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A COMBINATORIAL IDENTITY WITH APPLICATION TO CATALAN NUMBERS

Hao Pan and Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
haopan79@yahoo.com.cn zwsun@nju.edu.cn

Abstract. By a very simple argument, we prove that if $l, m, n \in \{0, 1, 2, \dots\}$ then

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m}=\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}.$$

On the basis of this identity, for $d, r \in \{0, 1, 2, \dots\}$ we construct explicit $F(d,r)$ and $G(d,r)$ such that for any prime $p > \max\{d, r\}$ we have

$$\sum_{k=1}^{p-1}k^rC_{k+d}\equiv\begin{cases}F(d,r) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ G(d,r) \pmod p & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

where C_n denotes the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. For example, when $p \geqslant 5$ is a prime, we have

$$\sum_{k=1}^{p-1}k^2C_k\equiv\begin{cases}-2/3 \pmod p & \text{if } p\equiv 1 \pmod 3,\\ -1/3 \pmod p & \text{if } p\equiv 2 \pmod 3;\end{cases}$$

and

$$\sum_{0<k<p-4}\frac{C_{k+4}}{k}\equiv\begin{cases}503/30 \pmod p & \text{if } p\equiv 1 \pmod 3,\\ -100/3 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

This paper also contains some new recurrence relations for Catalan numbers.

Key words and phrases. Binomial coefficient; Combinatorial identity; Catalan number.

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1. Introduction

As usual, for $k \in \mathbb{Z}$ we define the binomial coefficient $\binom{x}{k}$ as follows:

$$\binom{x}{k} = \begin{cases} \frac{1}{k!}\prod_{j=0}^{k-1}(x-j) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

There are many combinatorial identities involving binomial coefficients. (See, e.g., [GJ], [GKP] and [PWZ].) A nice identity of Dixon (cf. [PWZ, p. 43]) states that

$$\sum_{k\in\mathbb{Z}}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$$

for any $a, b, c \in \mathbb{N} = \{0, 1, 2, \dots\}$.

During the second author's visit (January–March, 2005) to the Institute of Camille Jordan at Univ. Lyon-I, Dr. Victor J. W. Guo told Sun that he had made the following "conjecture": *Given $l, m \in \mathbb{N}$ one has*

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l}\binom{2k}{k-2l+m} = \begin{cases} \binom{2m/3}{m/3}\binom{m/3}{l-m/3} & \text{if } 3 \mid m, \\ 0 & \text{otherwise;} \end{cases}$$

in other words,

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l}\binom{2k}{k-2l+m} = [3 \mid m]\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l}, \tag{1.0}$$

where $\lceil\cdot\rceil$ is the ceiling function, and for an assertion A we adopt the notation

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

The above conjecture is similar to Dixon's identity in some sense; of course it can be proved with the aid of computer via the WZ method or Zeilberger's algorithm (cf. [PWZ]). After we showed (1.0) in a preliminary version of this paper by Lagrange's inversion formula (cf. [GJ, p. 17]), Prof. C. Krattenthaler at Univ. Lyon-I kindly told us that (1.0) can also be proved by letting $a = m - 3l$, $b = 1/2 - l$ and $x \to 1$ in Bailey's hypergeometric series identity (cf. [B] or Ex. 38(a) of [AAR, p. 185])

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a,\ 2b-a-1,\ a-2b+2 \\ b,\ a-b+3/2 \end{matrix}; \frac{x}{4}\right) \\ =&\frac{1}{(1-x)^a}\ {}_3F_2\left(\begin{matrix} a/3,\ (a+1)/3,\ (a+2)/3 \\ b,\ a-b+3/2 \end{matrix}; -\frac{27x}{4(1-x)^3}\right). \end{aligned}$$

In this paper, by a simple argument we show the following combinatorial identity the special case $n = l$ of which yields (1.0).

Theorem 1.1. *Provided that $l, m, n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m}=\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}. \tag{1.1}$$

Remark 1.1. (a) The preceding hypergeometric series identity of Bailey does not imply (1.1) which involves three parameters l, m and n. However, Prof. C. Krattenthaler informed us that (1.1) can also be deduced by putting $a = m/3 - l$, $b = d = 1-2l+m$ and $e = 1-l+m-n$ in the complicated hypergeometric identity (3.26) of [KR] (which was obtained on the basis of Bailey's identity). Nevertheless, (1.1) has not been pointed out explicitly before, and our proof of (1.1) is very elementary and particularly simple.

(b) The identity (1.1) might have a combinatorial interpretation related to Callan's idea (cf. [C]) in his combinatorial proof of a curious identity due to Sun.

Corollary 1.1. *Let l and m be nonnegative integers. Then*

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l+1}\binom{2k}{k-2l+m}=(1-[3\mid m-1])\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l+1} \tag{1.2}$$

and

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l+2}\binom{2k}{k-2l+m}=(1+[3\mid m+1])\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l+2}. \tag{1.3}$$

Proof. Putting $n = l+j$ in (1.1) with $j \in \{1,2\}$, we get that

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l+j}\binom{2k}{k-2l+m}=\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{l+j}\binom{j}{j-(3k-m)}.$$

If $0 \leqslant 3k-m \leqslant j$, then $m/3 \leqslant k \leqslant (m+2)/3$ and hence $k = \lceil m/3\rceil$. Note that

$$\binom{1}{1-(3\lceil m/3\rceil-m)}=1-[3\mid m-1] \quad\text{and}\quad \binom{2}{2-(3\lceil m/3\rceil-m)}=1+[3\mid m+1].$$

So we have (1.2) and (1.3). $\square$

From (1.0), (1.2) and (1.3) we can deduce the following result.

Theorem 1.2. *Let p be a prime and $d \in \{0,\ldots,p\}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod{p}, \tag{1.4}$$

where the Legendre symbol $(\frac{a}{3})$ *coincides with the unique integer in* $\{0, \pm 1\}$ *satisfying* $a \equiv (\frac{a}{3}) \pmod 3$*. Also,*

$$\sum_{k=1}^{p-1} k\binom{2k}{k+d} \equiv \left([3 \mid p-d] - \frac{1}{3}\right)\left(2\left(\frac{p-d}{3}\right) - d\right) - [p=3] \pmod p, \tag{1.5}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k+d}}{k} \equiv \begin{cases} d^{-1}(-1+2(-1)^d+3[3 \mid p-d]) \pmod p & \text{if } d \neq 0, \\ -[p=3] \pmod p & \text{if } d = 0. \end{cases} \tag{1.6}$$

The well-known Catalan numbers given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} \qquad (n = 0, 1, 2, \ldots)$$

play important roles in combinatorics. For $n \in \mathbb{N}$ and $j = 0, 1, \ldots, n+1$, we define

$$C_{n,j} = 2\binom{2n}{n-j} - \binom{2n}{n-1-j} - \binom{2n}{n+1-j}$$

and view $C_{n,j}/2$ as a generalized Catalan number; it is clear that $C_{n,0}/2 = C_n$. From (1.6) we can deduce the following result.

