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MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS

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Abstract. By means of q-series, we prove that any natural number is a sum of an even square and two triangular numbers, and that each positive integer is a sum of a triangular number plus x^2+y^2 for some $x,y \in \mathbb{Z}$ with $x \not\equiv y \pmod 2$ or $x = y > 0$. The paper also contains some other results and open conjectures on mixed sums of squares and triangular numbers.

1. Introduction

A classical result of Fermat asserts that any prime $p \equiv 1 \pmod 4$ is a sum of two squares of integers. Fermat also conjectured that each $n \in \mathbb{N}$ can be written as a sum of three triangular numbers, where $\mathbb{N}$ is the set $\{0, 1, 2, \dots\}$ of natural numbers, and triangular numbers are those integers $t_x = x(x+1)/2$ with $x \in \mathbb{Z}$. An equivalent version of this conjecture states that $8n+3$ is a sum of three squares (of odd integers). This follows from the following profound theorem (see, e.g., [G, pp. 38–49] or [N, pp. 17-23]).

Gauss-Legendre Theorem. *$n \in \mathbb{N}$ can be written as a sum of three squares of integers if and only if n is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$.*

Building on some work of Euler, in 1772 Lagrange showed that every natural number is a sum of four squares of integers.

For problems and results on representations of natural numbers by various quadratic forms with coefficients in $\mathbb{N}$, the reader may consult [Du] and [G].

Motivated by Ramanujan's work [Ra], L. Panaitopol [P] proved the following interesting result in 2005.

Theorem A. *Let a, b, c be positive integers with $a \leqslant b \leqslant c$. Then every odd natural number can be written in the form $ax^2+by^2+cz^2$ with $x, y, z \in \mathbb{Z}$, if and only if the vector (a, b, c) is $(1,1,2)$ or $(1,2,3)$ or $(1,2,4)$.*

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According to L. E. Dickson [D2, p. 260], Euler already noted that any odd integer $n > 0$ is representable by $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$.

In 1862 J. Liouville (cf. [D2, p. 23]) proved the following result.

Theorem B. *Let a, b, c be positive integers with $a \leqslant b \leqslant c$. Then every $n \in \mathbb{N}$ can be written as $at_x + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$, if and only if (a, b, c) is among the following vectors:*

$$(1, 1, 1),\ (1, 1, 2),\ \ (1, 1, 4),\ (1, 1, 5),\ (1, 2, 2),\ (1, 2, 3),\ (1, 2, 4).$$

Now we turn to representations of natural numbers by mixed sums of squares (of integers) and triangular numbers.

Let $n \in \mathbb{N}$. By the Gauss-Legendre theorem, $8n + 1$ is a sum of three squares. It follows that $8n+1 = (2x)^2+(2y)^2+(2z+1)^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv y \pmod 2$; this yields the representation

$$n = \frac{x^2 + y^2}{2} + t_z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + t_z$$

as observed by Euler. According to Dickson [D2, p. 24], E. Lionnet stated, and V. A. Lebesgue [L] and M. S. Réalis [Re] proved that n can also be written in the form $x^2 + t_y + t_z$ with $x, y, z \in \mathbb{Z}$. Quite recently, this was reproved by H. M. Farkas [F] via the theory of theta functions.

Using the theory of ternary quadratic forms, in 1939 B. W. Jones and G. Pall [JP, Theorem 6] proved that for any $n \in \mathbb{N}$ we have $8n + 1 = ax^2 + by^2 + cz^2$ for some $x, y, z \in \mathbb{Z}$ if the vector (a, b, c) belongs to the set

$$\{(1, 1, 16),\ (1, 4, 16),\ (1, 16, 16),\ (1, 2, 32),\ (1, 8, 32),\ (1, 8, 64)\}.$$

As $(2z+1)^2 = 8t_z+1$, the result of Jones and Pall implies that each $n \in \mathbb{N}$ can be written in any of the following three forms with $x, y, z \in \mathbb{Z}$:

$$2x^2 + 2y^2 + t_z = (x + y)^2 + (x - y)^2 + t_z,\ x^2 + 4y^2 + t_z,\ x^2 + 8y^2 + t_z.$$

In this paper we establish the following result by means of q-series.

