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FINITE COVERS OF GROUPS BY COSETS OR SUBGROUPS

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Abstract. This paper deals with combinatorial aspects of finite covers of groups by cosets or subgroups. Let $a_1G_1, \ldots, a_kG_k$ be left cosets in a group G such that $\{a_iG_i\}_{i=1}^k$ covers each element of G at least m times but none of its proper subsystems does. We show that if G is cyclic, or G is finite and $G_1, \ldots, G_k$ are normal Hall subgroups of G, then the inequality $k \geqslant m + f([G : \bigcap_{i=1}^k G_i])$ holds, where $f(\prod_{t=1}^r p_t^{\alpha_t}) = \sum_{t=1}^r \alpha_t(p_t - 1)$ if $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are nonnegative integers. When all the a_i are the identity element of G and all the G_i are subnormal in G, we prove that there is a composition series from $\bigcap_{i=1}^k G_i$ to G whose factors are of prime orders. The paper also includes some other results and two challenging conjectures.

1. Introduction

Let G be a multiplicative group, and let

$$\mathcal{A} = \{a_iG_i\}_{i=1}^k \tag{1.1}$$

be a finite system of left cosets in G (where $a_1, \ldots, a_k \in G$, and $G_1, \ldots, G_k$ are subgroups of G).

For any $I \subseteq [1, k] = \{1, \ldots, k\}$ we define the *index map* I^* from G to the power set of $[1, k]$ as follows:

$$I^*(x) = \{i \in I\colon x \in a_iG_i\} \quad (x \in G). \tag{1.2}$$

$m(\mathcal{A}) = \inf_{x\in G} |[1, k]^*(x)|$ is said to be the *covering multiplicity* of (1.1).

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For a positive integer m, if $m(\mathcal{A}) \geqslant m$ (respectively, $|[1,k]^*(x)| = m$ for all $x \in G$) then we call (1.1) an *m-cover* (resp. *exact m-cover*) of G. If (1.1) forms an m-cover of G but none of its proper subsystems does, then it is said to be a *minimal* (or an *irredundant*) *m-cover* of G. A *cover* of G refers to a 1-cover of G, and a *disjoint cover* (or *partition*) of G means an exact 1-cover of G. Obviously an exact m-cover is a minimal m-cover and any minimal m-cover has covering multiplicity m. Disjoint covers of general groups were investigated by I. Korec [K74], M. M. Parmenter [P84] and R. Brandl [Br90], while exact m-covers of groups were studied by the author in [S01] and [S04b].

If (1.1) forms a minimal cover of a group G, then all the G_i have finite index in G by B. H. Neumann [N54a, N54b], and furthermore

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \leqslant k! \tag{1.3}$$

by M. J. Tomkinson [T87]. The author [S90] observed that this still holds if (1.1) is a minimal m-cover of a group G.

Here we give a simple way to explain why $[G : G_i] < \infty$ for all $i = 1, \ldots, k$ provided that (1.1) is a minimal m-cover of a group G. For any $j \in [1,k]$, there is an $x \in a_jG_j$ with $|[1,k]^*(x)| = m$. Choose a minimal $I \subseteq ([1,k] \setminus [1,k]^*(x)) \cup \{j\}$ such that $\{a_iG_i\}_{i\in I}$ is a cover of G, then we must have $j \in I$ and hence $[G : G_j] < \infty$ by Neumann's result.

Neumann's basic result has applications in Galois theory, group rings, Banach spaces, projective geometry and Riemann surfaces (cf. [SV]).

Concerning covers of groups by infinitely many cosets, Tomkinson [T86] proved that if a group G is irredundantly covered by $\kappa \geqslant \aleph_0$ left cosets a_iG_i then $[G : G_i] \leqslant 2^\kappa$ for each i and hence

$$\left[G : \bigcap_{i} G_i\right] \leqslant \prod_{i} [G : G_i] \leqslant (2^\kappa)^\kappa = 2^\kappa.$$

Any infinite cyclic group is isomorphic to the additive group $\mathbb{Z}$ of integers. The subgroups of $\mathbb{Z}$ different from $\{0\}$ are in the form $n\mathbb{Z} = \{nx : x \in \mathbb{Z}\}$ where $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For any positive integer n, the index of $n\mathbb{Z}$ in $\mathbb{Z}$ is n and a coset of $n\mathbb{Z}$ in $\mathbb{Z}$ is just a residue class

$$a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\} \quad (a \in \mathbb{Z}).$$

The study of cover of $\mathbb{Z}$ in the form

$$A = \{a_i + n_i\mathbb{Z}\}_{i=1}^{k} \tag{1.4}$$

was initiated by P. Erdös ([E50]) in the early 1930s, who viewed this topic as his most favorite one (cf. [E97]). Since then many researchers have investigated such covers of $\mathbb{Z}$ (cf. [G04] and [PS]). A simple example of cover of $\mathbb{Z}$ with distinct moduli is

$$\{2\mathbb{Z},\ 3\mathbb{Z},\ 1+4\mathbb{Z},\ 5+6\mathbb{Z},\ 7+12\mathbb{Z}\}.$$

M. Z. Zhang [Z91] showed that for each $m=2,3,\ldots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be split into an exact n-cover of $\mathbb{Z}$ (with $0<n<m$) and an exact $(m-n)$-cover of $\mathbb{Z}$. The author investigated m-covers of $\mathbb{Z}$ and exact m-covers of $\mathbb{Z}$ in a series of papers (see, e.g., [S97, S99, S03a, S04a]). Covering multiplicity plays an important role in the author's unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$ (cf. [S03b]).

Let H be a subnormal subgroup of a group G with finite index. Define

$$d(G,H)=\sum_{i=1}^{n}(|H_i/H_{i-1}|-1) \tag{1.5}$$

where $H=H_0\subset H_1\subset\cdots\subset H_n=G$ is a composition series from H to G. ($d(G,G)$ is regarded as 0.) By [S90, Theorem 6] we have

$$[G:H]-1\geqslant d(G,H)\geqslant f([G:H])\geqslant \log_2[G:H], \tag{1.6}$$

here the Mycielski function $f:\mathbb{Z}^+\to\mathbb{N}=\{0,1,2,\ldots\}$ is given by

$$f\bigg(\prod_{t=1}^{r}p_t^{\alpha_t}\bigg)=\sum_{t=1}^{r}\alpha_t(p_t-1) \tag{1.7}$$

where $p_1,\ldots,p_r$ are distinct primes and $\alpha_1,\ldots,\alpha_r\in\mathbb{N}$. In [S01] the author showed that $d(G,H)=f([G:H])$ if and only if G/H_G is solvable, where $H_G=\bigcap_{g\in G}gHg^{-1}$ is the largest normal subgroup of G contained in H.