Corollary 1.2. *Let* p *be a prime. Then, for any* $d = 0, \ldots, p-1$ *we have*

$$\sum_{k=1}^{p-1} \frac{C_{k+d}}{k} \equiv -[p=3]C_d + \sum_{j=1}^{d+1} \left(-1+2(-1)^j+3[3 \mid p-j]\right)\frac{C_{d,j}}{j} \pmod p. \tag{1.7}$$

Consequently, if $p \geqslant 5$ *then*

$$\sum_{k=1}^{p-1} \frac{C_k}{k} \equiv \frac{3}{2}\left(1 - \left(\frac{p}{3}\right)\right) \pmod p, \tag{1.8}$$

$$\sum_{k=1}^{p-2} \frac{C_{k+1}}{k} \equiv \frac{3}{4}\left(1 + \left(\frac{p}{3}\right)\right) \pmod p, \tag{1.9}$$

$$\sum_{k=1}^{p-3} \frac{C_{k+2}}{k} \equiv 3\left(\frac{p}{3}\right) \pmod p, \tag{1.10}$$

$$\sum_{k=1}^{p-4} \frac{C_{k+3}}{k} \equiv \frac{207(\frac{p}{3}) - 47}{24} \pmod p, \tag{1.11}$$

$$\sum_{k=1}^{p-5} \frac{C_{k+4}}{k} \equiv \frac{1503(\frac{p}{3}) - 497}{60} \pmod p. \tag{1.12}$$

Proof. Let $d \in \{0, \ldots, p-1\}$ and $k \in \mathbb{N}$. With the help of the Chu-Vandermonde identity (cf. [GKP, (5.27)]),

$$\begin{aligned}
C_{k+d} =& \binom{2k+2d}{k+d} - \binom{2k+2d}{k+d-1} \\
=& \sum_{j=-d}^{d} \binom{2d}{d-j}\binom{2k}{k+j} - \sum_{j=-d}^{d} \binom{2d}{d-j}\binom{2k}{k+j-1} \\
=& \sum_{j=-d}^{d} \binom{2d}{d-j}\binom{2k}{k+j} - \sum_{i=-d-1}^{d+1} \binom{2d}{d-1-i}\binom{2k}{k+i} \\
=& \sum_{0<j\leqslant d} \left(\binom{2d}{d-j}\binom{2k}{k+j} + \binom{2d}{d+j}\binom{2k}{k-j}\right) \\
&- \sum_{0<j\leqslant d+1} \left(\binom{2d}{d-1-j}\binom{2k}{k+j} + \binom{2d}{d-1+j}\binom{2k}{k-j}\right) \\
&+ \binom{2d}{d}\binom{2k}{k} - \binom{2d}{d-1}\binom{2k}{k}.
\end{aligned}$$

Thus

$$\begin{aligned}
&C_{k+d} + \left(\binom{2d}{d-1} - \binom{2d}{d}\right)\binom{2k}{k} \\
=& \sum_{0<j\leqslant d} 2\binom{2d}{d-j}\binom{2k}{k+j} - \sum_{j=1}^{d+1} \left(\binom{2d}{d-1-j} + \binom{2d}{d+1-j}\right)\binom{2k}{k+j}
\end{aligned}$$

and hence

$$C_{k+d} = C_d\binom{2k}{k} + \sum_{j=1}^{d+1} C_{d,j}\binom{2k}{k+j}. \tag{1.13}$$

In view of (1.13),

$$\sum_{k=1}^{p-1} \frac{C_{k+d}}{k} = C_d \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} + \sum_{j=1}^{d+1} C_{d,j} \sum_{k=1}^{p-1} \frac{\binom{2k}{k+j}}{k}.$$

Combining this with (1.6), we immediately get (1.7).

Observe that

$$C_{p+k} \equiv 2C_k \pmod p \qquad \text{for every } k = 0, \ldots, p-2; \tag{1.14}$$

in fact,

$$C_{p+k} = \frac{\binom{2p+2k}{p+k}}{p+k+1} = \frac{\binom{2p}{p}}{p+k+1} \cdot \frac{\prod_{0<j\leqslant 2k}(2p+j)}{\prod_{0<i\leqslant k}(p+i)^2} \equiv \frac{2}{k+1}\binom{2k}{k} \pmod p$$

since

$$\frac{1}{2}\binom{2p}{p}=\binom{2p-1}{p-1}=\prod_{j=1}^{p-1}\frac{2p-j}{j}\equiv(-1)^{p-1}\equiv 1 \pmod p.$$

Thus

$$\sum_{p-d\leqslant k<p}\frac{C_{k+d}}{k}=\sum_{0<k\leqslant d}\frac{C_{p-k+d}}{p-k}\equiv -2\sum_{0<k\leqslant d}\frac{C_{d-k}}{k} \pmod p.$$

(Note that if $0<k\leqslant d$ then $0\leqslant d-k<d\leqslant p-1$.)

Now assume that $p\geqslant 5$. Clearly

$$\begin{aligned}\sum_{j=1}^{d+1}[3\mid p-j]\frac{C_{d,j}}{j}=&\frac{1+(\frac{p}{3})}{2}\sum_{\substack{1\leqslant j\leqslant d+1\\ j\equiv 1\,(\mathrm{mod}\ 3)}}\frac{C_{d,j}}{j}+\frac{1-(\frac{p}{3})}{2}\sum_{\substack{1\leqslant j\leqslant d+1\\ j\equiv 2\,(\mathrm{mod}\ 3)}}\frac{C_{d,j}}{j}\\ =&\frac{1}{2}\sum_{\substack{j=1\\ 3\nmid j}}^{d+1}\frac{C_{d,j}}{j}+\left(\frac{p}{3}\right)\frac{1}{2}\sum_{j=1}^{d+1}\left(\frac{j}{3}\right)\frac{C_{d,j}}{j}.\end{aligned}$$

Therefore, by applying the above and (1.7) we obtain that

$$\begin{aligned}&\sum_{0<k<p-d}\frac{C_{k+d}}{k}-2\sum_{0<k\leqslant d}\frac{C_{d-k}}{k}\\ \equiv&\sum_{k=0}^{p-1}\frac{C_{k+d}}{k}\equiv\sum_{j=1}^{d+1}\left(2(-1)^j-1\right)\frac{C_{d,j}}{j}+\frac{3}{2}\sum_{\substack{j=1\\ 3\nmid j}}^{d+1}\frac{C_{d,j}}{j}+\left(\frac{p}{3}\right)\frac{3}{2}\sum_{j=1}^{d+1}\left(\frac{j}{3}\right)\frac{C_{d,j}}{j} \pmod p.\end{aligned}$$

When $d=0,1,2,3,4$, this yields (1.8)–(1.12) after some trivial computations. □

As usual we let $\lfloor\cdot\rfloor$ be the greatest integer function. On the basis of Theorem 1.1, we also establish the following general theorem concerning Catalan numbers.

