{2^j l})$ divides $(-q;q)_m=\prod_{j=1}^m(1+q^j)$ since $m\geqslant n/2=2^{k-1}l$. Thus $[2^s]_{q^t}=[2^st]_q/[t]_q$ divides

$$[l]_q(-q;q)_m\begin{bmatrix}2^st\\ n\end{bmatrix}_q=\frac{(-q;q)_m}{[2^k]_{q^l}}[2^st]_q\begin{bmatrix}2^st-1\\ n-1\end{bmatrix}_q.$$

Note that $[2^s]_{q^t}=\prod_{r=0}^{s-1}(1+q^{2^r t})$ is relatively prime to $[l]_q=(1-q^l)/(1-q)$ since $l\equiv 1 \pmod 2$. Therefore $[2^s]_{q^t}$ divides $(-q;q)_m\begin{bmatrix}2^st\\ n\end{bmatrix}_q$.

Clearly $(1+q)^{\lfloor (m+1)/2\rfloor}$ divides

$$\prod_{j=1}^{\lfloor (m+1)/2\rfloor}(1+q^{2j-1})\times\prod_{j=1}^{\lfloor m/2\rfloor}(1+q^{2j})=(-q;q)_m.$$

Since

$$[2^s]_{q^t}=\frac{1-q^{2t}}{1-q^t}\cdot\frac{1-q^{2^st}}{1-q^{2t}}=(1+q)\sum_{j=0}^{t-1}(-q)^j\sum_{r=0}^{2^{s-1}-1}q^{2rt}$$

and $\sum_{0\leqslant j<t}(-q)^j\sum_{0\leqslant r<2^{s-1}}q^{2rt}$ takes value $2^{s-1}t\neq 0$ at $q=-1$, the polynomial $[2^s]_{q^t}$ is divisible by $1+q$ but not by $(1+q)^2$. Therefore $(1+q)^{\lfloor (m-1)/2\rfloor}[2^s]_{q^t}$ divides $(-q;q)_m\begin{bmatrix}2^st\\ n\end{bmatrix}_q$ by the above. □

Proof of Theorem 1.1. The case $s=0$ is easy. In fact,

$$E_{2n}(q)-E_{2n+2^0t}(q)=E_{2n}(q)\equiv 1=[2^0]_{q^t} \pmod{(1+q)[2^0]_{q^t}}$$

by Corollary 2.1.

Below we handle the case $s>0$ and use induction on n. Assume that

$$E_{2m}(q)-E_{2m+2^st}(q)\equiv[2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}} \text{ whenever } 0\leqslant m<n. \quad (*)$$

(This holds trivially in the case $n=0$.) In view of Lemma 2.1, we have

$$\begin{aligned}&E_{2n}(q)-E_{2n+2^st}(q)\\ &=\sum_{k=1}^{n+2^{s-1}t}(-q;q)_{2k-1}\left(\begin{bmatrix}2n+2^st\\ 2k\end{bmatrix}_q E_{2n+2^st-2k}(q)-\begin{bmatrix}2n\\ 2k\end{bmatrix}_q E_{2n-2k}(q)\right),\end{aligned}$$

where we set $E_l(q)=0$ for $l<0$.

Let $0 < k \leqslant n+2^{s-1}t$. Applying a q-analogue of the Chu-Vandermonde identity (cf. [AAR, Exercise 10.4(b)]), we find that

$$\begin{aligned}&\begin{bmatrix}2n+2^st\\2k\end{bmatrix}_q E_{2n+2^st-2k}(q)-\begin{bmatrix}2n\\2k\end{bmatrix}_q E_{2n-2k}(q)\\=&E_{2n+2^st-2k}(q)\sum_{j=0}^{2k}q^{(2n-j)(2k-j)}\begin{bmatrix}2n\\j\end{bmatrix}_q\begin{bmatrix}2^st\\2k-j\end{bmatrix}_q-\begin{bmatrix}2n\\2k\end{bmatrix}_q E_{2n-2k}(q)\\=&E_{2n+2^st-2k}(q)\sum_{j=0}^{2k-1}q^{(2n-j)(2k-j)}\begin{bmatrix}2n\\j\end{bmatrix}_q\begin{bmatrix}2^st\\2k-j\end{bmatrix}_q\\&+\begin{bmatrix}2n\\2k\end{bmatrix}_q(E_{2n+2^st-2k}(q)-E_{2n-2k}(q)).\end{aligned}$$