Here is the main result of [S01] on exact m-covers of groups.

Theorem 1.1 (Z. W. Sun, 2001). *Let* (1.1) *be an exact m-cover of a group G by left cosets.*

(i) *Whenever $G/(G_i)_G$ is solvable, we have $k\geqslant m+f([G:G_i])$.*

(ii) *If all the G_i are subnormal in G, then we have the inequality*

$$k\geqslant m+d\bigg(G,\bigcap_{i=1}^{k}G_i\bigg), \tag{1.8}$$

where the lower bound is best possible.

We now give some historical remarks. When $m=1$ and G is abelian, part (i) was first conjectured by J. Mycielski (cf. [MS]) and confirmed by

Š. Znám [Zn66] in the case $G = \mathbb{Z}$. In 1988 M. A. Berger, A. Felzenbaum and A. S. Fraenkel [BFF] proved that if (1.1) forms a disjoint cover of a finite solvable group G then $k \geqslant 1 + f([G : G_i])$ for any $i = 1, \ldots, k$. When $m = 1$ and $G_1, \ldots, G_k$ are normal in G, part (ii) was essentially obtained by I. Korec [K74].

For general covers of groups, things become more subtle and complicated. By using algebraic number theory and characters of abelian groups, G. Lettl and the author [LS] recently obtained the following result.

Theorem 1.2 (G. Lettl and Z. W. Sun, 2004)**.** *Let* (1.1) *be a minimal* m*-cover of an abelian group* G*. Then* $k \geqslant m + f([G : G_i])$ *for all* $i = 1, \ldots, k$*.*

As Example 1.1 of [LS] shows, we cannot replace the lower bound of k in Theorem 1.2 by $m + f([G : \bigcap_{i=1}^k G_i])$ even if $G = C_p \times C_p$ where p is an odd prime and C_p is the cyclic group of order p.

Definition 1.1. Let (1.1) be a finite system of left cosets in a group G. If $[1,k]^*(G) = \{[1,k]^*(x) \colon x \in G\}$ contains $I^*(G)$ for no $\emptyset \neq I \subset [1,k]$ (i.e., whenever $\emptyset \neq I \subset [1,k]$ there is a $g \in G$ such that $I^*(g) \neq [1,k]^*(x)$ for all $x \in G$), then we call (1.1) a *regular system.* If (1.1) is regular, and $[1,k]^*(G) \not\supseteq \emptyset^*(G) = \{\emptyset\}$ (i.e., (1.1) is a cover of G), then we call (1.1) *a regular cover* of G.

When (1.1) is a minimal m-cover of a group G, it forms a regular cover of G because for any $I \subset [1,k]$ there is a $g \in G$ such that $|I^*(g)| < m$ while $|[1,k]^*(x)| \geqslant m$ for all $x \in G$. A regular system may not be a cover, e.g., we can easily check that $\{2\mathbb{Z}, 4\mathbb{Z}, 2 + 4\mathbb{Z}\}$ is a regular system of residue classes but it is not a cover of $\mathbb{Z}$. As we will see later, if (1.1) forms a regular cover of a group G then all the indices $[G : G_i]$ must be finite.

Now we introduce our first result in this paper.

Theorem 1.3. *If* (1.1) *is a regular cover of a group* G*, and for any* $i, j = 1, \ldots, k$ *either* G_i *and* G_j *are subnormal in* G *with* $[G : G_i]$ *relatively prime to* $[G_i : G_i \cap G_j]$*, or* G_i *and* G_j *are normal in* G *with* $G/(G_i \cap G_j)$ *cyclic, then we have the inequality*

$$k \geqslant m(\mathcal{A}) + d\left(G, \bigcap_{i=1}^{k} G_i\right). \tag{1.9}$$

Remark 1.1. (a) In view of Example 1.1 of [LS], the conditions of Theorem 1.3 are essentially indispensable. (b) By Example 1.2 of [S01], for any subnormal subgroup H of G with finite index, there does exist an exact m-cover (1.1) of G such that all the G_i are subnormal in G, $\bigcap_{i=1}^k G_i = H$ and $k = m + d(G, H)$.

A subgroup H of a finite group G is called a *Hall subgroup* of G if $|H|$ is relatively prime to $[G:H]$ (i.e., 1 is the greatest common divisor $(|H|,[G:H])$ of $|H|$ and $[G:H]$). By [S04b, Corollary 2.1], a subnormal Hall subgroup of G must be normal in G.

Corollary 1.1. *Let* (1.1) *be a regular cover of a group G. If G is cyclic, or G is finite and $G_1,\ldots,G_k$ are normal Hall subgroups of G, then* (1.9) *holds.*

Proof. This follows from Theorem 1.3 immediately. □

Remark 1.2. The following consequence of Corollary 1.1 was announced by the author in [S01]: Let (1.1) be a minimal m-cover of a group G. If G is cyclic, or $|G|$ is squarefree and all the G_i are normal in G, then we have the inequality

$$k \geqslant m + f\left(\left[G : \bigcap_{i=1}^{k} G_i\right]\right). \tag{1.10}$$

When $m=1$ and $G=\mathbb{Z}$, this was conjectured by Znám [Zn75] and confirmed by R. J. Simpson [Si85]; when $m=1$ and G is of squarefree order, this was proved by Berger, Felzenbaum and Fraenkel [BFF] in 1988.

Recall that a group G is said to be *perfect* if it coincides with its derived group G'. Here is our second theorem.

Theorem 1.4. *Suppose that $\{G_i\}_{i=1}^k$ is a minimal m-cover of a group G by subnormal subgroups. Then there is a composition series from $\bigcap_{i=1}^k G_i$ to G whose factors are of prime orders, and all the G_i contain every perfect subgroup of G.*

Remark 1.3. (a) Clearly Theorem 1.4 extends the following result of M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [BCK]: If $\{G_i\}_{i=1}^k$ is a minimal cover of a group G by finitely many normal subgroups, then $G/\bigcap_{i=1}^k G_i$ is solvable and all perfect normal subgroups of G are contained in each of $G_1,\ldots,G_k$. (b) Any finite non-cyclic group G can be covered by finitely many proper subgroups because $G=\bigcup_{x\in G}\langle x\rangle$, but no field can be covered by finitely many proper subfields (cf. [BBS]).