Theorem 1.3. *Let p be a prime and $d,r\in\{0,\ldots,p-1\}$. Then*

$$(-1)^r\sum_{k=0}^{p-1}\binom{k+r}{r}C_{k+d}\equiv\sum_{0\leqslant k<d}\binom{d-1-k}{r}C_k+\sum_{i=0}^{r}(-1)^i\binom{d}{r-i}f_i(\varepsilon_i) \pmod p, \tag{1.15}$$

where $\varepsilon_i=(\frac{p-i-1}{3})$ and

$$\begin{aligned}f_i(\varepsilon_i)=&\sum_{k=0}^{\lfloor(i+1-\varepsilon_i)/3\rfloor}(-1)^{k+\varepsilon_i}\binom{i+2}{3k+1+\varepsilon_i}\binom{k+(i-2+\varepsilon_i)/3}{i}\\ &+[\varepsilon_i=0\ \&\ 3\mid i+1]+[i=0](3[\varepsilon_i=-1]-1).\end{aligned}$$

Remark 1.2. (a) (1.15) in the case $d = 0$ yields the congruence

$$\sum_{k=0}^{p-1}\binom{k+r}{r}C_k$$
$$\equiv\begin{cases}\sum_{k=0}^{\lfloor (r+2)/3\rfloor}(-1)^{k-1}\binom{r+2}{3k}\binom{k+(r-3)/3}{r}\pmod p & \text{if } p-r\equiv 0 \pmod 3,\\ \sum_{k=0}^{\lfloor (r+1)/3\rfloor}(-1)^{k}\binom{r+2}{3k+1}\binom{k+(r-2)/3}{r}+[p=3]\pmod p & \text{if } p-r\equiv 1 \pmod 3,\\ \sum_{k=0}^{\lfloor r/3\rfloor}(-1)^{k-1}\binom{r+2}{3k+2}\binom{k+(r-1)/3}{r}\pmod p & \text{if } p-r\equiv 2 \pmod 3.\end{cases}$$

(b) Let p be a prime and $d\in\{0,\ldots,p-1\}$. For each $r=0,\ldots,p-1$, clearly

$$\begin{aligned}(-1)^r\sum_{k=1}^{p}k^rC_{k+d-1} &= \sum_{k=0}^{p-1}(-k-1)^rC_{k+d}=\sum_{k=0}^{p-1}C_{k+d}\sum_{s=0}^{r}s!S(r,s)\binom{-k-1}{s}\\ &=\sum_{s=0}^{r}(-1)^s s!S(r,s)\sum_{k=0}^{p-1}\binom{k+s}{s}C_{k+d},\end{aligned}$$

where

$$S(r,s)=\frac{1}{s!}\sum_{t=0}^{s}(-1)^{s-t}\binom{s}{t}t^r\quad (0\leqslant s\leqslant r)$$

are Stirling numbers of the second kind (cf. [GKP]). This, together with Theorem 1.3, shows that if $P(x)$ is a polynomial of degree at most $p-1$ with p-adic integer coefficients then

$$\sum_{k=0}^{p-1}P(k)C_{k+d}\equiv\psi_P(d;\,p \bmod 3)\pmod p$$

for a suitable function ψ_P which can be constructed explicitly. This is general enough, because any integer r can be written in the form $(p-1)q+r_0$ with $q\in\mathbb{Z}$ and $r_0\in\{0,\ldots,p-2\}$, and by Fermat's little theorem we have $k^r\equiv k^{r_0}\pmod p$ for all $k=1,\ldots,p-1$.

Corollary 1.3. *Let p be a prime, and let $d\in\{0,1,\ldots,p-1\}$. Then we have*

$$\sum_{k=0}^{p-1}C_{k+d}\equiv\frac{3(\frac{p}{3})-1}{2}+\sum_{0\leqslant k<d}C_k\pmod p, \tag{1.16}$$

$$\sum_{k=0}^{p-1}kC_{k+d}\equiv\frac{d+1}{2}\left(1-\left(\frac{p}{3}\right)\right)-\left(\frac{p}{3}\right)d-\sum_{k=0}^{d}kC_{d-k}\pmod p, \tag{1.17}$$

$$\sum_{k=0}^{p-1}k^2C_{k+d}\equiv\frac{9d^2+6d-1}{6}\left(\frac{p}{3}\right)-\frac{(d+1)^2}{2}-[p=3]+\sum_{0<k\leqslant d}k^2C_{d-k}\pmod p. \tag{1.18}$$

Proof. For $i=0,1,2$ let ε_i and $f_i(\varepsilon_i)$ be as in Theorem 1.3. It is easy to verify that

$$f_0(\varepsilon_0)=(-1)^{\varepsilon_0}\binom{2}{1+\varepsilon_0}+3[\varepsilon_0=-1]-1\equiv\frac{3(\frac{p}{3})-1}{2}\pmod p$$

and $f_1(\varepsilon_1)=(\frac{p}{3})$ and $f_2(\varepsilon_2)=\frac{2}{3}(\frac{p}{3})+[p=3]$. Thus (1.15) in the case $r=0$ is actually equivalent to (1.16). Putting $r=1$ in (1.15) we get that

$$\begin{aligned}-\sum_{k=0}^{p-1}(k+1)C_{k+d}&\equiv\sum_{0\leqslant k<d}(d-1-k)C_k+df_0(\varepsilon_0)-f_1(\varepsilon_1)\\&\equiv-\sum_{0\leqslant k<d}C_k+\sum_{k=0}^{d}(d-k)C_k+d\frac{3(\frac{p}{3})-1}{2}-\left(\frac{p}{3}\right)\pmod p.\end{aligned}$$

This, together with (1.16), yields (1.17). By (1.15) in the case $r=2$,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(k+1)(k+2)}{2}C_{k+d}\equiv&\sum_{0\leqslant k<d}\frac{(d-k-1)(d-k-2)}{2}C_k\\&+\frac{d(d-1)}{2}f_0(\varepsilon_0)-df_1(\varepsilon_1)+f_2(\varepsilon_2)\pmod p\end{aligned}$$

and hence

$$\begin{aligned}\sum_{k=0}^{p-1}(k^2+3k+2)C_{k+d}\equiv&\sum_{0\leqslant k<d}((d-k)^2-3(d-k)+2)C_k+d(d-1)\frac{3(\frac{p}{3})-1}{2}\\&-2d\left(\frac{p}{3}\right)+\frac{4}{3}\left(\frac{p}{3}\right)+2[p=3]\pmod p.\end{aligned}$$

Combining this with (1.16) and (1.17) we immediately get (1.18). □

The Catalan numbers can also be defined by $C_0=1$ and the recursion $C_{n+1}=\sum_{k=0}^{n}C_kC_{n-k}$ $(n=0,1,2,\ldots)$. Below we provide some new recursions for Catalan numbers by using our previous congruences.

Theorem 1.4. *Let $d\in\mathbb{N}$ and $\delta\in\{0,1\}$. Then we have*

$$\begin{aligned}C_d=&(1-2\delta)\sum_{0\leqslant k<d}C_k+(-1)^{\delta}\sum_{i=0}^{d}\left(\frac{i-\delta}{3}\right)C_{d,i+1}+1+\delta\\=&\frac{1}{2}\sum_{j=1}^{d+1}(1-3[3\mid j])C_{d,j}+\frac{3}{2}.\end{aligned}\tag{1.19}$$

Also,

$$\begin{aligned}\sum_{k=0}^{d} kC_{d-k} =&\sum_{j=1}^{d+1}\left(\frac{j-1}{3}\right)\left(2\binom{2d}{d-j}-(d+1)C_{d,j}\right)-d\\ =&\sum_{j=1}^{d+1}\left(\frac{j+1}{3}\right)\left(2\binom{2d}{d-j}-(d+1)C_{d,j}\right)+2d+1\end{aligned}\tag{1.20}$$

and

$$\begin{aligned}&\sum_{0<k\leqslant d}\left(k-\frac{2}{3}\right)C_{d-k}+\frac{1}{3}\sum_{j=1}^{d+1} jC_{d,j}\\ =&\sum_{\substack{1\leqslant j\leqslant d+1\\ j\equiv 1\ (\mathrm{mod}\ 3)}} jC_{d,j}-d+\frac{2}{3}=\sum_{\substack{1\leqslant j\leqslant d+1\\ j\equiv 2\ (\mathrm{mod}\ 3)}} jC_{d,j}+2d-\frac{1}{3}.\end{aligned}\tag{1.21}$$

Remark 1.3. A referee of this paper noted that some identities in Theorem 1.4, such as (1.19), can also be established by generating function manipulations and the observation

$$C_{d,j}=C_{d-j}^{(2j)}-C_{d-(j-1)}^{(2j-2)}\qquad (j=1,\dots,d+1),$$

where $C_n^{(k)}=\binom{2n+k}{n}-\binom{2n+k}{n-1}$ for $k,n\in\mathbb{Z}$.