In view of the hypothesis $(*)$,

$$(-q;q)_{2k-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q(E_{2n+2^st-2k}(q)-E_{2(n-k)}(q))\equiv 0 \pmod{(1+q)[2^s]_{q^t}}.$$

By Lemma 2.2, if $0\leqslant j<2k$ then $(-q;q)_{2k-1}\left[{2^st \atop 2k-j}\right]_q$ is divisible by $(1+q)^{k-1}[2^s]_{q^t}$. Therefore, if $k>1$ then $(1+q)[2^s]_{q^t}$ divides

$$(-q;q)_{2k-1}\left(\begin{bmatrix}2n+2^st\\2k\end{bmatrix}_q E_{2n+2^st-2k}(q)-\begin{bmatrix}2n\\2k\end{bmatrix}_q E_{2n-2k}(q)\right)$$

by the above. In the case $k=1$,

$$(-q;q)_{2k-1}\begin{bmatrix}2^st\\2k-1\end{bmatrix}_q=(1+q)[2^st]_q=(1+q)[2^s]_{q^t}[t]_q$$

and hence

$$\begin{aligned}&(-q;q)_1\left(\begin{bmatrix}2n+2^st\\2\end{bmatrix}_q E_{2n+2^st-2}(q)-\begin{bmatrix}2n\\2\end{bmatrix}_q E_{2n-2}(q)\right)\\\equiv&(1+q)E_{2n+2^st-2}(q)q^{(2n-0)(2-0)}\begin{bmatrix}2n\\0\end{bmatrix}_q\begin{bmatrix}2^st\\2\end{bmatrix}_q \pmod{(1+q)[2^s]_{q^t}}\\\equiv&E_{2n+2^st-2}(q)q^{4n}\frac{1+q}{[2]_q}[2^st]_q[2^st-1]_q \pmod{(1+q)[2^s]_{q^t}}\\\equiv&E_{2n+2^st-2}(q)q^{4n}[2^s]_{q^t}[t]_q(1+q[2^st-2]_q)\equiv[2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}};\end{aligned}$$

in the last step we have noted that $q^{4n}-1, [t]_q-1, [2^st-2]_q$ are divisible by $1+q$, and $E_{2n+2^st-2}(q)\equiv 1 \pmod{1+q}$ by Corollary 2.1.

Combining the above we obtain that

$$E_{2n}(q) - E_{2n+2^s t}(q) \equiv \sum_{k=1}^{n+2^{s-1}t} \delta_{k,1}[2^s]_{q^t} = [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}}.$$

This concludes the induction.

The proof of Theorem 1.1 is now complete. □

Remark 2.1. With a bit more efforts we can prove the following more general result: For $k = 1, 2, 3, \ldots$ let

$$\sum_{n=0}^{\infty} E_n^{(k)}(q) \frac{x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} q^{\binom{kn}{2}} \frac{x^{kn}}{(q;q)_{kn}} \right)^{-1}.$$

Given positive integers k, s, t with $2 \nmid t$, we have

$$E_{2k'n}^{(2k')}(q) - E_{2k'(n+2^{s-1}t)}^{(2k')}(q) \equiv (2k'-1)[2^s]_{q^{k't}} \left(\bmod\ (1+q^{k'})[2^s]_{q^{k't}}\right)$$

for all $n \in \mathbb{N}$, where $k' = 2^{k-1}$. This is a q-analogue of Conjecture 5.5 in [GZ].

3. An Auxiliary Theorem

Theorem 3.1. *For all $m, n \in \mathbb{N}$, both*

$$S_n^m := \sum_{k=0}^{n} (-1)^k q^{k(k-1)+2m(n-k)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \tag{3.1}$$

and

$$T_n^m := \sum_{0 \leqslant k < n} (-1)^k q^{k(k-1)+2m(n-1-k)} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q \tag{3.2}$$

are divisible by $(-q;q)_n = \prod_{0<k\leqslant n}(1+q^k)$ in the ring $\mathbb{Z}[q]$. Also, for any $m, n \in \mathbb{N}$ and $\delta \in \{0, 1\}$ we have the congruence

$$\sum_{k=0}^{n} (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k+\delta \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n}. \tag{3.3}$$

Proof. (i) We use induction on n to prove the first part.