Theorem 1. (i) *Any $n \in \mathbb{N}$ is a sum of an even square and two triangular numbers. Moreover, if $n/2$ is not a triangular number then*

$$\begin{aligned} &|\{(x, y, z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} :\ x^2 + t_y + t_z = n \text{ and } 2 \nmid x\}| \\ =&|\{(x, y, z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} :\ x^2 + t_y + t_z = n \text{ and } 2 \mid x\}|. \end{aligned} \tag{1}$$

(ii) *If $n \in \mathbb{N}$ is not a triangular number, then*

$$\begin{aligned} &|\{(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} :\ x^2 + y^2 + t_z = n \text{ and } x \not\equiv y \pmod 2\}| \\ =&|\{(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} :\ x^2 + y^2 + t_z = n \text{ and } x \equiv y \pmod 2\}| > 0. \end{aligned} \tag{2}$$

(iii) *A positive integer n is a sum of an odd square, an even square and a triangular number, unless it is a triangular number t_m $(m > 0)$ for which all prime divisors of $2m+1$ are congruent to 1 mod 4 and hence $t_m = x^2 + x^2 + t_z$ for some integers $x > 0$ and z with $x \equiv m/2 \pmod 2$.*

Remark. Note that $t_2 = 1^2 + 1^2 + t_1$ but we cannot write $t_2 = 3$ as a sum of an odd square, an even square and a triangular number.

Here are two more theorems of this paper.

Theorem 2. *Let a, b, c be positive integers with $a \leqslant b$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + by^2 + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a, b, c) is among the following vectors:*

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

Theorem 3. *Let a, b, c be positive integers with $b \geqslant c$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a, b, c) is among the following vectors:*

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

Theorem 1 and Theorems 2–3 will be proved in Sections 2 and 3 respectively. In Section 4, we will pose three conjectures and discuss the converses of Theorems 2 and 3.

2. Proof of Theorem 1

Given two integer-valued quadratic polynomials $f(x, y, z)$ and $g(x, y, z)$, by $f(x, y, z) \sim g(x, y, z)$ we mean

$$\{f(x, y, z) : x, y, z \in \mathbb{Z}\} = \{g(x, y, z) : x, y, z \in \mathbb{Z}\}.$$

Clearly $\sim$ is an equivalence relation on the set of all integer-valued ternary quadratic polynomials.

The following lemma is a refinement of Euler's observation $t_y + t_z \sim y^2 + 2t_z$ (cf. [D2, p. 11]).

Lemma 1. *For any $n \in \mathbb{N}$ we have*

$$|\{(y, z) \in \mathbb{N}^2 : t_y + t_z = n\}| = |\{(y, z) \in \mathbb{Z} \times \mathbb{N} : y^2 + 2t_z = n\}|. \qquad (3)$$

Proof. Note that $t_{-y-1} = t_y$. Thus

$$\begin{aligned}
&|\{(y,z)\in\mathbb{N}^2:\ t_y+t_z=n\}| = \frac{1}{4}|\{(y,z)\in\mathbb{Z}^2:\ t_y+t_z=n\}|\\
=&\frac{1}{4}|\{(y,z)\in\mathbb{Z}^2:\ 4n+1=(y+z+1)^2+(y-z)^2\}|\\
=&\frac{1}{4}|\{(x_1,x_2)\in\mathbb{Z}^2:\ 4n+1=x_1^2+x_2^2 \text{ and } x_1\not\equiv x_2 \pmod 2\}|\\
=&\frac{2}{4}|\{(y,z)\in\mathbb{Z}^2:\ 4n+1=(2y)^2+(2z+1)^2\}|\\
=&\frac{1}{2}|\{(y,z)\in\mathbb{Z}^2:\ n=y^2+2t_z\}| = |\{(y,z)\in\mathbb{Z}\times\mathbb{N}:\ n=y^2+2t_z\}|.
\end{aligned}$$