Example 1.1. Let G be a group with $G/Z(G)$ finite, where $Z(G)$ is the center of G. For $x,y\in G$, if $xy\neq yx$ then $(x^{-1}y)x\neq x(x^{-1}y)$ and hence $xZ(G)\neq yZ(G)$. Let $X=\{x_1,\ldots,x_k\}$ be a maximal set of pairwise non-commuting elements of G. Then $k=|X|\leqslant|G/Z(G)|$, and $\{C_G(x_i)\}_{i=1}^k$ forms a minimal cover of G by centralizers with $\bigcap_{i=1}^k C_G(x_i)=Z(G)$ (cf. [T87, Theorem 5.1]), and $|G/Z(G)|\leqslant c^k$ for some absolute constant $c>0$ (see [Py87]). If $\overline{G}=G/Z(G)$ is not solvable, then not all the $C_G(x_i)$ are subnormal in G by Theorem 1.4. When $\overline{G}$ is solvable, Tomkinson [T97] provided a lower bound of k (conjectured by Cohn [C94]) in terms of a chief factor of $\overline{G}$. D. R. Mason [M78] proved that $|G| \geqslant 2k-2$, which was conjectured by Erdös and E. G. Straus [ES].

Concerning covers of a group by subnormal subgroups, the author and his student Song Guo have made the following conjecture.

Conjecture 1.1 (S. Guo and Z. W. Sun, 2004)**.** *Let $\{G_i\}_{i=1}^k$ be a minimal m-cover of a group G by finitely many subnormal subgroups. Assume that $[G : \bigcap_{i=1}^k G_i] = \prod_{t=1}^r p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Then*

$$k > m + \sum_{t=1}^{r} (\alpha_t - 1)(p_t - 1).$$

Let H be a subgroup of a group G. When $X \subseteq G$ is a union of finitely many left cosets of H, we use $[X : H]$ to represent the number of left cosets of H contained in X, which agrees with the index of H in X if X is a subgroup of G. Below is our third result in this paper.

Theorem 1.5. *Let H be a subgroup of a group G with finite index. Suppose that $G_1, \ldots, G_k$ are subnormal subgroups of G containing H with $([G : G_i], [G_i : H]) = 1$ for all $i = 1, \ldots, k$. Given $a_1, \ldots, a_k \in G$ we have the inequality*

$$\Big[\bigcup_{i=1}^{k} a_i G_i : H\Big] \geqslant \Big[\bigcup_{i=1}^{k} G_i : H\Big]. \tag{1.11}$$

Here is an obvious consequence of Theorem 1.5.

Corollary 1.2. *Let G be a finite group and let $G_1, \ldots, G_k$ be normal Hall subgroups of G. Then, for any $a_1, \ldots, a_k \in G$, we have*

$$\Big|\bigcup_{i=1}^{k} a_i G_i\Big| \geqslant \Big|\bigcup_{i=1}^{k} G_i\Big|. \tag{1.12}$$

Remark 1.4. In [S90] it was asked whether for any finitely many left cosets $a_1G_1, \ldots, a_kG_k$ in a finite group G we always have (1.12). Later Tomkinson pointed out that this is not true for the Klein group $C_2 \times C_2$.

The following conjecture of the author seems very challenging.

Conjecture 1.2 (Z. W. Sun, 2004). *Let $a_1G_1,\ldots,a_kG_k$ $(k>1)$ be finitely many pairwise disjoint left cosets in a group G with $[G:G_i]<\infty$ for all $i=1,\ldots,k$. Then $([G:G_i],[G:G_j])\geqslant k$ for some $1\leqslant i<j\leqslant k$.*

Remark 1.5. (a) The case $k=2$ will be handled in Remark 2.2. The conjecture remains open even for the additive cyclic group $\mathbb{Z}$.

(b) Under the condition of Conjecture 1.2, clearly

$$\Big[G:\bigcap_{j=1}^k G_j\Big]\geqslant\Big[\bigcup_{i=1}^k a_iG_i:\bigcap_{j=1}^k G_j\Big]=\sum_{i=1}^k\Big[G_i:\bigcap_{j=1}^k G_j\Big]$$

and hence $\sum_{i=1}^k[G:G_i]^{-1}\leqslant 1$. Suppose that $[G:G_1]\leqslant\cdots\leqslant[G:G_k]$. Since $\sum_{i=1}^{k-1}[G:G_i]^{-1}<1$, there is an $i\in[1,k-1]$ such that $[G:G_i]\not\leqslant k-1$ and hence $[G:G_k]\geqslant[G:G_i]\geqslant k$. If $[G:G_k]$ is divisible by all those $[G:G_1],\ldots,[G:G_{k-1}]$ (this happens if G is a p-group with p a prime), then $([G:G_i],[G:G_k])=[G:G_i]\geqslant k$.

We will derive some combinatorial properties of regular systems in the next section, and prove Theorems 1.3-1.5 in the third section.

2. Combinatorial Properties of Regular Systems

Recall that a *monoid* is a semigroup containing an identity element.

In order to unify disjoint cover and minimal cover, the author [S90] proposed the following general notion.

Let M be a commutative monoid (considered as an additive one) and S a set. A finite system $\{X_i\}_{i=1}^k$ of nonempty subsets of a set $X\neq\emptyset$ is called an (M,S)-*cover* of X if there exist $m_1,\ldots,m_k\in M$ such that

$$\Big\{\sum_{\substack{i=1\\ x\in X_i}}^k m_i:\ x\in X\Big\}\subseteq S$$

but

$$\Big\{\sum_{\substack{i\in I\\ x\in X_i}} m_i:\ x\in X\Big\}\not\subseteq S\qquad\text{for all } I\subset[1,k].$$

(Without any loss of generality we may let S be a subset of M.)

Though useful in [S90], this concept involving a monoid and a set seems cumbersome. We can avoid it by using regular covers of a nonempty set by nonempty subsets. (The definition is similar to that of regular cover of a group by cosets.)

Let X be a nonempty set and $\{X_i\}_{i=1}^k$ a finite system of nonempty subsets of X. If $\{X_i\}_{i=1}^k$ forms an (M,S)-cover of X for some commutative monoid M and its subset S, then it is easy to see that $\{X_i\}_{i=1}^k$ is a regular cover of X. Conversely, if $\{X_i\}_{i=1}^k$ is a regular cover of X then it is an (M_0, S_0)-cover of X, where M_0 is the multiplicative monoid of positive integers, and

$$S_0 = \Big\{ \prod_{i\in[1,k]^*(x)} p_i \colon x \in X \Big\}$$

with $p_1, \ldots, p_k$ distinct primes.

Theorem 2.1. *Let* (1.1) *be a regular system of left cosets in a group* G.

(i) *Let* $\emptyset \ne I \subseteq [1,k]$, *and* $I \ne [1,k]$ *if* (1.1) *is not a cover of* G. *Set* $\bar{I} = [1,k] \setminus I$, *and regard* $\bigcap_{i\in\emptyset} G_i$ *as* G.