For any $m \in \mathbb{N}$, clearly both $S_0^m = 1$ and $T_0^m = 0$ are divisible by $(-q;q)_0 = 1$, also both $S_1^m = q^{2m} - 1$ and $T_1^m = [2]_q = 1 + q$ are multiples of $(-q;q)_1 = 1 + q$.

Now let $n > 1$ be an integer and assume that $(-q;q)_{n-1}$ divides both S_{n-1}^m and T_{n-1}^m for all $m \in \mathbb{N}$.

For each $m \in \mathbb{Z}$ we have

$$\begin{aligned} S_n^m =& \sum_{l=0}^{n} (-1)^{n-l} q^{(n-l)(n-l-1)+2ml} \begin{bmatrix} 2n \\ 2(n-l) \end{bmatrix}_q \\ =&(-1)^n q^{n(n-1)} \sum_{l=0}^{n} (-1)^l q^{l(l+1)-2ln+2lm} \begin{bmatrix} 2n \\ 2l \end{bmatrix}_q \\ =&(-1)^n q^{n(n-1)-2n(n-1-m)} S_n^{n-1-m} = (-1)^n q^{n(2m-n+1)} S_n^{n-1-m}. \end{aligned}$$

In particular,

$$S_n^n = (-1)^n q^{n(n+1)} S_n^{-1} \quad \text{and} \quad S_n^{n-1} = (-1)^n q^{n(n-1)} S_n^0.$$

Similarly, for every $m \in \mathbb{Z}$ we have

$$\begin{aligned} T_n^m =& \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{(n-1-l)(n-l-2)+2ml} \begin{bmatrix} 2n \\ 2(n-1-l)+1 \end{bmatrix}_q \\ =&(-1)^{n-1} q^{(n-1)(n-2)} \sum_{l=0}^{n-1} (-1)^l q^{l(l+1)-2l(n-1)+2lm} \begin{bmatrix} 2n \\ 2l+1 \end{bmatrix}_q \\ =&(-1)^{n-1} q^{(n-1)(2m-n+2)} T_n^{n-2-m}. \end{aligned}$$

In particular,

$$T_n^{n-1} = (-1)^{n-1} q^{n(n-1)} T_n^{-1} \quad \text{and} \quad T_n^{n-2} = (-1)^{n-1} q^{(n-1)(n-2)} T_n^0.$$

For any $m \in \mathbb{N}$, clearly

$$\begin{aligned} S_n^{m+1} - S_n^m =& \sum_{k=0}^{n} (-1)^k q^{k(k-1)+2m(n-k)} (q^{2(n-k)} - 1) \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \\ =& \sum_{k=0}^{n} (-1)^k q^{k(k-1)+2m(n-k)} (q^{2n} - 1) \begin{bmatrix} 2n-1 \\ 2k \end{bmatrix}_q \\ =&(q^{2n}-1) \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} q^{2k} \begin{bmatrix} 2n-2 \\ 2k \end{bmatrix}_q \\ &+ (q^{2n}-1) \sum_{k=1}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} \begin{bmatrix} 2n-2 \\ 2k-1 \end{bmatrix}_q \\ =&(q^{2n}-1) q^{2(m+n-1)} S_{n-1}^{m-1} - (q^{2n}-1) q^{2(m+n-2)} T_{n-1}^{m-1} \\ =&(q^{2n}-1) q^{2(m+n-2)} (q^2 S_{n-1}^{m-1} - T_{n-1}^{m-1}) \end{aligned}$$

and

$$\begin{aligned}qT_n^{m+1}-T_n^m=&\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}(q^{2(n-1-k)+1}-1)\begin{bmatrix}2n\\2k+1\end{bmatrix}_q\\=&\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}(q^{2n}-1)\begin{bmatrix}2n-1\\2k+1\end{bmatrix}_q\\=&(q^{2n}-1)\sum_{k=0}^{n-2}(-1)^k q^{k(k-1)+2m(n-1-k)}q^{2k+1}\begin{bmatrix}2n-2\\2k+1\end{bmatrix}_q\\&+(q^{2n}-1)\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}\begin{bmatrix}2n-2\\2k\end{bmatrix}_q\\=&(q^{2n}-1)q^{2m+2n-3}T_{n-1}^{m-1}+(q^{2n}-1)S_{n-1}^m,\end{aligned}$$

therefore by the induction hypothesis we have

$$S_n^{m+1}\equiv S_n^m \pmod{(-q;q)_n}\quad\text{and}\quad qT_n^{m+1}\equiv T_n^m \pmod{(-q;q)_n}.$$

(Note that both $q^{n(n-1)}S_{n-1}^{-1}=(-1)^{n-1}S_{n-1}^{n-1}$ and $q^{(n-1)(n-2)}T_{n-1}^{-1}=(-1)^nT_{n-1}^{n-2}$ are divisible by $(-q;q)_{n-1}$ by the induction hypothesis.) Thus, if $(-q;q)_n$ divides both S_n^0 and T_n^0 then it divides both S_n^m and T_n^m for every $m=0,1,2,\ldots$.