This concludes the proof. □

Lemma 1 is actually equivalent to the following observation of Ramanujan (cf. Entry 25(iv) of [B, p. 40]): $\psi(q)^2 = \varphi(q)\psi(q^2)$ for $|q| < 1$, where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad\text{and}\quad \psi(q) = \sum_{n=0}^{\infty} q^{t_n}. \tag{4}$$

Let $n \in \mathbb{N}$ and define

$$r(n) = |\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2+t_y+t_z=n\}|, \tag{5}$$

$$r_0(n) = |\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2+t_y+t_z=n \text{ and } 2\mid x\}|, \tag{6}$$

$$r_1(n) = |\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2+t_y+t_z=n \text{ and } 2\nmid x\}|. \tag{7}$$

Clearly $r_0(n) + r_1(n) = r(n)$. In the following lemma we investigate the difference $r_0(n) - r_1(n)$.

Lemma 2. *For $m = 0, 1, 2, \ldots$ we have*

$$r_0(2t_m) - r_1(2t_m) = (-1)^m(2m+1). \tag{8}$$

Also, $r_0(n) = r_1(n)$ if $n \in \mathbb{N}$ is not a triangular number times 2.

Proof. Let $|q| < 1$. Recall the following three known identities implied by Jacobi's triple product identity (cf. [AAR, pp. 496–501]):

$$\varphi(-q) = \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n}) \quad \text{(Gauss)},$$

$$\psi(q) = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}} \quad \text{(Gauss)},$$

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{t_n} \quad \text{(Jacobi)}.$$

Observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(r_0(n)-r_1(n))q^n\\
=&\Big(\sum_{x=-\infty}^{\infty}(-1)^x q^{x^2}\Big)\Big(\sum_{y=0}^{\infty}q^{t_y}\Big)\Big(\sum_{z=0}^{\infty}q^{t_z}\Big)=\varphi(-q)\psi(q)^2\\
=&\Big(\prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n})\Big)\Big(\prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}}\Big)^2\\
=&\prod_{n=1}^{\infty}(1-q^{2n})^3=\sum_{m=0}^{\infty}(-1)^m(2m+1)(q^2)^{t_m}.
\end{aligned}$$

Comparing the coefficients of q^n on both sides, we obtain the desired result. $\square$

The following result was discovered by Hurwitz in 1907 (cf. [D2, p. 271]); an extension was established in [HS] via the theory of modular forms of half integer weight.

Lemma 3. *Let $n>0$ be an odd integer, and let $p_1,\ldots,p_r$ be all the distinct prime divisors of n congruent to 3 mod 4. Write $n=n_0\prod_{0<i\leqslant r}p_i^{\alpha_i}$, where $n_0,\alpha_1,\ldots,\alpha_r$ are positive integers and n_0 has no prime divisors congruent to 3 mod 4. Then*

$$|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}|=6n_0\prod_{0<i\leqslant r}\left(p_i^{\alpha_i}+2\frac{p_i^{\alpha_i}-1}{p_i-1}\right). \quad (9)$$

Proof. We deduce (9) in a new way and use some standard notations in number theory.