(a) $\bigcup_{i\in I} a_iG_i$ *contains a left coset of* $\bigcap_{j\in\bar{I}} G_j$, *and* $\bigcap_{j\in\bar{I}} G_j$ *contains at most* $|I|!$ *left cosets of* $\bigcap_{i=1}^k G_i$. *For each* $x \in \bigcap_{j\in\bar{I}} G_j$ *there are positive integers* $n_1 \leqslant 1, \ldots, n_{|I|} \leqslant |I|$ *such that* $x^{n_1\cdots n_{|I|}} \in \bigcap_{i=1}^k G_i$.

(b) *If* $[G : G_i] < \infty$ *for all* $i = 1, \ldots, k$, *then*

$$\sum_{i\in I} \frac{1}{[G:G_i]} \geqslant \sum_{i\in I} \frac{1}{[G:G_i\cap\bigcap_{j\in\bar{I}} G_j]} \geqslant \frac{1}{[G:\bigcap_{j\in\bar{I}} G_j]} \geqslant \frac{1}{\prod_{j\in\bar{I}}[G:G_j]} \tag{2.1}$$

where an empty product is regarded as 1. *Also,*

$$\Big[\bigcup_{i\in I} G_i : \bigcap_{i=1}^k G_i\Big] \leqslant \sum_{l=1}^{|I|} (-1)^{l-1}(k-l)!\binom{|I|}{l}. \tag{2.2}$$

(ii) *For any subgroup* H *of* G *with* $J = \{1 \leqslant i \leqslant k \colon G_i \not\subseteq H\} \ne \emptyset, [1,k]$, *we have*

$$|\{C \in G/H : C \supseteq a_iG_i \text{ for some } 1 \leqslant i \leqslant k\}| \geqslant \Big[\Big(\bigcap_{j\in J} G_j\Big)H : H\Big], \tag{2.3}$$

where G/H *refers to the set* $\{gH : g \in G\}$.

Proof. (i) Let us prove part (i) first.

(a) Since (1.1) is regular and $\bar{I} \ne [1,k]$, there exists a $g \in G$ such that

$$\bar{I}^*(g) = \{j \in \bar{I} : g \in a_jG_j\} \notin [1,k]^*(G).$$

For each $x \in \bigcap_{j\in\bar{I}} G_j$, as $\bar{I}^*(gx) = \bar{I}^*(g) \ne [1,k]^*(gx)$, we must have $gx \in \bigcup_{i\in I} a_iG_i$. So

$$g\Big(\bigcap_{j\in\bar{I}} G_j\Big) \subseteq \bigcup_{i\in I} a_iG_i.$$

For each $i \in I$, $\{i\} \cup \bar{I}$ is the complement of $I \setminus \{i\}$ in $[1, k]$; if $a_iG_i \cap g(\bigcap_{j\in\bar{I}} G_j)$ is nonempty then it contains exactly $[G_i \cap \bigcap_{j\in\bar{I}} G_j : F]$ left cosets of $F = G_1 \cap \cdots \cap G_k$. As

$$g\Big(\bigcap_{j\in\bar{I}} G_j\Big) = \bigcup_{i\in I}\Big(a_iG_i \cap g\Big(\bigcap_{j\in\bar{I}} G_j\Big)\Big),$$

we have

$$\Big[\bigcap_{j\in\bar{I}} G_j : F\Big] \leqslant \sum_{i\in I}\Big[\bigcap_{j\in\overline{I\setminus\{i\}}} G_j : F\Big]. \tag{2.4}$$

In the case $I = \{i\}$, this yields that $[\bigcap_{j\in\bar{I}} G_j : F] \leqslant 1 = |I|!$. By induction on $|I|$ we see that $[\bigcap_{j\in\bar{I}} G_j : F] \leqslant |I|!$.

Let $x \in \bigcap_{j\in\bar{I}} G_j$. Since

$$\Big\{g, gx, \ldots, gx^{|I|}\Big\} \subseteq g\Big(\bigcap_{j\in\bar{I}} G_j\Big) \subseteq \bigcup_{i\in I} a_iG_i,$$

for some $0 \leqslant s < t \leqslant |I|$ both gx^s and gx^t belong to a certain a_iG_i with $i \in I$ and therefore $x^{t-s} = (gx^s)^{-1}gx^t \in G_i$. This shows that

$$x^n \in \bigcup_{i\in I} G_i \cap \bigcap_{j\in\bar{I}} G_j = \bigcup_{i\in I} \bigcap_{j\in\overline{I\setminus\{i\}}} G_j$$

where $n = t - s \in \{1, \ldots, |I|\}$. Recall that if $|I| = 1$ then $\bigcap_{j\in\bar{I}} G_j = F$. Again, by induction on $|I|$, we find that $x^{n_1\cdots n_{|I|}} \in F$ for some positive integers $n_1 \leqslant 1, \ldots, n_{|I|} \leqslant |I|$.

(b) When $G_1, \ldots, G_k$ are of finite index in G, dividing both sides of the inequality (2.4) by $[G : F] < \infty$ we obtain (2.1).

For each nonempty subset J of I, if $J = [1, k]$ then

$$\Big[\bigcap_{j\in J} G_j : F\Big] - \sum_{\substack{i\in I\\ i>\max J}}\Big[G_i \cap \bigcap_{j\in J} G_j : F\Big] = 1 - 0 = (k - |J|)!$$

where $\max J$ is the maximal element of J; otherwise, by (2.4) and part (a) we have

$$\begin{aligned}
&\Big[\bigcap_{j\in J}G_j:F\Big]-\sum_{\substack{i\in I\\ i>\max J}}\Big[G_i\cap\bigcap_{j\in J}G_j:F\Big]\\
\leqslant&\sum_{\substack{i=1\\ i\notin J}}^{k}\Big[G_i\cap\bigcap_{j\in J}G_j:F\Big]-\sum_{\substack{i\in I\\ i>\max J}}\Big[G_i\cap\bigcap_{j\in J}G_j:F\Big]\\
=&\sum_{\substack{1\leqslant i\leqslant k,\ i\notin J\\ i\notin I\ \text{or}\ i\leqslant\max J}}\Big[G_i\cap\bigcap_{j\in J}G_j:F\Big]\\
\leqslant&\sum_{\substack{1\leqslant i\leqslant k,\ i\notin J\\ i\notin I\ \text{or}\ i\leqslant\max J}}(k-(|J|+1))!\\
=&\sum_{\substack{i=1\\ i\notin J}}^{k}(k-(|J|+1))!-\sum_{\substack{i\in I\\ i>\max J}}(k-(|J|+1))!\\
=&(k-|J|)!-\sum_{\substack{i\in I\\ i>\max J}}(k-(|J|+1))!.
\end{aligned}$$