Observe that

$$\begin{aligned}S_n^0=&\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix}2n\\2n-2k\end{bmatrix}_q\\=&\sum_{k=1}^{n}(-1)^k q^{k(k-1)+2n-2k}\begin{bmatrix}2n-1\\2n-2k\end{bmatrix}_q+\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)}\begin{bmatrix}2n-1\\2n-2k-1\end{bmatrix}_q\\=&\sum_{k=1}^{n}(-1)^k q^{k(k-1)}q^{2(2n-2k)}\begin{bmatrix}2n-2\\2n-2k\end{bmatrix}_q\\&+\sum_{k=1}^{n-1}(-1)^k q^{k(k-1)}(q^{2n-2k}+q^{2n-2k-1})\begin{bmatrix}2n-2\\2n-2k-1\end{bmatrix}_q\\&+\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)}\begin{bmatrix}2n-2\\2n-2k-2\end{bmatrix}_q\\=&-q^{2n-2}S_{n-1}^1-q^{2n-3}(1+q)T_{n-1}^0+S_{n-1}^0\end{aligned}$$

and hence $(-q;q)_{n-1}$ divides S_n^0 by the induction hypothesis. Similarly, $(-q;q)_{n-1}$ divides $T_n^0=-q^{2n-2}T_{n-1}^1+(1+q)S_{n-1}^1+T_{n-1}^0$.

Since

$$(-1)^n q^{n(n-1)} S_n^0 = S_n^{n-1} \equiv S_n^0 \pmod{(-q;q)_n}$$

and

$$1-(-1)^n q^{n(n-1)} \equiv 1-(-1)^n(-1)^{n-1} = 2 \pmod{1+q^n},$$

we must have $S_n^0/(-q;q)_{n-1} \equiv 0 \pmod{1+q^n}$ and hence $(-q;q)_n \mid S_n^0$. Similarly, as

$$q^{n-2}(-1)^{n-1}q^{(n-1)(n-2)}T_n^0 = q^{n-2}T_n^{n-2} \equiv T_n^0 \pmod{(-q;q)_n}$$

and $1-(-1)^{n-1}q^{n(n-2)} \equiv 2 \pmod{1+q^n}$, we have $T_n^0/(-q;q)_{n-1} \equiv 0 \pmod{1+q^n}$ and hence $(-q;q)_n \mid T_n^0$. This concludes our induction step and proves the first part.

(ii) Now fix $m, n \in \mathbb{N}$ and $\delta \in \{0,1\}$. We can verify (3.3) directly if $n < 2$.

Below we assume $n \geqslant 2$. By a previous argument,

$$(-1)^n S_n^{m+n-1} = q^{n(2m+n-1)} S_n^{-m} = q^{n(n-1)} \sum_{k=0}^{n} (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$$

and

$$\begin{aligned}(-1)^{n-1}T_n^{m+n-2} =& q^{(n-1)(2m+n-2)}T_n^{-m} \\ =& q^{(n-1)(n-2)} \sum_{k=0}^{n-1} (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q.\end{aligned}$$

Thus, applying the first part we immediately get (3.3).

The proof of Theorem 3.1 is now complete. □

Remark 3.1. Theorem 3.1 is somewhat difficult and sophisticated, however it is easy to evaluate the sums

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k + (-i)^k}{2}$$

and

$$\sum_{0 \leqslant k < n} (-1)^k \binom{2n}{2k+1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k - (-i)^k}{2i}.$$

Now let us explain why (1.8) holds for any $l \in \mathbb{Z}$ and $n \in \mathbb{N}$. Write $l = 2m + \delta$ with $m \in \mathbb{Z}$ and $\delta \in \{0, 1\}$. Then

$$\begin{aligned}
&\sum_{\substack{k\in\mathbb{Z}\\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \\
=&\sum_{k+m\in\mathbb{N}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2(k+m)+\delta \end{bmatrix}_q \\
=&\sum_{k\in\mathbb{N}} (-1)^{k-m} q^{(k-m)(k-m-1)} \begin{bmatrix} 2n \\ 2k+\delta \end{bmatrix}_q \\
=&(-1)^m \sum_{k=0}^{n-\delta} (-1)^k q^{k(k-1)-2km+m(m+1)} \begin{bmatrix} 2n \\ 2k+\delta \end{bmatrix}_q .
\end{aligned}$$