By (4.8) and (4.10) of [G],

$$\begin{aligned}
&|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}|\\
=&\sum_{d\mid n}\frac{24}{\pi}d\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{-4d^2}{m}\right)\\
=&\frac{24}{\pi}\sum_{d\mid n}d\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\sum_{c\mid\gcd(2k-1,d)}\mu(c)\\
=&\frac{24}{\pi}\sum_{d\mid n}d\sum_{c\mid d}\mu(c)\sum_{k=1}^{\infty}\frac{(-1)^{((2k-1)c-1)/2}}{(2k-1)c}\\
=&\frac{24}{\pi}\sum_{d\mid n}d\sum_{c\mid d}\frac{\mu(c)}{c}(-1)^{(c-1)/2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\\
=&6\sum_{c\mid n}(-1)^{(c-1)/2}\frac{\mu(c)}{c}\sum_{q\mid\frac{n}{c}}cq=6\sum_{c\mid n}(-1)^{(c-1)/2}\mu(c)\sigma\left(\frac{n}{c}\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{1}{6}|\{(x,y,z)\in\mathbb{Z}^3:\,x^2+y^2+z^2=n^2\}|\\
=&\sum_{d_0\mid n_0}\ \sum_{d_1\mid p_1^{\alpha_1},\ldots,d_r\mid p_r^{\alpha_r}}\left(\frac{-1}{d_0d_1\cdots d_r}\right)\mu(d_0d_1\cdots d_r)\sigma\left(\frac{n_0}{d_0}\prod_{0<i\leqslant r}\frac{p_i^{\alpha_i}}{d_i}\right)\\
=&\sum_{d_0\mid n_0}\left(\frac{-1}{d_0}\right)\mu(d_0)\sigma\left(\frac{n_0}{d_0}\right)\times\prod_{0<i\leqslant r}\sum_{d_i\mid p_i^{\alpha_i}}\left(\frac{-1}{d_i}\right)\mu(d_i)\sigma\left(\frac{p_i^{\alpha_i}}{d_i}\right)\\
=&\sum_{d_0\mid n_0}\mu(d_0)\sigma\left(\frac{n_0}{d_0}\right)\times\prod_{0<i\leqslant r}\left(\sigma(p_i^{\alpha_i})+\left(\frac{-1}{p_i}\right)\mu(p_i)\sigma(p_i^{\alpha_i-1})\right)\\
=&n_0\prod_{0<i\leqslant r}\left(p_i^{\alpha_i}+2\sigma(p_i^{\alpha_i-1})\right)=n_0\prod_{0<i\leqslant r}\left(p_i^{\alpha_i}+2\frac{p_i^{\alpha_i}-1}{p_i-1}\right).
\end{aligned}$$

This completes the proof. □

`Proof of Theorem 1`. (i) By the Gauss-Legendre theorem, $4n+1$ is a sum of three squares and hence $4n+1=(2x)^2+(2y)^2+(2z+1)^2$ (i.e., $n=x^2+y^2+2t_z$) for some $x,y,z\in\mathbb{Z}$. Combining this with Lemma 1 we obtain a simple proof of the known result

$$r(n)=|\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\,x^2+t_y+t_z=n\}|>0.$$

Recall that $r_0(n)+r_1(n)=r(n)$. If $n/2$ is not a triangular number, then $r_0(n)=r_1(n)=r(n)/2>0$ by Lemma 2. If $n=2t_m$ for some $m\in\mathbb{N}$, then we also have $r_0(n)>0$ since $n=0^2+t_m+t_m$.

(ii) Note that

$$n=x^2+y^2+t_z\iff 2n=2(x^2+y^2)+2t_z=(x+y)^2+(x-y)^2+2t_z.$$

From this and Lemma 1, we get

$$\begin{aligned}
&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\,x^2+y^2+t_z=n\}|\\
=&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\,x^2+y^2+2t_z=2n\}|\\
=&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\,x^2+t_y+t_z=2n\}|=r(2n)>0;
\end{aligned}$$

in the language of generating functions, it says that

$$\varphi(q)\psi(q)^2+\varphi(-q)\psi(-q)^2=2\varphi(q^2)^2\psi(q^2).$$

Similarly,

$$\begin{aligned}
&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\,x^2+y^2+t_z=n\text{ and }x\equiv y\pmod 2\}|\\
=&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\,x^2+y^2+2t_z=2n\text{ and }2\mid x\}|\\
=&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\,x^2+t_y+t_z=2n\text{ and }2\mid x\}|=r_0(2n)>0
\end{aligned}$$

and

$$|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2+y^2+t_z=n \text{ and } x\not\equiv y \pmod 2\}|=r_1(2n). \tag{10}$$

If n is not a triangular number, then $r_0(2n)=r_1(2n)=r(2n)/2>0$ by Lemma 2, and hence (2) follows from the above.