In Sections 2-5 we are going to show Theorems 1.1-1.4 respectively.

2. Proof of Theorem 1.1

Let R be a commutative ring with identity. For a formal power series $f(t)\in R[\![t]\!]$ and a nonnegative integer n, by $[t^n]f(t)$ we mean the coefficient of t^n in $f(t)$.

Proof of Theorem 1.1. We fix $l,n\in\mathbb{N}$. By the Chu-Vandermonde identity, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}s^m\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m}\\ =&\sum_{m=0}^{\infty}s^m\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\sum_{j=0}^{n}\binom{2(l-k)}{n-j}\binom{k-2l+m}{j}\binom{2k}{k-2l+m}\\ =&\sum_{k=0}^{l}(-1)^k\binom{l}{k}\sum_{j=0}^{n}\binom{2(l-k)}{n-j}\binom{2k}{j}\sum_{m=0}^{\infty}\binom{2k-j}{k-2l+m-j}(-s)^m\\ =&\sum_{k=0}^{l}(-1)^k\binom{l}{k}\sum_{j=0}^{n}\binom{2(l-k)}{n-j}\binom{2k}{j}(-s)^{2l-k+j}(1-s)^{2k-j}\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{m=0}^{\infty} s^m \sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m}\\
=&\sum_{k=0}^{l}(-1)^k\binom{l}{k}(-s)^{2l-k}[t^n](1+t)^{2(l-k)}((1-s)-st)^{2k}\\
=&[t^n]s^l\sum_{k=0}^{l}\binom{l}{k}\left(s(1+t)^2\right)^{l-k}(1-s(1+t))^{2k}\\
=&[t^n]s^l\left(s(1+t)^2+(1-s(1+t))^2\right)^l\\
=&[t^n]\left((s+st)^2+s(1-s-st)^2\right)^l.
\end{aligned}$$

Clearly

$$\begin{aligned}
&[t^n](1+s)^l\left((s+t)^2+s(1-s-t)^2\right)^l\\
=&[t^n]\left((1+s)((1+s)t^2+2s^2t+s^2+s(1-s)^2)\right)^l\\
=&[t^n]\left(((1+s)t+s^2)^2+s\right)^l\\
=&\sum_{k=0}^{l}\binom{l}{k}[t^n]\left((1+s)t+s^2\right)^{2k}s^{l-k}\\
=&\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}(1+s)^n(s^2)^{2k-n}s^{l-k}.
\end{aligned}$$

Replacing t by st we obtain that

$$\begin{aligned}
&[t^n]\left((s+st)^2+s(1-s-st)^2\right)^l\\
=&s^n\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}s^{3k+l-2n}(1+s)^{n-l}\\
=&\sum_{m=0}^{\infty}s^m\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}.
\end{aligned}$$

In view of the above, we immediately get (1.1) for any $m\in\mathbb{N}$ by equating coefficients of s^m. $\square$

3. Proof of Theorem 1.2

Lemma 3.1. *Let p be any prime, and $k,r\in\{0,\ldots,p-1\}$. Then*

$$\binom{p-1}{k}\equiv(-1)^k\pmod p\ \text{ and }\ \binom{p+k+r}{p+r}\equiv\binom{p+k+r}{r}\equiv\binom{k+r}{r}\pmod p.$$

Proof. Clearly

$$\binom{p-1}{k} = \prod_{0<j\leqslant k} \frac{p-j}{j} = \prod_{0<j\leqslant k} \left(\frac{p}{j}-1\right) \equiv (-1)^k \pmod p$$

and

$$\begin{aligned}\binom{p+k+r}{p+r} &= \binom{p+k+r}{k} = \prod_{0<j\leqslant k} \left(\frac{p}{j}+1\right) \times \prod_{0<s\leqslant r} \frac{p+k+s}{p+s} \\ &\equiv \prod_{0<s\leqslant r} \frac{p+k+s}{s} = \binom{p+k+r}{r} \equiv \prod_{0<s\leqslant r} \frac{k+s}{s} = \binom{k+r}{r} \pmod p.\end{aligned}$$

So we have the desired congruences. □

Proof of Theorem 1.2. In the case $d=p$, (1.4)–(1.6) hold trivially. Below we assume $d<p$.

(i) Let $m=2(p-1)+d$. Applying (1.2) and (1.3) with $l=p-1$ we obtain that

$$\sum_{k=0}^{p-1} (-1)^{d-k} \binom{p-1}{k} \binom{m-k}{p} \binom{2k}{k+d} = (1-[3\mid m-1]) \binom{p-1}{\lceil m/3\rceil} \binom{2\lceil m/3\rceil}{p}$$

and

$$\sum_{k=0}^{p-1} (-1)^{d-k} \binom{p-1}{k} \binom{m-k}{p+1} \binom{2k}{k+d} = (1+[3\mid m+1]) \binom{p-1}{\lceil m/3\rceil} \binom{2\lceil m/3\rceil}{p+1}.$$

If $d\leqslant k\leqslant p-1$, then $0\leqslant (m-k)-p=p-2-(k-d)<p$ unless $d=0$ and $k=p-1$, in which case $\binom{2k}{k+d}=\binom{2p-2}{p-1}\equiv 0 \pmod p$; also, $m-k=p\equiv 0 \pmod p$ when $m-k-1<p$. Thus, by applying Lemma 3.1, we have

$$\sum_{k=0}^{p-1} (-1)^{k} \binom{p-1}{k} \binom{m-k}{p} \binom{2k}{k+d} \equiv \sum_{k=0}^{p-1} \binom{2k}{k+d} \pmod p$$

and

$$\sum_{k=0}^{p-1} (-1)^{k} \binom{p-1}{k} \binom{m-k}{p+1} \binom{2k}{k+d} \equiv \sum_{k=0}^{p-1} (m-k) \binom{2k}{k+d} \pmod p.$$

Let $\varepsilon=(\frac{p-d}{3})$. Then $p-\varepsilon\equiv d \pmod 3$. Clearly

$$0<\frac{m}{3}\leqslant \left\lceil\frac{m}{3}\right\rceil = \frac{2(p-\varepsilon)+d}{3} - [3\mid p-d+1] \leqslant \frac{m+2}{3} = \frac{2p+d}{3} < p.$$