So (1.8) follows from Theorem 3.1. Note also that

$$\begin{aligned}
&\sum_{\substack{k\in\mathbb{Z}\\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+l \end{bmatrix}_q \\
&- \sum_{\substack{k\in\mathbb{Z}\\ 2k+l-1\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l-1 \end{bmatrix}_q \\
=&\sum_{\substack{k\in\mathbb{Z}\\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)+2k+l} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \\
=&q^l \sum_{\substack{k\in\mathbb{Z}\\ 2k+l-2\geqslant 0}} (-1)^{k-1} q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l-2 \end{bmatrix}_q
\end{aligned}$$

and thus

$$\sum_{\substack{k\in\mathbb{Z}\\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n}. \tag{3.4}$$

4. Proofs of Theorems 1.2 and 1.3

Lemma 4.1. *We have*

$$1 + \sum_{n=1}^{\infty} (-q;q)_{2n-1} \frac{(-1)^n x^{2n}}{(q;q)_{2n}} = \sum_{k=0}^{\infty} q^{\binom{2k}{2}} \frac{(-1)^k x^{2k}}{(q;q)_{2k}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q;q)_{2l}} \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} (-q;q)_{2n} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}} = \sum_{k=0}^{\infty} q^{\binom{2k+1}{2}} \frac{(-1)^k x^{2k+1}}{(q;q)_{2k+1}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q;q)_{2l}}. \tag{4.2}$$

Proof. Let $\delta \in \{0,1\}$. Then

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{2k+\delta}{2}}x^{2k+\delta}}{(q;q)_{2k+\delta}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}}\\ =&\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+\delta}}{(q;q)_{2n+\delta}}\sum_{k=0}^{n}q^{\binom{2k+\delta}{2}}\begin{bmatrix}2n+\delta\\2k+\delta\end{bmatrix}_q.\end{aligned}$$

By the q-binomial theorem (cf. [AAR, Corollary 10.2.2(c)]),

$$(x;q)_m=\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}x^k \quad \text{for any } m\in\mathbb{N}.$$

Thus

$$\begin{aligned}2\sum_{k=0}^{n}q^{\binom{2k+\delta}{2}}\begin{bmatrix}2n+\delta\\2k+\delta\end{bmatrix}_q=&\sum_{l=0}^{2n+\delta}q^{\binom{l}{2}}\begin{bmatrix}2n+\delta\\l\end{bmatrix}_q+\sum_{l=0}^{2n+\delta}(-1)^{\delta+l}q^{\binom{l}{2}}\begin{bmatrix}2n+\delta\\l\end{bmatrix}_q\\ =&(-1;q)_{2n+\delta}+(-1)^{\delta}(1;q)_{2n+\delta}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{2k+\delta}{2}}x^{2k+\delta}}{(q;q)_{2k+\delta}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}}\\ =&\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+\delta}}{(q;q)_{2n+\delta}}\cdot\frac{(-1;q)_{2n+\delta}+(-1)^{\delta}(1;q)_{2n+\delta}}{2}\\ =&\begin{cases}1+\sum_{n=1}^{\infty}(-q;q)_{2n-1}\frac{(-1)^n x^{2n}}{(q;q)_{2n}} & \text{if } \delta=0,\\ \sum_{n=0}^{\infty}(-q;q)_{2n}\frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}} & \text{if } \delta=1.\end{cases}\end{aligned}$$

We are done. □

Remark 4.1. (4.1) and (4.2) are q-analogues of the trigonometric identities

$$\frac{1+\cos(2x)}{2}=\cos^2 x \quad\text{and}\quad \frac{\sin(2x)}{2}=\sin x\cos x$$

respectively.