If $\emptyset\ne J'\subseteq I$ and $2\mid |J'|$, then $J'=J\cup\{i\}$ and $2\nmid |J|$ where $i=\max J'$ and $J=J'\setminus\{i\}$. Thus, in view of the above and the inclusion–exclusion principle,

$$\begin{aligned}
\Big[\bigcup_{i\in I}G_i:F\Big]=&\sum_{\emptyset\ne J\subseteq I}(-1)^{|J|-1}\Big[\bigcap_{j\in J}G_j:F\Big]\\
=&\sum_{\substack{J\subseteq I\\ 2\nmid |J|}}\bigg(\Big[\bigcap_{j\in J}G_j:F\Big]-\sum_{\substack{i\in I\\ i>\max J}}\Big[G_i\cap\bigcap_{j\in J}G_j:F\Big]\bigg)\\
\leqslant&\sum_{\substack{J\subseteq I\\ 2\nmid |J|}}\bigg((k-|J|)!-\sum_{\substack{i\in I\\ i>\max J}}(k-(|J|+1)!\bigg)\\
=&\sum_{\emptyset\ne J\subseteq I}(-1)^{|J|-1}(k-|J|)!=\sum_{l=1}^{|I|}(-1)^{l-1}\binom{|I|}{l}(k-l)!.
\end{aligned}$$

So (2.2) is valid.

(ii) By part (i) there exists $g\in G$ such that $g(\bigcap_{j\in J}G_j)\subseteq\bigcup_{i\in\bar J}a_iG_i$. Therefore

$$g\Big(\bigcap_{j\in J}G_j\Big)H\subseteq\bigcup_{i\in\bar J}a_iG_iH=\bigcup_{i\in\bar J}a_iH$$

and hence

$$\begin{aligned}&|\{C\in G/H:\, C\supseteq a_iG_i \text{ for some } 1\leqslant i\leqslant k\}|\\ &\quad=|\{a_iH\colon 1\leqslant i\leqslant k \text{ and } a_iG_i\subseteq a_iH\}|\\ &\quad\geqslant\Big[\Big(\bigcap_{j\in J}G_j\Big)H:H\Big].\end{aligned}$$

We are done. □

Remark 2.1. (i) By Theorem 2.1(i), for any regular cover (1.1) of a group G we have

$$\Big[G:\bigcap_{i=1}^k G_i\Big]=\Big[\bigcap_{i\in\emptyset}G_i:\bigcap_{i=1}^k G_i\Big]\leqslant k!<\infty.$$

(ii) Let (1.1) be a minimal cover of a group G. That $\sum_{i\in I}[G:G_i]^{-1}\geqslant\prod_{j\in\bar I}[G:G_j]^{-1}$ for all $\emptyset\neq I\subseteq[1,k]$ was first observed by Neumann [N54b]. When all the a_i are the identity element e of G, McCoy [Mc57] obtained the existence of a positive integer n such that $x^n\in\bigcap_{i=1}^k G_i$ for all $x\in G=\bigcap_{i\in\emptyset}G_i$, and S. Wigzell [W95] deduced the inequality $[G:\bigcap_{i=1}^k G_i]\leqslant k!\sum_{l=1}^k(-1)^{l-1}/l!$. In contrast with McCoy's result, J. Backelin [B94] proved that if a (not necessarily unitary) ring R is irredundantly covered by finitely many ideals $I_1,\ldots,I_k$ then

$$R^{k-1}:=\Big\{\sum_{j=1}^l r_{j,1}\cdots r_{j,k-1}:\, l\in\mathbb{Z}^+ \text{ and } r_{j,1},\ldots,r_{j,k-1}\in R\Big\}$$

is contained in $\bigcap_{i=1}^k I_i$.

The following example shows that those inequalities in Theorem 2.1(i) are essentially sharp.

Example 2.1. Let G be the symmetric group S_k on $\{1,\ldots,k\}$ $(k>1)$ and H be the stabilizer of 1. Then

$$\{G_1,\ (12)G_2,\ \ldots,\ (1k)G_k\}=\{H,\ H(12),\ H(13),\ \ldots,\ H(1k)\}$$

forms a partition of G, where G_i is the stabilizer of i for each $i=1,\ldots,k$. Let $\emptyset\neq I\subseteq[1,k]$ and $\bar I=[1,k]\setminus I$. Clearly $\bigcap_{j\in\bar I}G_j\cong S_{|I|}$ and hence $\left|\bigcap_{j\in\bar I}G_j\right|=|I|!$, also $x^{\prod_{s=1}^{|I|}s}\in\bigcap_{i=1}^k G_i=\{e\}$ for all $x\in\bigcap_{j\in\bar I}G_j$. Let $a_1=e$, and $a_i=(1i)$ for $i\in[2,k]$. If $\sigma\in G=S_k$ and $\sigma^{-1}(1)\in I$, then $\sigma\bigcap_{j\in\bar I}G_j\subseteq\bigcup_{i\in I}a_iG_i$, because for any $\tau\in\bigcap_{s\in\bar I}G_s$ and $j\in\bar I$ we have $\sigma\tau(j)=\sigma(j)\neq 1$ and hence $\sigma\tau\notin a_jG_j$. Note also that

$$\sum_{i\in I}\frac{1}{[G:G_i\cap\bigcap_{j\in\bar I}G_j]}=\sum_{i\in I}\frac{(|I|-1)!}{k!}=\frac{1}{[G:\bigcap_{j\in\bar I}G_j]}$$

and

$$\Big|\bigcup_{i\in I}G_i\Big| = \sum_{l=1}^{|I|}(-1)^{l-1}\sum_{\substack{J\subseteq I\\ |J|=l}}\Big|\bigcap_{j\in J}G_j\Big| = \sum_{l=1}^{|I|}(-1)^{l-1}\binom{|I|}{l}(k-l)!.$$

Lemma 2.1. *Let H and K be subgroups of a group G. If $[G:H]$ and $[G:K]$ are finite and relatively prime, then HK coincides with G.*

Proof. Use $[m,n]$ to denote the least common multiple of $m,n\in\mathbb{Z}^+$. Then

$$[[G:H],[G:K]]\mid[G:H\cap K] \ \text{ and } \ [G:H\cap K]\leqslant[G:H][G:K].$$

As $([G:H],[G:K])=1$, we have $[G:H\cap K]=[G:H][G:K]$, i.e., $[K:H\cap K]=[G:H]$. Since HK contains exactly $[K:H\cap K]$ right cosets of H, by the above we must have $HK=G$. □

Remark 2.2. Combining Lemma 2.1 with [S01, Lemma 2.1(i)] we find that for two subgroups H and K of G with finite index if $aH\cap bK=\emptyset$ for some $a,b\in G$ then $([G:H],[G:K])>1$. This confirms Conjecture 1.2 in the case $k=2$.