(iii) By Theorem 1(ii), if n is not a triangular number then $n=x^2+y^2+t_z$ for some $x,y,z\in\mathbb{Z}$ with $2\mid x$ and $2\nmid y$.

Now assume that $n=t_m$ $(m>0)$ is not a sum of an odd square, an even square and a triangular number. Then $r_1(2t_m)=0$ by (10). In view of (ii) and (8),

$$\begin{aligned}&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2+y^2+t_z=t_m\}|\\=&r_0(2t_m)+r_1(2t_m)=r_0(2t_m)-r_1(2t_m)=(-1)^m(2m+1).\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^m(2m+1)=\frac{1}{2}|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+t_z=t_m\}|\\=&\frac{1}{2}|\{(x,y,z)\in\mathbb{Z}^3:\ 2(2x)^2+2(2y)^2+(2z+1)^2=8t_m+1\}|\\=&\frac{1}{2}|\{(x,y,z)\in\mathbb{Z}^3:(2x+2y)^2+(2x-2y)^2+(2z+1)^2=8t_m+1\}|\\=&\frac{1}{2}|\{(x_1,y_1,z)\in\mathbb{Z}^3:\ 4(x_1^2+y_1^2)+(2z+1)^2=(2m+1)^2\}|\\=&\frac{1}{6}|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=(2m+1)^2\}|.\end{aligned}$$

Since

$$(-1)^m 6(2m+1)=|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=(2m+1)^2\}|\not> 6(2m+1),$$

by Lemma 3 the odd number $2m+1$ cannot have a prime divisor congruent to 3 mod 4. So all the prime divisors of $2m+1$ are congruent to 1 mod 4, and hence

$$|\{(x,y)\in\mathbb{N}^2:\ x>0 \text{ and } x^2+y^2=2m+1\}|=\sum_{d\mid 2m+1}1>1$$

by Proposition 17.6.1 of [IR, p. 279]. Thus $2m+1$ is a sum of two squares of positive integers. Choose positive integers x and y such that $x^2+y^2=2m+1$ with $2\mid x$ and $2\nmid y$. Then

$$8t_m+1=(2m+1)^2=(x^2-y^2)^2+4x^2y^2=8t_{(x^2-y^2-1)/2}+1+16\left(\frac{x}{2}\right)^2y^2$$

and hence

$$t_m = t_{(x^2-y^2-1)/2} + \left(\frac{x}{2}y\right)^2 + \left(\frac{x}{2}y\right)^2.$$

As $x^2 = 2m + 1 - y^2 \equiv 2m \pmod 8$, m is even and

$$\frac{m}{2} \equiv \left(\frac{x}{2}\right)^2 \equiv \frac{x}{2} \equiv \frac{x}{2}y \pmod 2.$$

We are done. □

3. Proofs of Theorems 2 and 3

`Proof of Theorem 2.` We distinguish four cases.

Case 1. $a = c = 1$. Write $8 = x_0^2 + by_0^2 + t_{z_0}$ with $x_0, y_0, z_0 \in \mathbb{Z}$, then $y_0 \neq 0$ and hence $8 \geqslant b$. Since $x^2 + 5y^2 + t_z \neq 13$, $x^2 + 6y^2 + t_z \neq 47$ and $x^2 + 7y^2 + t_z \neq 20$, we must have $b \in \{1, 2, 3, 4, 8\}$.