Lemma 4.2. *Let $n \geqslant k \geqslant 1$ be integers. Then both $(-q;q)_k\begin{bmatrix}2n\\2k\end{bmatrix}_q$ and $(-q;q)_k\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q$ are divisible by*

$$(-q^{n-k+1};q)_k=\prod_{j=1}^{k}(1+q^{n-j+1}).$$

Proof. Observe that

$$\begin{aligned}\begin{bmatrix}2n\\2k\end{bmatrix}_q &= \prod_{j=1}^{2k}\frac{1-q^{2n-j+1}}{1-q^j} = \prod_{j=1}^{k}\frac{(1-q^{2n-2j+1})(1-q^{2n-(2j-1)+1})}{(1-q^{2j})(1-q^{2j-1})}\\ &= \prod_{j=1}^{k}\frac{(1-q^{n-j+1})(1+q^{n-j+1})(1-q^{2n-2j+1})}{(1-q^j)(1+q^j)(1-q^{2j-1})}\\ &= \begin{bmatrix}n\\k\end{bmatrix}_q \frac{\prod_{j=1}^k(1+q^{n-j+1})}{(-q;q)_k}\prod_{j=1}^{k}\frac{1-q^{2n-2j+1}}{1-q^{2j-1}}\end{aligned}$$

and hence

$$(-q^{n-k+1};q)_k \,\Big|\, (-q;q)_k\begin{bmatrix}2n\\2k\end{bmatrix}_q\prod_{j=1}^{k}(1-q^{2j-1}).$$

Recall that $(-q^{n-k+1};q)_k = \prod_{n-k<l\leqslant n}(1+q^l)$ is relatively prime to $\prod_{j=1}^k(1-q^{2j-1})$. Therefore $(-q^{n-k+1};q)_k \mid (-q;q)_k\begin{bmatrix}2n\\2k\end{bmatrix}_q$.

Since $[2k+1]_q$ is also relatively prime to $(-q^{n-k+1};q)_k$, we have

$$(-q;q)_k\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q = (-q;q)_k\frac{[2n+1]_q}{[2k+1]_q}\begin{bmatrix}2n\\2k\end{bmatrix}_q \equiv 0 \pmod{(-q^{n-k+1};q)_k}.$$

This concludes the proof. □

Remark 4.2. Lemma 4.2 yields a trivial result as $q\to 1$.

Proof of Theorem 1.2. Clearly

$$\begin{aligned}f(x) &:= \left(\sum_{n=0}^{\infty}S_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}}\right)\left(1+\sum_{n=1}^{\infty}(-q;q)_{2n-1}\frac{(-1)^n x^{2n}}{(q;q)_{2n}}\right)\\ &= \sum_{n=0}^{\infty}S_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}} + \sum_{n=1}^{\infty}\frac{x^{2n}}{(q;q)_{2n}}\sum_{k=1}^{n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q S_{2n-2k}(q).\end{aligned}$$

On the other hand, by (4.1) we have

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty}\frac{q^{k(k-1)}x^{2k}}{(q;q)_{2k}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(q;q)_{2n}}\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix}2n\\2k\end{bmatrix}_q.\end{aligned}$$

Therefore

$$\begin{aligned}&S_{2n}(q)+\sum_{0<k\leqslant n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q S_{2n-2k}(q)\\=&(-1)^n\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix}2n\\2k\end{bmatrix}_q\equiv 0 \pmod{(-q;q)_n}\end{aligned}$$

with the help of (1.8) or Theorem 3.1. If $(-q;q)_l \mid S_{2l}(q)$ for all $0\leqslant l<n$, then

$$S_{2n}(q)\equiv-\sum_{0<k\leqslant n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q S_{2n-2k}(q)\equiv 0 \pmod{(-q;q)_n}$$

since $\prod_{0<j\leqslant n-k}(1+q^j)$ divides $S_{2n-2k}(q)$ and $\prod_{n-k<j\leqslant n}(1+q^j)$ divides $(-q;q)_{2k-1}\left[{2n \atop 2k}\right]_q$ by Lemma 4.2. Thus we have the desired result by induction. □

Remark 4.3. As $q\to 1$ our new recursion for q-Salié numbers yields a useful recursion for Salié numbers:

$$S_{2n}+\sum_{0<k\leqslant n}(-1)^k 2^{2k-1}\binom{2n}{2k}S_{2n-2k}=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n}{2k},$$

from which the Carlitz result $2^n \mid S_{2n}$ follows by induction.