Recall that for a subgroup H of a group G we use G/H to denote the set of all left cosets of H in G even if H may not be normal in G.

Theorem 2.2. *Let $G_1,\ldots,G_k,H$ be subgroups of a group G such that either $G_1,\ldots,G_k$ are normal and H is maximal in G, or $G_1,\ldots,G_k$ are subnormal and H is maximal normal in G. Suppose that* (1.1) *is a regular system for some $a_1,\ldots,a_k\in G$ and $[G:G_i]<\infty$ for all $i=1,\ldots,k$. If for any $i,j=1,\ldots,k$, either $[G:G_i]$ is relatively prime to $[G_i:G_i\cap G_j]$, or G_i and G_j are normal in G with $G/(G_i\cap G_j)$ cyclic, then we have*

$$\{C\in G/H: C\supseteq a_iG_i \text{ for some } i=1,\ldots,k\}=\emptyset \text{ or } G/H \tag{2.5}$$

provided that (1.1) *is a cover of G or not all the G_i are contained in H.*

Proof. Let $I=\{1\leqslant i\leqslant k\colon G_i\not\subseteq H\}$. If $I=[1,k]$, then the left-hand side of (2.5) is empty. (Note that if $aH\supseteq a_iG_i$ for some $a\in G$ then $a_iH=aH\supseteq a_iG_i$ and hence $H\supseteq G_i$.) When I is empty and (1.1) is a cover of G, the left-hand side of (2.5) coincides with G/H because $xH\supseteq a_iG_i$ if $x\in a_iG_i$.

Now assume that $\emptyset\neq I\neq[1,k]$ and fix a $j\in[1,k]\setminus I$. Since $H\supseteq G_j$, $[G:H]\leqslant[G:G_j]<\infty$. By Theorem 2.1(ii), the left-hand side of (2.5) contains at least $[(\bigcap_{i\in I}G_i)H:H]$ left cosets of H. So it suffices to show that $(\bigcap_{i\in I}G_i)H=G$.

Let $i \in I$. As $G_i \not\subseteq H$, we have $G_iH = HG_i = G$ by [S01, Lemma 2.1(ii)]. Observe that $[G : H] = [G_iH : H] = [G_i : G_i \cap H]$ divides $[G_i : G_i \cap G_j]$ and that

$$[G : G_i \cap H] = [G : G_i][G_i : G_i \cap H] = [G : G_i][G_iH : H] = [G : G_i][G : H].$$

If $([G : G_i], [G_i : G_i \cap G_j]) = 1$, then $([G : G_i], [G : H]) = 1$. When G_i, G_j are normal in G and $G/(G_i \cap G_j)$ is cyclic, we have

$$\begin{aligned}&[G/(G_i \cap G_j) : G_i/(G_i \cap G_j) \cap H/(G_i \cap G_j)]\\ =&[[G/(G_i \cap G_j) : G_i/(G_i \cap G_j)], [G/(G_i \cap G_j) : H/(G_i \cap G_j)]],\end{aligned}$$

hence $[G : G_i][G : H] = [G : G_i \cap H] = [[G : G_i], [G : H]]$ and therefore $([G : G_i], [G : H]) = 1$.

In light of [S01, Lemma 3.1(ii)], $[G : \bigcap_{i\in I} G_i]$ divides $\prod_{i\in I}[G : G_i]$. So $([G : \bigcap_{i\in I} G_i], [G : H]) = 1$ by the above. Hence $\left(\bigcap_{i\in I} G_i\right) H = G$ by Lemma 2.1, and this concludes the proof. □

Corollary 2.1. *Let* (1.1) *be a regular cover of a group G such that for any $i, j = 1, \ldots , k$ either G_i and G_j are normal in G with $G/(G_i \cap G_j)$ cyclic, or G_i and G_j are subnormal in G with $([G : G_i], [G_i : G_i \cap G_j]) = 1$. If $G_j \neq G$, then for some $i \neq j$ with $G_i \neq G$ we have $a_iG_i \cap a_jG_j = \emptyset$.*

Proof. Suppose that $G_j \neq G$ where $1 \leqslant j \leqslant k$. As G_j is subnormal and of finite index in G it is contained in a proper maximal normal subgroup H of G. Clearly $a_jG_j \subseteq a_jH \subset G$. Choose $x \in G \setminus a_jH$. By Theorem 2.2, $xH \supseteq a_iG_i$ for some $i \in [1, k]$. Since xH is disjoint from a_jH, we have $a_iG_i \cap a_jG_j = \emptyset$. Obviously $i \neq j$ and $G_i \neq G$. We are done. □

3. Proofs of Theorems 1.3-1.5

Proof of Theorem 1.3. We use induction on the finite index $[G : \bigcap_{i=1}^k G_i]$.

If $[G : \bigcap_{i=1}^k G_i] = 1$, then $G_1 = \cdots = G_k = G$, $m(\mathcal{A}) = k$ and $d(G, \bigcap_{i=1}^k G_i) = 0$, so (1.9) holds trivially.

Now let's proceed to the induction step and assume that $[G : \bigcap_{i=1}^k G_i] > 1$.

The following observations are important in our induction step. Let K be any subgroup of G. Then $G_i \cap K$ is subnormal in K for any $i = 1, \ldots , k$. Let $i, j \in [1, k]$. If $([G : G_i], [G_i : G_i \cap G_j]) = 1$, then $[K : G_i \cap K]$ is relatively prime to $[G_i \cap K : G_i \cap G_j \cap K]$ because $[K : G_i \cap K] \mid [G : G_i]$ and $[G_i \cap K : (G_i \cap G_j) \cap (G_i \cap K)] \mid [G_i : G_i \cap G_j]$ by [S01, Lemma 3.1]. If both G_i and G_j are normal in G with $G/(G_i \cap G_j)$ cyclic, then both $G_i \cap K$ and $G_j \cap K$ are normal in K and $K/(G_i \cap G_j \cap K) \cong (G_i \cap G_j)K/(G_i \cap G_j)$ is cyclic.