If $p \geqslant 5$ then $\lceil m/3 \rceil \geqslant m/3 > p/2$; if $p = 3$ then $\lceil m/3 \rceil \geqslant \lceil 4/3 \rceil = 2 > p/2$. So

$$0 < 2\left\lceil \frac{m}{3} \right\rceil - p < 2p - p = p$$

unless $p = 2$ in which case $\lceil m/3 \rceil = 1$. Therefore, with the help of Lemma 3.1,

$$\begin{aligned}&(1 - [3 \mid m-1])\binom{p-1}{\lceil m/3 \rceil}\binom{2\lceil m/3 \rceil}{p}\\ \equiv&(1 - [3 \mid p-d])(-1)^{\lceil m/3 \rceil} = (1 - [3 \mid p-d])(-1)^{d-[3|p-d+1]} = (-1)^d \varepsilon \pmod p\end{aligned}$$

and

$$\begin{aligned}&(1 + [3 \mid m+1])\binom{p-1}{\lceil m/3 \rceil}\binom{2\lceil m/3 \rceil}{p+1}\\ \equiv&(1 + [3 \mid p-d+1])(-1)^{\lceil m/3 \rceil}\binom{2\lceil m/3 \rceil}{1}[p \neq 2]\\ \equiv&(1 + [3 \mid p-d+1])(-1)^{d-[3|p-d+1]}2\left(\frac{2(p-\varepsilon)+d}{3} - [3 \mid p-d+1]\right)[p \neq 2]\\ \equiv&\begin{cases} (-1)^d \frac{4}{3}(1-d) + (-1)^d[p=3] \pmod p & \text{if } \varepsilon = -1 \text{ (i.e., } 3 \mid p-d+1),\\ (-1)^d \frac{2}{3}(d-2\varepsilon) + (-1)^d[p=3] \pmod p & \text{otherwise.}\end{cases}\end{aligned}$$

In view of the above,

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv (-1)^d(1 - [3 \mid m-1])\binom{p-1}{\lceil m/3 \rceil}\binom{2\lceil m/3 \rceil}{p} \equiv \varepsilon \pmod p,$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1} k\binom{2k}{k+d}\\ =&m\sum_{k=0}^{p-1}\binom{2k}{k+d} - \sum_{k=0}^{p-1}(m-k)\binom{2k}{k+d}\\ \equiv&m\varepsilon - (-1)^d(1 + [3 \mid m+1])\binom{p-1}{\lceil m/3 \rceil}\binom{2\lceil m/3 \rceil}{p+1}\\ \equiv&\begin{cases}(d-2)\varepsilon - \frac{4}{3}(1-d) - [p=3] = \frac{d+2}{3} - [p=3] & \text{if } \varepsilon = -1,\\ (d-2)\varepsilon - \frac{2}{3}(d-2\varepsilon) - [p=3] = \varepsilon d - \frac{2}{3}(d+\varepsilon) - [p=3] & \text{otherwise,}\end{cases}\\ \equiv&\left([3 \mid p-d] - \frac{1}{3}\right)(2\varepsilon - d) - [p=3] \pmod p.\end{aligned}$$

This proves (1.4) and (1.5).

(ii) Our strategy to deduce (1.6) is to compute S mod p^2 in two different ways, where

$$S=\sum_{k=0}^{p}(-1)^k\binom{p}{k}\binom{2p+d-k}{p}\binom{2k}{k+d}.$$

Observe that $\binom{2p}{p}\equiv 2 \pmod{p^2}$. In the case $p\neq 2$, this is because

$$\begin{aligned}\frac{1}{2}\binom{2p}{p}=&\binom{2p-1}{p-1}=\prod_{k=1}^{p-1}\frac{2p-k}{k}=\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)\\ \equiv&1-2p\sum_{k=1}^{p-1}\frac{1}{k}=1-p\sum_{k=1}^{p-1}\left(\frac{1}{k}+\frac{1}{p-k}\right)\equiv 1 \pmod{p^2}.\end{aligned}$$

(Moreover, by Wolstenholme's theorem, $\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ if $p>3$.) Therefore

$$\begin{aligned}S=&\sum_{k=d}^{p}(-1)^k\binom{p}{k}\binom{2p+d-k}{p}\binom{2k}{k+d}\\ \equiv&2(-1)^d\binom{p}{d}-\binom{p+d}{p}\binom{2p}{p+d}+\sum_{d<k<p}(-1)^k\binom{p}{k}\binom{2k}{k+d} \pmod{p^2}.\end{aligned}$$

(Note that if $d<k<p$ then $p\mid\binom{p}{k}$ and $\binom{p+(p-(k-d))}{p}\equiv 1 \pmod{p}$.) If $d\neq 0$, then

$$\binom{p}{d}=\frac{p}{d}\binom{p-1}{d-1}\equiv(-1)^{d-1}\frac{p}{d} \pmod{p^2}$$

and

$$\begin{aligned}\binom{p+d}{p}\binom{2p}{p+d}=&\binom{p+d}{p}\frac{2p}{p-d}\binom{2p-1}{p-d-1}\\ =&\frac{2p}{p-d}\binom{p+d}{p}\prod_{0<j<p-d}\left(\frac{2p}{j}-1\right)\\ \equiv&\frac{2p}{p-d}\binom{d}{0}(-1)^{p-d-1}\equiv(-1)^{d-1}\frac{2p}{d} \pmod{p^2}.\end{aligned}$$

Thus

$$\begin{aligned}&S-\sum_{k=d}^{p-1}(-1)^k\binom{p}{k}\binom{2k}{k+d}\\ \equiv&(-1)^d\binom{p}{d}-\binom{p+d}{d}\binom{2p}{p+d}\\ \equiv&\begin{cases}-\frac{p}{d}-(-1)^{d-1}\frac{2p}{d}=(2(-1)^d-1)\frac{p}{d} \pmod{p^2} & \text{if } d\neq 0,\\ 1-2=-1 \pmod{p^2} & \text{if } d=0.\end{cases}\end{aligned}$$

Set $m = 2p + d$. Applying (1.0) with $l = p$ we get that

$$(-1)^d S = \begin{cases} \binom{p}{m/3}\binom{2m/3}{p} & \text{if } 3 \mid m \text{ (i.e., } p \equiv d \pmod 3\text{)},\\ 0 & \text{otherwise.}\end{cases}$$

In the case $3 \mid m$,

$$\begin{aligned}\binom{p}{m/3}\binom{2m/3}{p} =& \frac{p}{m/3}\binom{p-1}{m/3-1}\binom{p+(p+2d)/3}{p}\\ \equiv& \frac{p}{m/3}(-1)^{m/3-1} \equiv (-1)^{d-1}\frac{3p}{m} \pmod{p^2}.\end{aligned}$$

Therefore

$$S \equiv \begin{cases} -3p/m \pmod{p^2} & \text{if } p \equiv d \pmod 3,\\ 0 \pmod{p^2} & \text{otherwise.}\end{cases}$$

Comparing the two congruences for S mod p^2, we finally obtain that

$$\sum_{k=0}^{p-1}(-1)^k\binom{p}{k}\binom{2k}{k+d} \equiv \begin{cases} -[3 \mid p-d]\frac{3p}{d} - (2(-1)^d - 1)\frac{p}{d} \pmod{p^2} & \text{if } d \neq 0,\\ [p=3] - (-1) \pmod{p^2} & \text{if } d = 0.\end{cases}$$

This is equivalent to (1.6) since $\binom{p}{k} \equiv \frac{p}{k}(-1)^{k-1} \pmod{p^2}$ for $k = 1, \ldots, p-1$.

The proof of Theorem 1.2 is now complete. $\square$

4. Proof of Theorem 1.3

Lemma 4.1. *Let r be a positive integer, and let $p \geqslant 4r + 7$ be a prime. Then $\sum_{k=0}^{p-1}\binom{k+r}{r}C_k$ is congruent to*

$$\sum_{k=0}^{\lfloor (r+1-\varepsilon_r)/3\rfloor}(-1)^{k+\varepsilon_r}\binom{r+2}{3k+1+\varepsilon_r}\binom{k+(r-2+\varepsilon_r)/3}{r}$$

modulo p with $\varepsilon_r = (\frac{p-r-1}{3})$.