Proof of Theorem 1.3. It is apparent that

$$\begin{aligned}g(x):=&\left(\sum_{n=0}^{\infty}C_{2n}(q)\frac{x^{2n}}{(q;q)_{2n}}\right)\left(\sum_{n=0}^{\infty}(-q;q)_{2n}\frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}}\right)\\=&\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(q;q)_{2n+1}}\sum_{k=0}^{n}(-1)^k(-q;q)_{2k}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q C_{2n-2k}(q).\end{aligned}$$

On the other hand, (4.2) implies that

$$\begin{aligned}g(x)=&\sum_{k=0}^{\infty}\frac{q^{k(k-1)}x^{2k+1}}{(q;q)_{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}}\\=&\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(q;q)_{2n+1}}\sum_{k=0}^{n}(-1)^{n-k}q^{k(k-1)}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q.\end{aligned}$$

Therefore we have the recurrence relation

$$\sum_{k=0}^{n}(-1)^k(-q;q)_{2k}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q C_{2n-2k}(q)=\sum_{k=0}^{n}(-1)^{n-k}q^{k(k-1)}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q.$$

The right-hand side of the last equality is a multiple of $(-q;q)_n$ by (3.4). So we have

$$\sum_{k=0}^{n}(-1)^k(-q;q)_{2k}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q C_{2n-2k}(q)\equiv 0 \pmod{(-q;q)_n}.$$

Assume that $(-q;q)_l$ divides the numerator of $C_{2l}(q)$ for each $0\leqslant l<n$. Then $(-q;q)_n$ divides the numerator of $(-q;q)_{2k}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q C_{2n-2k}(q)$ for each $0<k\leqslant n$, because $\prod_{0<j\leqslant n-k}(1+q^j)$ divides the numerator of $C_{2n-2k}(q)$ and $\prod_{n-k<j\leqslant n}(1+q^j)$ divides $(-q;q)_{2k}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q$ by Lemma 4.2. Thus $(-q;q)_n$ must also divide the numerator of $\begin{bmatrix}2n+1\\1\end{bmatrix}_q C_{2n}(q)=[2n+1]_q C_{2n}(q)$. Recall that $[2n+1]_q$ is relatively prime to $(-q;q)_n$. So the numerator of $C_{2n}(q)$ is divisible by $(-q;q)_n$.

In view of the above, the desired result follows by induction on n. □

Remark 4.4. As $q\to 1$ our new recursion for q-Carlitz numbers yields the following recurrence relation for Carlitz numbers:

$$\sum_{k=0}^{n}(-1)^k 2^{2k}\binom{2n+1}{2k+1}C_{2n-2k}=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}.$$

From this one can easily deduce the Carlitz congruence $C_{2n}\equiv 0 \pmod{2^n}$.

Acknowledgments. The second author is indebted to Prof. Jiang Zeng at University of Lyon-I for showing Carlitz's paper [C2] and the preprint [GZ] during Sun's visit to the Institute of Camille Jordan. This paper was finished during Sun's visit to the University of California at Irvine; he would like to thank Prof. Daqing Wan for the invitation.

References

[AAR] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[C1] L. Carlitz, *The coefficients of* $\sinh x/\sin x$, Math. Mag. **29** (1956), 193-197.

[C2] L. Carlitz, *The coefficients of* $\cosh x/\cos x$, Monatsh. Math. **69** (1965), 129-135.

[E] R. Ernvall, *Generalized Bernoulli numbers, generalized irregular primes, and class number*, Ann. Univ. Turku. Ser. A, I(178), 1979, 72 pp.

[G] J. M. Gandhi, *The coefficients of* $\cosh x/\cos x$ *and a note on Carlitz's coefficients of* $\sinh x/\sin x$, Math. Mag. **31** (1958), 185-191.

[GZ] Victor J. W. Guo and J. Zeng, *Some arithmetic properties of the q-Euler numbers and q-Salié numbers*, European J. Combin. **27** (2006), 884–895.

[St] M. A. Stern, *Zur Theorie der Eulerschen Zahlen*, J. Reine Angew. Math. **79** (1875), 67-98.

[Su] Z. W. Sun, *On Euler numbers modulo powers of two*, J. Number Theory **115** (2005), 371–380.

[W] S. S. Wagstaff, Jr., *Prime divisors of the Bernoulli and Euler numbers*, in: Number Theory for the Millennium, III (Urbana, IL, 2000), 357–374, A K Peters, Natick, MA, 2002.

Department of Mathematics
Nanjing University
Nanjing 210093
People's Republic of China
E-mail: (Hao Pan) haopan79@yahoo.com.cn
(Zhi-Wei Sun) zwsun@nju.edu.cn