Suppose $G_{i_0} \neq G$ where $1 \leqslant i_0 \leqslant k$. As G_{i_0} is subnormal in G there is a maximal normal subgroup H of G with $G_{i_0} \subseteq H \subset G$. Let I_0 be a minimal subset of $[1,k]$ such that $|\{i \in I_0\colon x \in a_iG_i\}| \geqslant m(\mathcal{A})$ for all $x \in H$. Clearly such an I_0 exists and $a_iG_i \cap H \neq \emptyset$ for all $i \in I_0$. Observe that the nonempty system $\mathcal{A}_0 = \{a_iG_i \cap H\}_{i\in I_0}$ forms a minimal $m(\mathcal{A})$-cover of H by left cosets of subnormal subgroups $G_i \cap H$ $(i \in I_0)$ of H. Also,

$$\Big[H : \bigcap_{i\in I_0}(G_i \cap H)\Big] \leqslant \Big[H : \bigcap_{i=1}^{k} G_i \cap H\Big] = \Big[H : \bigcap_{i=1}^{k} G_i\Big] < \Big[G : \bigcap_{i=1}^{k} G_i\Big].$$

Since a minimal $m(\mathcal{A})$-cover is also a regular cover, by the induction hypothesis we have

$$|I_0| \geqslant m(\mathcal{A}_0) + d\Big(H, \bigcap_{i\in I_0}(G_i \cap H)\Big) = m(\mathcal{A}) + d\Big(H, H \cap \bigcap_{i\in I_0} G_i\Big).$$

In the case $I_0 \neq [1,k]$, by Theorem 2.1(i) there exists a $g_1 \in G$ such that

$$\bigcup_{\substack{j=1\\ j\notin I_0}}^{k} a_jG_j \supseteq g_1 \bigcap_{i\in I_0} G_i,$$

hence we can choose a minimal $I_1 \subseteq [1,k]$ with $I_1 \cap I_0 = \emptyset$ such that

$$\bigcup_{j\in I_1} a_jG_j \cap g_1H \supseteq g_1\Big(\bigcap_{i\in I_0} G_i \cap H\Big).$$

Since $\mathcal{A}_1 = \{g_1^{-1}a_jG_j \cap \bigcap_{i\in I_0} G_i \cap H\}_{j\in I_1}$ forms a minimal cover of $\bigcap_{i\in I_0} G_i \cap H$ by left cosets of subnormal subgroups $G_j \cap \bigcap_{i\in I_0} G_i \cap H$ $(j \in I_1)$ of $\bigcap_{i\in I_0} G_i \cap H$, again by the induction hypothesis we have

$$\begin{aligned}|I_1| \geqslant& m(\mathcal{A}_1) + d\Big(\bigcap_{i\in I_0} G_i \cap H, \bigcap_{j\in I_1}\Big(G_j \cap \bigcap_{i\in I_0} G_i \cap H\Big)\Big)\\ =&1 + d\Big(H \cap \bigcap_{i\in I_0} G_i, H \cap \bigcap_{i\in I_0\cup I_1} G_i\Big).\end{aligned}$$

Continue the above procedure until we obtain a partition $\{I_s\}_{s=0}^{n}$ (with $0 \leqslant n < k$) of $[1,k]$ such that for each $s \in [1,n]$ there is a $g_s \in G$ with $a_iG_i \cap g_sH \neq \emptyset$ for all $i \in I_s$ and

$$|I_s| \geqslant 1 + d\Big(H \cap \bigcap_{i\in I_0\cup\cdots\cup I_{s-1}} G_i, H \cap \bigcap_{i\in I_0\cup\cdots\cup I_s} G_i\Big).$$

Observe that

$$\begin{aligned}k=\sum_{s=0}^{n}|I_s| \geqslant& m(\mathcal{A})+d\Big(H,H\cap\bigcap_{i\in I_0}G_i\Big)\\ &+\sum_{0<s\leqslant n}\Big(1+d\Big(H\cap\bigcap_{i\in I_0\cup\cdots\cup I_{s-1}}G_i,H\cap\bigcap_{i\in I_0\cup\cdots\cup I_s}G_i\Big)\Big)\\ =&m(\mathcal{A})+n+d\Big(H,H\cap\bigcap_{i\in I_0\cup\cdots\cup I_n}G_i\Big)\\ =&m(\mathcal{A})+n+d\Big(H,\bigcap_{i=1}^{k}G_i\Big).\end{aligned}$$

As $a_{i_0}H\supseteq a_{i_0}G_{i_0}$, by Theorem 2.2 each $C\in G/H$ contains a_iG_i for some $i\in[1,k]=I_0\cup\cdots\cup I_n$. Recall that $a_iG_i\cap g_0H\neq\emptyset$ for all $i\in I_0$ where $g_0=e$. For any $g\in G$, if $gH\supseteq a_iG_i$ and $i\in I_s$ then $gH\cap g_sH\supseteq a_iG_i\cap g_sH\neq\emptyset$ and hence $gH=g_sH$. Therefore

$$|G/H|=|\{g_sH\colon 0\leqslant s\leqslant n\}|\leqslant n+1$$

and hence

$$\begin{aligned}k-m(\mathcal{A})\geqslant& n+d\Big(H,\bigcap_{i=1}^{k}G_i\Big)\geqslant|G/H|-1+d\Big(H,\bigcap_{i=1}^{k}G_i\Big)\\ =&d(G,H)+d\Big(H,\bigcap_{i=1}^{k}G_i\Big)=d\Big(G,\bigcap_{i=1}^{k}G_i\Big).\end{aligned}$$

This completes the induction proof. □

Remark 3.1. In view of Theorem 2.2, by modifying the complicated proof of [S90, Theorem 8], we can obtain the following result for minimal m-covers (*not* for regular covers): Let (1.1) be a minimal m-cover of a group G, and suppose that for any $i,j=1,\ldots,k$ either G_i and G_j are subnormal in G with $([G:G_i],[G_i:G_i\cap G_j])=1$, or G_i and G_j are normal in G with $G/(G_i\cap G_j)$ cyclic. Then, for each subgroup K of G with $I(K)=\{1\leqslant i\leqslant k:\ K\not\subseteq G_i\}\neq\emptyset$, we can find an $r\in I(K)$ and $x_i\in K\setminus G_i$ (for $i\in I(K)\setminus\{r\}$) such that

$$\Big|\Big\{x_i\Big(K\cap\bigcap_{s=1}^{k}G_s\Big)\ :\ i\in I(K)\setminus\{r\}\Big\}\Big|\geqslant d\Big(K,K\cap\bigcap_{s=1}^{k}G_s\Big).$$

If $K\supseteq G_j$ for some $j\in[1,k]$, or $K\supseteq H=(\bigcap_{s=1}^{k}G_s)_G$ and K/H is a Hall subgroup of G/H, then $I(K)=\{1\leqslant i\leqslant k:\ [G:G_i]\nmid[G:K]\}$ by Lemma 3.1 given later.

Proof of Theorem 1.4. Let M be any perfect normal subgroup of G. Fix $x \in G$ and set $M_x = \langle M, x\rangle = \langle x\rangle M$. Since $M_x/M \cong \langle x\rangle/(\langle x\rangle \cap M)$ is cyclic and hence abelian, $M_x' \subseteq M = M' \subseteq M_x'$ and therefore $M_x' = M$.