Case 2. $a = 1$ and $c = 2$. Write $5 = x_0^2 + by_0^2 + 2t_{z_0}$ with $x_0, y_0, z_0 \in \mathbb{Z}$. Then $y_0 \neq 0$ and hence $5 \geqslant b$. Observe that $x^2 + 3y^2 + 2t_z \neq 8$ and $x^2 + 5y^2 + 2t_z \neq 19$. Therefore $b \in \{1, 2, 4\}$.

Case 3. $a = 1$ and $c \geqslant 3$. Since $2 = x^2 + by^2 + ct_z$ for some $x, y, z \in \mathbb{Z}$, we must have $b \leqslant 2$. If $b = 1$, then there are $x_0, y_0, z_0 \in \mathbb{Z}$ such that $3 = x_0^2 + y_0^2 + ct_{z_0} \geqslant c$ and hence $c = 3$. But $x^2 + y^2 + 3t_z \neq 6$, therefore $b = 2$. For some $x, y, z \in \mathbb{Z}$ we have $5 = x^2 + 2y^2 + ct_z \geqslant c$. Since $x^2 + 2y^2 + 3t_z \neq 23$ and $x^2 + 2y^2 + 5t_z \neq 10$, c must be 4.

Case 4. $a > 1$. As $b \geqslant a \geqslant 2$ and $ax^2 + by^2 + ct_z = 1$ for some $x, y, z \in \mathbb{Z}$, we must have $c = 1$. If $a > 2$, then $ax^2 + by^2 + t_z \neq 2$. Thus $a = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $4 = 2x_0^2 + by_0^2 + t_{z_0} \geqslant b$. Note that $2x^2 + 3y^2 + t_z \neq 7$ and $2x^2 + 4y^2 + t_z \neq 20$. Therefore $b = 2$.

In view of the above, Theorem 2 has been proven. □

`Proof of Theorem 3.` Let us first consider the case $c > 1$. Since $1 = ax^2 + bt_y + ct_z$ for some $x, y, z \in \mathbb{Z}$, we must have $a = 1$. Clearly $x^2 + bt_y + ct_z \neq 2$ if $c \geqslant 3$. So $c = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $5 = x_0^2 + bt_{y_0} + 2t_{z_0} \geqslant b$. It is easy to check that $x^2 + 3t_y + 2t_z \neq 8$. Therefore $b \in \{2, 4, 5\}$.

Below we assume that $c = 1$. If a and b are both greater than 2, then $ax^2 + bt_y + t_z \neq 2$. So $a \leqslant 2$ or $b \leqslant 2$.

Case 1. $a = 1$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $8 = x_0^2 + bt_{y_0} + t_{z_0} \geqslant b$. Note that $x^2 + 5t_y + t_z \neq 13$ and $x^2 + 7t_y + t_z \neq 20$. So $b \in \{1, 2, 3, 4, 6, 8\}$.

Case 2. $a = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $4 = 2x_0^2 + bt_{y_0} + t_{z_0} \geqslant b$. Thus $b \in \{1, 2, 4\}$ since $2x^2 + 3t_y + t_z \neq 7$.

Case 3. $a > 2$. In this case $b \leqslant 2$. If $b = 1$, then for some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $5 = ax_0^2 + t_{y_0} + t_{z_0} \geqslant a$, and hence $a = 4$ since $3x^2 + t_y + t_z \neq 8$ and $5x^2 + t_y + t_z \neq 19$. If $b = 2$, then for some $x, y, z \in \mathbb{Z}$ we have $4 = ax^2 + 2t_y + t_z \geqslant a$ and so $a \in \{3, 4\}$.

The proof of Theorem 3 is now complete. □

4. Some conjectures and related discussion

In this section we raise three related conjectures.

Conjecture 1. *Any positive integer n is a sum of a square, an odd square and a triangular number. In other words, each natural number can be written in the form $x^2 + 8t_y + t_z$ with $x, y, z \in \mathbb{Z}$.*

We have verified Conjecture 1 for $n \leqslant 15,000$. By Theorem 1(iii), Conjecture 1 is valid when $n \neq t_4, t_8, t_{12}, \ldots$.