Proof. Let $l = p - r - 1$ and $\delta \in \{0, 1\}$. Applying (1.1) with $m = 2l - \delta$ and $n = p$, we obtain that

$$\sum_{k=0}^{l}(-1)^{k+\delta}\binom{l}{k}\binom{2l-\delta-k}{p}\binom{2k}{k-\delta} = \sum_{k=0}^{l}\binom{l}{k}\binom{2k}{p}\binom{r+1}{l+p-\delta-3k}.$$

For $k = 0, \ldots, l$ it is apparent that

$$\begin{aligned}\binom{l}{k} =& \binom{p-r-1}{k} = \frac{(p-1)\cdots(p-k-r)}{(k+r)!} \times \frac{(k+1)\cdots(k+r)}{(p-1)\cdots(p-r)}\\ \equiv& (-1)^{k+r}(-1)^r\frac{(k+1)\cdots(k+r)}{r!} = (-1)^k\binom{k+r}{r} \pmod p.\end{aligned}$$

Thus

$$
\begin{aligned}
&(-1)^{\delta}\sum_{k=0}^{l}\binom{k+r}{r}\binom{2l-\delta-k}{p}\binom{2k}{k-\delta}\\
&\equiv\sum_{k=0}^{l}(-1)^k\binom{k+r}{r}\binom{2k}{p}\binom{r+1}{l+p-\delta-3k}\pmod p.
\end{aligned}
$$

If $0\leqslant k\leqslant 2l-\delta-p=p-2r-2-\delta$, then $2l-\delta-k\in[p,2p)$ and hence

$$\binom{2l-\delta-k}{p}\equiv\binom{2l-\delta-k}{0}=1\pmod p$$

by Lemma 3.1. For $k\in\{0,\dots,l\}$, clearly

$$r+1\geqslant l+p-\delta-3k\iff 3k\geqslant 2l-\delta\iff k\geqslant\frac{2l-\delta}{3}.$$

If $l\geqslant k\geqslant\lceil(2l-\delta)/3\rceil$, then

$$2p>2k\geqslant\frac{2}{3}(2p-2r-2-\delta)=p+\frac{p-4r-4-2\delta}{3}>p+\frac{p-4r-7}{3}\geqslant p$$

and hence $\binom{2k}{p}\equiv\binom{2k}{0}=1\pmod p$ by Lemma 3.1. Therefore

$$
\begin{aligned}
&(-1)^{\delta}\sum_{k=0}^{p-2r-2-\delta}\binom{k+r}{r}\binom{2k}{k-\delta}\\
&\equiv\sum_{\lceil\frac{2l-\delta}{3}\rceil\leqslant k\leqslant\lfloor\frac{l+p}{3}\rfloor}(-1)^k\binom{k+r}{r}\binom{r+1}{l+p-\delta-3k}\pmod p.
\end{aligned}
$$

When $p-2r-2\leqslant k\leqslant p-2$, we have $2k\geqslant 2(p-2r-2)>p>k+1$ and hence

$$C_k=\frac{(2k)!}{k!(k+1)!}\equiv 0\pmod p\quad\text{and}\quad\binom{2k}{k}=(k+1)C_k\equiv 0\pmod p.$$

Note also that $\binom{p-1+r}{r}=p\cdots(p+r-1)/r!\equiv 0\pmod p$. So, by the above,

$$
\begin{aligned}
\sum_{k=0}^{p-1}\binom{k+r}{r}C_k&\equiv\sum_{k=0}^{p-2r-3}\binom{k+r}{r}C_k\\
&\equiv\sum_{k=0}^{p-2r-2}\binom{k+r}{r}\binom{2k}{k}-\sum_{k=0}^{p-2r-3}\binom{k+r}{r}\binom{2k}{k-1}\\
&\equiv\sum_{2l\leqslant 3k\leqslant l+p}(-1)^k\binom{k+r}{r}\binom{r+1}{l+p-3k}\\
&\quad+\sum_{2l-1\leqslant 3k\leqslant l+p}(-1)^k\binom{k+r}{r}\binom{r+1}{l+p-1-3k}\pmod p
\end{aligned}
$$

and hence
$$\sum_{k=0}^{p-1}\binom{k+r}{r}C_k \equiv S(r) \pmod p,$$
where
$$\begin{aligned}S(r)=&\sum_{2l-1\leqslant 3k\leqslant l+p}(-1)^k\binom{k+r}{r}\binom{r+2}{l+p-3k}\\ =&\sum_{3k\geqslant 2l-1}(-1)^k\binom{k+r}{r}\binom{r+2}{3k-2l+1}.\end{aligned}$$
Clearly
$$\left\lceil\frac{2l-1}{3}\right\rceil=\frac{2l-(\frac{2l}{3})}{3}=\frac{2l+\varepsilon_r}{3}$$
and thus
$$\begin{aligned}S(r)=&\sum_{k\in\mathbb{N}}(-1)^{k+(2l+\varepsilon_r)/3}\binom{k+(2l+\varepsilon_r)/3+r}{r}\binom{r+2}{3k+2l+\varepsilon_r-2l+1}\\ =&\sum_{k\in\mathbb{N}}(-1)^{k+\varepsilon_r}\binom{r+2}{3k+1+\varepsilon_r}\binom{k+(2p+r-2+\varepsilon_r)/3}{r}\\ \equiv&\sum_{k=0}^{\lfloor(r+1-\varepsilon_r)/3\rfloor}(-1)^{k+\varepsilon_r}\binom{r+2}{3k+1+\varepsilon_r}\binom{k+(r-2+\varepsilon_r)/3}{r} \pmod p.\end{aligned}$$
(Note that $\binom{x}{r}$ is a polynomial in x with p-adic integer coefficients.) So we have the desired result. $\square$

Proof of Theorem 1.3. With the help of (1.14),
$$\begin{aligned}\sum_{k=0}^{p-1}\binom{k+r}{r}C_{k+d}=&\sum_{k=d}^{p-1}\binom{k-d+r}{r}C_k+\sum_{0\leqslant k<d}\binom{p+k-d+r}{r}C_{p+k}\\ \equiv&\sum_{k=d}^{p-1}\binom{k-d+r}{r}C_k+2\sum_{0\leqslant k<d}\binom{k-d+r}{r}C_k \pmod p.\end{aligned}$$
By the transformation $(-1)^r\binom{-x}{r}=\binom{x+r-1}{r}$ and the Chu-Vandermonde identity, for any $k\in\{0,\ldots,p-1\}$ we have
$$\begin{aligned}(-1)^r\binom{k-d+r}{r}=&\binom{d-k-1}{r}=\sum_{i=0}^{r}\binom{d}{r-i}\binom{-k-1}{i}\\ =&\sum_{i=0}^{r}\binom{d}{r-i}(-1)^i\binom{k+i}{i}.\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^r\sum_{k=0}^{p-1}\binom{k+r}{r}C_{k+d}\\ &\equiv\sum_{i=0}^{r}(-1)^i\binom{d}{r-i}\sum_{k=0}^{p-1}\binom{k+i}{i}C_k+\sum_{0\leqslant k<d}\binom{d-1-k}{r}C_k \pmod p.\end{aligned}$$

So, it suffices to show that $\sum_{k=0}^{p-1}\binom{k+i}{i}C_k\equiv f_i(\varepsilon_i) \pmod p$ for all $i=0,\ldots,p-1$. As r is an arbitrarily chosen element of $\{0,\ldots,p-1\}$, below we only need to show the congruence

$$\sum_{k=0}^{p-1}\binom{k+r}{r}C_k\equiv f_r(\varepsilon_r) \pmod p. \tag{4.1}$$

To prove (4.1) we further extend the idea in the proofs of (1.4) and (1.5).