As $\{G_i\}_{i=1}^k$ is an m-cover of G, we can choose a minimal $I_x \subseteq [1,k]$ such that $|I_x^*(g)| \geqslant m$ for all $g \in M_x$. Clearly $\mathcal{A}_x = \{G_i \cap M_x\}_{i\in I_x}$ forms a minimal m-cover of M_x by subnormal subgroups of M_x. We claim that $J = \{i \in I_x : G_i \not\supseteq M_x\}$ is empty. Assume on the contrary that $J \neq \emptyset$. Then $|\{i \in I_x : G_i \supseteq M_x\}| < m$. Since $\{G_i \cap M_x\}_{i\in J}$ is a cover of M_x by proper subnormal subgroups, M_x possesses a cover by proper normal subgroups. Applying a result of Brodie, Chamberlain and Kappe [BCK], we find that M_x has a normal subgroup H such that $M_x/H \cong C_p \times C_p$ for some prime p. As M_x/H is abelian, we have $M = M_x' \subseteq H$ and hence $M_x/H \cong (M_x/M)/(H/M)$ is cyclic. This contradiction shows that our claim is true.

Let $I = \bigcup_{x\in G} I_x$. For each $x \in G$, clearly $|I^*(x)| \geqslant |I_x^*(x)| \geqslant m$. Since $\{G_i\}_{i=1}^k$ is a minimal m-cover of G, we must have $I = [1,k]$. For any $i \in [1,k]$, there is an $x \in G$ such that $i \in I_x$ and hence $G_i \supseteq M_x \supseteq M$.

By the above, $F = \bigcap_{i=1}^k G_i$ contains any perfect normal subgroup of G. As $\{G_i/F_G\}_{i=1}^k$ is a minimal m-cover of G/F_G, $F/F_G = \bigcap_{i=1}^k G_i/F_G$ contains any perfect normal subgroup of the finite group G/F_G. In light of parts (i)and (v) of Theorem 3.1 of [S01], there exists a composition series from F/F_G to G/F_G whose factors are of prime orders. Thus there is also a composition series from F to G whose factors are of prime orders, and hence F contains all perfect subgroups of G by [S01, Theorem 3.1]. This concludes our proof. □

Lemma 3.1. *Let H and K be subgroups of a group G with finite index. Suppose that H or K is subnormal in G, and* $([G : K], [K : H \cap K]) = 1$. *Then*

$$[G : H \cap K] = [[G : H], [G : K]]; \tag{3.1}$$

hence

$$H \supseteq K \iff [G : H] \mid [G : K], \quad \textit{and} \quad K \supseteq H \iff [G : K] \mid [G : H].$$

Proof. As $[G : K]$ is relatively prime to $[K : H \cap K]$,

$$[G : H \cap K] = [G : K][K : H \cap K] = [[G : K], [K : H \cap K]].$$

By [S01, Lemma 3.1(i)], $[K : H \cap K] \mid [G : H]$ and therefore $[G : H \cap K] \mid [[G : H], [G : K]]$. On the other hand, $[G : H \cap K]$ is obviously divisible by both $[G : H]$ and $[G : K]$. So we have (3.1).

Observe that

$$\begin{aligned} H \supseteq K &\iff [G : H \cap K]/[G : K] = [K : H \cap K] = 1 \\ &\iff [[G : H], [G : K]] = [G : K] \text{ (i.e., } [G : H] \mid [G : K]). \end{aligned}$$

Similarly, $K \supseteq H$ if and only if $[G:K] \mid [G:H]$. This ends the proof. $\square$

Proof of Theorem 1.5. For any $\emptyset \neq J \subseteq [1,k]$, we assert that $[G:\bigcap_{j\in J} G_j]$ is the least common multiple $[n_j]_{j\in J}$ of those $n_j = [G:G_j]$ with $j \in J$. When $|J| = 1$, this is trivial. Now assume $|J| > 1$ and $[G:\bigcap_{j\in J\setminus\{j_0\}} G_j] = [n_j]_{j\in J\setminus\{j_0\}}$, where $j_0 = \max J$. Observe that $[G_{j_0} : G_{j_0} \cap \bigcap_{j\in J\setminus\{j_0\}} G_j]$ divides $[G_{j_0} : H]$ and hence it is relatively prime to $[G:G_{j_0}]$. With the help of Lemma 3.1, we find that

$$\begin{aligned}\Big[G:\bigcap_{j\in J} G_j\Big] =& \Big[[G:G_{j_0}], \Big[G:\bigcap_{j\in J\setminus\{j_0\}} G_j\Big]\Big]\\ =&[n_{j_0}, [n_j]_{j\in J\setminus\{j_0\}}] = [n_j]_{j\in J}.\end{aligned}$$

This proves the assertion by induction.

In view of the above and the inclusion-exclusion principle,

$$\begin{aligned}\Big[\bigcup_{i=1}^k G_i : H\Big] &= \sum_{\emptyset\neq J\subseteq[1,k]} (-1)^{|J|-1}\Big[\bigcap_{j\in J} G_j : H\Big]\\ &= \sum_{\emptyset\neq J\subseteq[1,k]} (-1)^{|J|-1}\frac{[G:H]}{[n_j]_{j\in J}}\\ &= \sum_{\emptyset\neq J\subseteq[1,k]} (-1)^{|J|-1}\Big|\Big\{0\leqslant x<[G:H]:\, x\in\bigcap_{j\in J} n_j\mathbb{Z}\Big\}\Big|\\ &= \sum_{\emptyset\neq J\subseteq[1,k]} (-1)^{|J|-1}\Big|\bigcap_{j\in J} X_j\Big| = \Big|\bigcup_{i=1}^k X_i\Big|\end{aligned}$$

where $X_i = \{0 \leqslant x < [G:H] :\, x \in n_i\mathbb{Z}\}$ for $i = 1,\dots,k$. Applying [S04b, Theorem 3.1], we finally obtain that

$$\begin{aligned}\Big[\bigcup_{i=1}^k a_iG_i : H\Big] \geqslant&\Big|\Big\{0\leqslant x<[G:H]:\, x\in\bigcup_{i=1}^k n_i\mathbb{Z}\Big\}\Big|\\ =&\Big|\bigcup_{i=1}^k X_i\Big| = \Big[\bigcup_{i=1}^k G_i : H\Big].\end{aligned}$$

The proof of Theorem 1.5 is now complete. $\square$

Remark 3.2. If $a_1G_1,\dots,a_kG_k$ are left cosets in a finite cyclic group G, then we also have (1.12) because $[G:H\cap K] = [[G:H],[G:K]]$ (i.e., $|H\cap K| = (|H|,|K|)$) for any subgroups H and K of G.

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