Conjecture 2. *Each $n \in \mathbb{N}$ can be written in any of the following forms with $x, y, z \in \mathbb{Z}$:*

$$x^2 + 3y^2 + t_z,\ x^2 + 3t_y + t_z,\ x^2 + 6t_y + t_z,\ 3x^2 + 2t_y + t_z,\ 4x^2 + 2t_y + t_z.$$

Conjecture 3. *Every $n \in \mathbb{N}$ can be written in the form $x^2+2y^2+3t_z$ (with $x, y, z \in \mathbb{Z}$) except $n = 23$, in the form $x^2 + 5y^2 + 2t_z$ (or the equivalent form $5x^2 + t_y + t_z$) except $n = 19$, in the form $x^2 + 6y^2 + t_z$ except $n = 47$, and in the form $2x^2 + 4y^2 + t_z$ except $n = 20$.*

Both Conjectures 2 and 3 have been verified for $n \leqslant 10,000$.

The second statement in Conjecture 3 is related to an assertion of Ramanujan confirmed by Dickson [D1] which states that even natural numbers not of the form $4^k(16l + 6)$ (with $k, l \in \mathbb{N}$) can be written as $x^2 + y^2 + 10z^2$ with $x, y, z \in \mathbb{Z}$. Observe that

$$\begin{aligned} &n = x^2 + 5y^2 + 2t_z \text{ for some } x, y, z \in \mathbb{Z} \\ \Longleftrightarrow\ &4n + 1 = x^2 + 5y^2 + z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid z \\ \Longleftrightarrow\ &8n + 2 = 2(x^2 + y^2) + 10z^2 = (x + y)^2 + (x - y)^2 + 10z^2 \\ &\qquad\qquad\qquad\qquad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid y \\ \Longleftrightarrow\ &8n + 2 = x^2 + y^2 + 10z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } x \not\equiv y \pmod 4. \end{aligned}$$

Below we reduce the converses of Theorems 2 and 3 to Conjectures 1 and 2. For convenience, we call a ternary quadratic polynomial $f(x, y, z)$ *essential* if $\{f(x, y, z) : x, y, z \in \mathbb{Z}\} = \mathbb{N}$. (Actually, in 1748 Goldbach (cf. [D2, p. 11]) already stated that $x^2 + y^2 + 2t_z$, $x^2 + 2y^2 + t_z$, $x^2 + 2y^2 + 2t_z$ and $2x^2 + 2t_y + t_z$ are essential.)

`Step I`. We show that the 10 quadratic polynomials listed in Theorem 2 are essential except for the form $x^2 + 3y^2 + t_z$ appearing in Conjecture 2.

As $4x^2 + y^2 + 2t_z \sim 4x^2 + t_y + t_z$, the form $x^2 + (2y)^2 + 2t_z$ is essential by Theorem 1(i). Both $x^2 + (2y)^2 + t_z$ and $2x^2 + 2y^2 + t_z = (x+y)^2 + (x-y)^2 + t_z$ are essential by Theorem 1(ii) and the trivial fact $t_z = 0^2 + 0^2 + t_z$. We have pointed out in Section 1 that $x^2+2(2y)^2+t_z$ is essential by [JP, Theorem 6], and we don't have an easy proof of this deep result.

Since

$$x^2+2y^2+4t_z \sim x^2+2(t_y+t_z) \sim t_x+t_y+2t_z,$$

the form $x^2+2y^2+4t_z$ is essential by Theorem B (of Liouville). By the Gauss-Legendre theorem, for each $n\in\mathbb{N}$ we can write $8n+2=(4x)^2+(2y+1)^2+(2z+1)^2$ (i.e., $n=2x^2+t_y+t_z$) with $x,y,z\in\mathbb{Z}$. Thus the form $x^2+2y^2+2t_z$ is essential since $2x^2+y^2+2t_z\sim 2x^2+t_y+t_z$.