Let $\delta\in\{0,1\}$. Applying (1.1) with $l=p-1$, $m=2p-1-\delta$ and $n=p+r$ we get that

$$(-1)^{\delta+1}S_\delta=\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{p+r}\binom{r+1}{2p-\delta+r-3k},$$

where

$$S_\delta=\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}\binom{2p-1-\delta-k}{p+r}\binom{2k}{k+1-\delta}.$$

By Lemma 3.1, $(-1)^k\binom{p-1}{k}\equiv 1 \pmod p$ for all $k=0,\ldots,p-1$, and $\binom{K}{p+r}\equiv\binom{K}{r} \pmod p$ for any integer $K\in[p+r,2p+r)$. Thus

$$S_\delta\equiv\sum_{0\leqslant k<p-r-\delta}\binom{2p-1-\delta-k}{r}\binom{2k}{k+1-\delta} \pmod p$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{p+r}\binom{r+1}{2p-\delta+r-3k}\\ &\equiv\sum_{\lceil\frac{p+r}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r}\binom{r+1}{2p-\delta+r-3k} \pmod p.\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^{\delta+1}\sum_{0\leqslant k<p-r-\delta}\binom{2p-1-\delta-k}{r}\binom{2k}{k+1-\delta}\\ &\equiv\sum_{\lceil\frac{p+r}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r}\binom{r+1}{2p-\delta+r-3k} \pmod p.\end{aligned} \tag{4.2}$$

If $r\in\{1,\dots,p-1\}$ then
$$\binom{2p-2-(p-1-r)}{r}=\binom{p-1+r}{r}=\frac{(p+r-1)!}{(p-1)!r!}\equiv 0 \pmod p;$$
if $r=0$ then
$$\binom{2(p-1-r)}{p-1-r}=\frac{(p+p-2)!}{(p-1)!(p-1)!}\equiv 0 \pmod p.$$
Thus, when $k=p-1-r$ we have
$$\binom{2p-2-k}{r}\binom{2k}{k}\equiv 0 \pmod p.$$
In view of this and (4.2),
$$\begin{aligned}&\sum_{k=0}^{p-1-r}\binom{2p-1-k}{r}C_k\\ =&\sum_{k=0}^{p-1-r}\binom{2p-1-k}{r}\left(\binom{2k}{k}-\binom{2k}{k+1}\right)\\ =&-\sum_{k=0}^{p-1-r}\binom{2p-1-k}{r}\binom{2k}{k+1}\\ &+\sum_{k=0}^{p-1-r}\left(\binom{2p-2-k}{r}+\binom{2p-2-k}{r-1}\right)\binom{2k}{k}\\ \equiv&\sum_{\lceil\frac{p+r}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r}\left(\binom{r+1}{2p+r-3k}+\binom{r+1}{2p-1+r-3k}\right)\\ &+\sum_{\lceil\frac{p+r-1}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r-1}\binom{r}{2p-2+r-3k} \pmod p.\end{aligned}$$
Since
$$\binom{2p-1-k}{r}=\prod_{0<s\leqslant r}\frac{2p-k-s}{s}\equiv(-1)^r\prod_{0<s\leqslant r}\frac{k+s}{s}=(-1)^r\binom{k+r}{r} \pmod p$$
for every $k=0,\dots,p-1-r$, we have
$$\begin{aligned}&(-1)^r\sum_{k=0}^{p-1-r}\binom{k+r}{r}C_k\\ \equiv&\sum_{\lceil\frac{p+r}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r}\binom{r+2}{2p+r-3k}\\ &+\sum_{\lceil\frac{p+r-1}{2}\rceil\leqslant k<p}(-1)^k\binom{2k}{r-1}\binom{r}{2p-2+r-3k} \pmod p.\end{aligned}$$

When $0 \leqslant 2p+r-3k \leqslant r+2$ (i.e., $2p-2 \leqslant 3k \leqslant 2p+r$), if $k < (p+r)/2$ then $p-1 < (4p-4)/3 \leqslant 2k \leqslant p+r-1$ and hence

$$\binom{2k}{r} = \frac{2k(2k-1)\cdots(2k-r+1)}{r!} \equiv 0 \pmod{p}.$$

Similarly, when $0 \leqslant 2p-2+r-3k \leqslant r$ (i.e., $2p-2 \leqslant 3k \leqslant 2p-2+r$), if $k < (p+r-1)/2$ then $p \leqslant 2k \leqslant p+r-2$ and hence

$$\binom{2k}{r-1} = \frac{2k(2k-1)\cdots(2k-r+2)}{(r-1)!} \equiv 0 \pmod{p}.$$

Therefore

$$\begin{aligned}
&(-1)^r \sum_{k=0}^{p-1-r} \binom{k+r}{r} C_k \\
\equiv &\sum_{2p-2\leqslant 3k\leqslant 2p+r} (-1)^k \binom{2k}{r}\binom{r+2}{2p+r-3k} \\
&+ \sum_{2p-2\leqslant 3k\leqslant 2p-2+r} (-1)^k \binom{2k}{r-1}\binom{r}{2p-2+r-3k} \\
\equiv &\sum_{k\geqslant h}(-1)^k\left(\binom{2k}{r}\binom{r+2}{3k-2p+2} + \binom{2k}{r-1}\binom{r}{3k-2p+2}\right) \pmod{p},
\end{aligned}$$

where

$$h = \left\lceil \frac{2p-2}{3} \right\rceil = \frac{2p-1+\varepsilon_1}{3}.$$

(Recall that $\varepsilon_1 = (\frac{p-2}{3}) \equiv p-2 \pmod{3}$.)

If $p-r \leqslant k \leqslant p-1$, then $k+1 \leqslant p \leqslant k+r$ and hence

$$\binom{k+r}{r} = \prod_{s=1}^{r} \frac{k+s}{s} \equiv 0 \pmod{p}.$$

Note also that $2h \equiv 2(\varepsilon_1-1)/3 + [p=3] \pmod{p}$. Thus, by the above we have

$$\begin{aligned}
&(-1)^r \sum_{k=0}^{p-1} \binom{k+r}{r} C_k \\
\equiv &\sum_{k\in\mathbb{N}}(-1)^{k+h}\left(\binom{2k+2h}{r}\binom{r+2}{3k+3h-2p+2} + \binom{2k+2h}{r-1}\binom{r}{3k+3h-2p+2}\right) \\
\equiv &\Psi_r(p \bmod 3) \pmod{p},
\end{aligned}$$

where

$$\Psi_r(p \bmod 3) = \sum_{k\in\mathbb{N}} (-1)^{k+\varepsilon_1-1}\binom{2k+2(\varepsilon_1-1)/3+[p=3]}{r}\binom{r+2}{3k+1+\varepsilon_1}$$
$$+\sum_{k\in\mathbb{N}} (-1)^{k+\varepsilon_1-1}\binom{2k+2(\varepsilon_1-1)/3+[p=3]}{r-1}\binom{r}{3k+1+\varepsilon_1}.$$

(Note that both ε_1 and $[p=3]$ only depend on p mod 3.)