`Step II`. We analyze the 15 quadratic polynomials listed in Theorem 3.

By Theorem 1(i), $(2x)^2+t_y+t_z$ and $x^2+t_y+t_z$ are essential. Since

$$\begin{aligned}
x^2+2t_y+t_z &\sim t_x+t_y+t_z,\\
x^2+2t_y+2t_z &\sim t_x+t_y+2t_z,\\
x^2+4t_y+2t_z &\sim t_x+4t_y+t_z,\\
x^2+5t_y+2t_z &\sim t_x+5t_y+t_z,\\
2x^2+4t_y+t_z &\sim 2t_x+2t_y+t_z,
\end{aligned}$$

the forms

$$x^2+2t_y+t_z,\ x^2+2t_y+2t_z,\ x^2+4t_y+2t_z,\ x^2+5t_y+2t_z,\ 2x^2+4t_y+t_z$$

are all essential by Liouville's theorem. For $n\in\mathbb{N}$ we can write $2n=x^2+4t_y+2t_z$ with $x,y,z\in\mathbb{Z}$, and hence $n=2x_0^2+2t_y+t_z$ with $x_0=x/2\in\mathbb{Z}$. So the form $2x^2+2t_y+t_z$ is also essential.

Recall that $2x^2+t_y+t_z$ and $2x^2+y^2+2t_z$ are essential by the last two sentences of Step I. For each $n\in\mathbb{N}$ we can choose $x,y,z\in\mathbb{Z}$ such that $2n+1=2x^2+(2y+1)^2+2t_z$ and hence $n=x^2+4t_y+t_z$. So the form $x^2+4t_y+t_z$ is essential.

The remaining forms listed in Theorem 3 are $x^2+8t_y+t_z$ and four other forms, which appear in Conjectures 1 and 2 respectively. We are done.

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Added in proof. The second conjecture in Section 4 has been confirmed by Song Guo, Hao Pan and the author.

References

[AAR] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[B] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
[D1] L. E. Dickson, *Integers represented by positive ternary quadratic forms*, Bull. Amer. Math. Soc. **33** (1927), 63–70.
[D2] L. E. Dickson, *History of the Theory of Numbers,* Vol. II, AMS Chelsea Publ., 1999.
[Du] W. Duke, *Some old problems and new results about quadratic forms*, Notice Amer. Math. Soc. **44** (1997), 190–196.
[F] H. M. Farkas, *Sums of squares and triangular numbers*, Online J. Anal. Combin. **1** (2006), #1, 11 pp. (electronic).
[G] E. Grosswald, *Representation of Integers as Sums of Squares*, Springer, New York, 1985.
[HS] M. D. Hirschhorn and J. Sellers, *On representations of a number as a sum of three squares*, Discrete Math. **199** (1999), 85–101.
[IR] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Grad. Texts in Math. 84, Springer, New York, 1990.
[JP] B. W. Jones and G. Pall, *Regular and semi-regular positive ternary quadratic forms*, Acta Math. **70** (1939), 165–191.
[L] V. A. Lebesque, *Questions 1059,1060,1061 (Lionnet)*, Nouv. Ann. Math. **11** (1872), 516–519.
[N] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Grad. Texts in Math. 164, Springer, New York, 1996.
[P] L. Panaitopol, *On the representation of natural numbers as sums of squares*, Amer. Math. Monthly **112** (2005), 168–171.
[Ra] S. Ramanujan, *On the expression of a number in the form* $ax^2+by^2+cz^2+du^2$, in: Collected Papers of Srinivasa Ramanujan, Cambridge Univ. Press, 1927, 169–178.
[Re] M. S. Réalis, *Scolies pour un théoreme d'arithmétique*, Nouv. Ann. Math. **12** (1873), 212–217.

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