As

$$\Psi_0(p \bmod 3) = (-1)^{\varepsilon_1-1}\binom{2}{1+\varepsilon_1} = f_0(\varepsilon_0),$$

(4.1) holds when $r=0$. If $p=3$ then

$$-\Psi_1(p \bmod 3) = -3 \equiv 0 = f_1(\varepsilon_1) \pmod 3 \quad\text{and}\quad \Psi_2(p \bmod 3) = 1 = f_2(\varepsilon_2).$$

So (4.1) is also valid in the case $p=3$.

Below we assume that $r\neq 0$ and $p\neq 3$. Recall that

$$\sum_{k=0}^{p-1}\binom{k+r}{r}C_k \equiv (-1)^r\Psi_r(p \bmod 3) \pmod p.$$

If $p'\geqslant 4r+7$ is a prime with $p'\equiv p \pmod 3$, then

$$(-1)^r\Psi_r(p \bmod 3) = (-1)^r\Psi_r(p' \bmod 3) \equiv \sum_{k=0}^{p'-1}\binom{k+r}{r}C_k \equiv f_r(\varepsilon_r) \pmod{p'}$$

with the help of Lemma 4.1. By Dirichlet's theorem (cf. [IR, p. 251]), there are infinitely many primes p' with $p'\equiv p \pmod 3$. So we must have $(-1)^r\Psi_r(p \bmod 3) = f_r(\varepsilon_r)$ and hence (4.1) follows. We are done. □

5. Proof of Theorem 1.4

In this section we let p be an arbitrary prime greater than d.

In view of (1.13),

$$\sum_{k=0}^{p-1} C_{k+d} = C_d\sum_{k=0}^{p-1}\binom{2k}{k} + \sum_{j=1}^{d+1} C_{d,j}\sum_{k=0}^{p-1}\binom{2k}{k+j}$$

and

$$\sum_{k=0}^{p-1} kC_{k+d} = C_d\sum_{k=0}^{p-1}k\binom{2k}{k} + \sum_{j=1}^{d+1} C_{d,j}\sum_{k=0}^{p-1}k\binom{2k}{k+j}.$$

Combining these with Theorem 1.2, we immediately get the congruences

$$\sum_{k=0}^{p-1} C_{k+d} \equiv \left(\frac{p}{3}\right) C_d + \sum_{j=1}^{d+1} \left(\frac{p-j}{3}\right) C_{d,j} \pmod p \tag{5.1}$$

and

$$\sum_{k=0}^{p-1} kC_{k+d} \equiv -\left(\frac{p}{3}\right)\frac{2}{3}C_d + \sum_{j=1}^{d+1} C_{d,j}\left([3 \mid p-j] - \frac{1}{3}\right)\left(2\left(\frac{p-j}{3}\right) - j\right) \pmod p. \tag{5.2}$$

(It is easy to check that $C_d + \sum_{j=1}^{d+1} C_{d,j} = 0$ if $p = 3$ (and hence $d \in \{0, 1, 2\}$).)

By (1.16) and (5.1),

$$\frac{3(\frac{p}{3}) - 1}{2} + \sum_{0 \leqslant k < d} C_k \equiv \left(\frac{p}{3}\right) C_d + \sum_{j=1}^{d+1} \left(\frac{p-j}{3}\right) C_{d,j} \pmod p.$$

If p is congruent to 1 or 2 modulo 3, this gives

$$1 + \sum_{0 \leqslant k < d} C_k \equiv C_d - \sum_{j=1}^{d+1} \left(\frac{j-1}{3}\right) C_{d,j} \pmod p$$

and

$$-2 + \sum_{0 \leqslant k < d} C_k \equiv -C_d - \sum_{j=1}^{d+1} \left(\frac{j-2}{3}\right) C_{d,j} \pmod p$$

respectively. Note that both sides of these two congruences are independent of p. Thus we have the first equality in (1.19) since the residue classes $1 \pmod 3$ and $2 \pmod 3$ both contain infinitely many primes by Dirichlet's theorem. The second equality in (1.19) also holds because

$$\begin{aligned}&\sum_{\delta=0}^{1}\left((1-2\delta)\sum_{0 \leqslant k < d} C_k + (-1)^\delta \sum_{i=0}^{d}\left(\frac{i-\delta}{3}\right) C_{d,i+1} + 1 + \delta\right)\\ =&\sum_{i=0}^{d}\left(\left(\frac{i}{3}\right) - \left(\frac{i-1}{3}\right)\right) C_{d,i+1} + 1 + 2 = \sum_{i=0}^{d}(1 - 3[3 \mid i+1])C_{d,i+1} + 3.\end{aligned}$$

Observe that

$$\begin{aligned}&\sum_{k=0}^{p-1} kC_{k+d} + (d+1)\sum_{k=0}^{p-1} C_{k+d}\\ =&\sum_{k=0}^{p-1}\binom{2k+2d}{k+d} = \sum_{j=-d}^{d}\binom{2d}{d-j}\sum_{k=0}^{p-1}\binom{2k}{k+j}\\ =&\binom{2d}{d}\sum_{k=0}^{p-1}\binom{2k}{k} + 2\sum_{0<j\leqslant d}\binom{2d}{d-j}\sum_{k=0}^{p-1}\binom{2k}{k+j}.\end{aligned}$$

Applying Theorem 1.2 and (5.1), we get that

$$\begin{aligned}\sum_{k=0}^{p-1} kC_{k+d} \equiv& \left(\frac{p}{3}\right)\binom{2d}{d} + 2\sum_{0<j\leqslant d}\left(\frac{p-j}{3}\right)\binom{2d}{d-j} \\ &- (d+1)\left(\frac{p}{3}\right)C_d - (d+1)\sum_{j=1}^{d+1}\left(\frac{p-j}{3}\right)C_{d,j} \\ \equiv& \sum_{j=1}^{d+1}\left(\frac{p-j}{3}\right)\left(2\binom{2d}{d-j} - (d+1)C_{d,j}\right) \pmod{p}.\end{aligned}$$

Comparing this with (1.17) we obtain the identity (1.20) by applying Dirichlet's theorem.

It follows from (5.1) and (5.2) that

$$\sum_{k=0}^{p-1}\left(k+\frac{2}{3}\right)C_{k+d} \equiv \sum_{j=1}^{d+1}\left(\frac{1}{3} - [3 \mid p-j]\right) jC_{d,j} \pmod{p}. \tag{5.3}$$

On the other hand, by (1.16) and (1.17) we have

$$\begin{aligned}\sum_{k=1}^{p-1}\left(k+\frac{2}{3}\right)C_{k+d} \equiv& \frac{d+1}{2}\left(1-\left(\frac{p}{3}\right)\right) - \left(\frac{p}{3}\right)d - \sum_{k=0}^{d} kC_{d-k} \\ &+ \frac{2}{3}\cdot\frac{3(\frac{p}{3})-1}{2} + \frac{2}{3}\sum_{0\leqslant k<d} C_k \pmod{p}.\end{aligned}$$

Comparing this with (5.3) we finally get (1.21) by applying Dirichlet's theorem.

The proof of Theorem 1.4 is now complete.

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