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On the number of zero-sum subsequences

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Abstract

For a sequence S of elements from an additive abelian group G, let $f(S)$ denote the number of subsequences of S the sum of whose terms is zero. In this paper we characterize all sequences S in G with $f(S) > 2^{|S|-2}$, where $|S|$ denotes the number of terms of S.

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1 Introduction

Let G be an additive abelian group. Let m be a positive integer. Throughout this paper, we use Ω_m to denote the set of sequences of elements from G of length m. For $S = (a_1, a_2, \ldots, a_m) \in \Omega_m$, set $\sigma(S) = \sum_{i=1}^{m} a_i$. We say S is *zero-sum* if $\sigma(S) = 0$. Let $f(S)$ denote the number of zero-sum subsequences of S. Obviously we have $f(S) \leq 2^m$ with equality if and only if each term of S is zero. Bulman-Fleming and Wang [1] proved that if $S \in \Omega_m$ and $f(S) < 2^m$ then $f(S) \leq 2^{m-1}$, and characterized all sequences for which the equality holds. Guichard [3] proved the following theorem.

Theorem 1. *Let $m > 2$ and $S \in \Omega_m$. If $f(S) < 2^{m-1}$ then $f(S) \leq 3 \times 2^{m-3}$ with equality if and only if there is an element a with $2a \neq 0$ such that S is an arrangement of $(0, \ldots, 0, a, a, -a)$ or $(0, \ldots, 0, a, a, -a, -a)$.*

Thus all sequences S with $f(S) \geq 3 \times 2^{|S|-3}$ were determined completely. In [4] F. Li and W. D. Gao obtained the following result.

Theorem 2. *Let $m > 2$. If $S \in \Omega_m$ and $\frac{5}{16} \times 2^m < f(S) < 2^{m-1}$ then there is an arrangement of S of the form*

$$(0, \ldots, 0, \overbrace{a, \ldots, a}^{e}, \overbrace{-a, \ldots, -a}^{f})$$

with $e \geq f \geq 0, 2a \neq 0$ and $3a = 0$ or $e \leq 2$.

In this paper we shall characterize all sequences S with $2^{|S|-2} < f(S) < 3 \times 2^{|S|-3}$ in the following theorem.

Theorem *Let $m > 2$ and $S \in \Omega_m$ with $2^{m-2} < f(S) < 3 \times 2^{m-3}$. Then there is an arrangement of S of the form $(0, \ldots, 0, \overbrace{a, \ldots, a}^{e}, \overbrace{-a, \ldots, -a}^{f})$ with $e \geq f \geq 0$, $e > 0$ and $a \neq 0$. Let d be the order of a. Then*

(1) $\frac{5}{16} \times 2^m < f(S) < \frac{3}{8} \times 2^m$ if and only if $d = 3$, $e \geq 3$ and $(e, f) \notin \{(3, 0), (3, 1), (4, 0), (4, 1)\}$;

(2) $f(S) = \frac{5}{16} \times 2^m$ if and only if $d = 3, (e, f) \in \{(3, 1), (4, 0), (4, 1)\}$ or $d \geq 4, (e, f) \in \{(3, 2), (3, 3)\}$;

(3) $\frac{35}{128} \times 2^m < f(S) < \frac{5}{16} \times 2^m$ if and only if $f(S) = \frac{9}{32} \times 2^m$ and $d = 4, (e, f) \in \{(4, 3), (4, 4), (7, 0), (8, 0)\}$;

(4) $f(S) = \frac{35}{128} \times 2^m$ if and only if $d \geq 5$ and $(e, f) \in \{(4, 3), (4, 4)\}$;

(5) $2^{m-2} < f(S) < \frac{35}{128} \times 2^m$ if and only if $d = 4$ and $e - f \equiv 0, \pm 1 \pmod 8, e + f > 8$.

2 Proof of Theorem

Lemma 1. (Li and Gao [4]) *Let $S \in \Omega_m$ with $f(S) > 2^{m-2}$. Then there is an arrangement of S of the form $(0, \ldots, 0, a, \ldots, a, -a, \ldots, -a)$.*

Proof. It is true for $m = 1, 2$. Now assume that $m \geq 3$. Suppose that there is an arrangement of S of the form $(a_1, a_2, \ldots, a_{m-2}, x, y)$ where $0, x, y, x + y$ are distinct. Set $T = (a_1, a_2, \ldots, a_{m-2})$. Then

$$\begin{aligned} f(S) &= |\{W : W \text{ is a subsequence of } T \ \&\ \sigma(W) = 0, -x, -y \text{ or } -x-y\}| \\ &\leq |\{W : W \text{ is a subsequence of } T\}| \\ &= 2^{m-2}. \end{aligned}$$

This is a contradiction. $\square$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, \ldots\}$. Following Z. W. Sun [5], for $d \in \mathbb{Z}^+, n \in \mathbb{N}$ and $r \in \mathbb{Z}$, we set

$$\begin{bmatrix} n \\ r \end{bmatrix}_d = \sum_{\substack{0 \le k \le n \\ k \equiv r \pmod d}} \binom{n}{k}.$$

Using

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{and} \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

we have

$$\begin{bmatrix} n \\ r \end{bmatrix}_d = \begin{bmatrix} n \\ n-r \end{bmatrix}_d \quad \text{and} \quad \begin{bmatrix} n+1 \\ r \end{bmatrix}_d = \begin{bmatrix} n \\ r \end{bmatrix}_d + \begin{bmatrix} n \\ r-1 \end{bmatrix}_d \tag{1}$$

as observed in Sun [5]. The following formula can be found in H. W. Gould's book (cf.[2]).

$$\begin{bmatrix} n \\ r \end{bmatrix}_d = \frac{1}{d} \sum_{j=1}^{d} \left(2 \cos \frac{j\pi}{d} \right)^n \cos \frac{(n-2r)j\pi}{d}.$$

Thus, in the case $d = 2, 3, 4$, the combinatorial sum $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_d$ can be obtained.

$$\begin{bmatrix} n \\ r \end{bmatrix}_2 = 2^{n-1} \tag{2}$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_3 = \frac{2^n}{3} \left(1 + 2^{1-n} \cos \frac{(n-2r)\pi}{3} \right) \tag{3}$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_4 = \frac{2^n}{4} \left(1 + 2^{1-n/2} \cos \frac{(n-2r)\pi}{4} \right) \tag{4}$$

Furthermore, Z. W. Sun [5] even determined $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_{12}$ in terms of linear recurrences.

Lemma 2. *Let d and n_0 be positive integers. If $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d < 2^{n_0-2}$ for $0 \le r \le n_0/2$ then $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_d < 2^{n-2}$ for all $n \ge n_0$ and $r \in \mathbb{Z}$.*

Proof. Since $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d = \left[\begin{smallmatrix} n_0 \\ n_0-r \end{smallmatrix}\right]_d$, we have $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d < 2^{n_0-2}$ for $0 \le r \le n_0$. Observe that $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d = \binom{n_0}{r} = 0$ for $n_0 < r \le d-1$ and $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d = \left[\begin{smallmatrix} n_0 \\ t \end{smallmatrix}\right]_d$ for $r \equiv t \pmod d$. As a result, $\left[\begin{smallmatrix} n_0 \\ r \end{smallmatrix}\right]_d < 2^{n_0-2}$ for $r \in \mathbb{Z}$. Using (1), the inequality can be proved by induction on n immediately. $\square$

Lemma 3. *Let* $S = (0, \ldots, 0, \overbrace{a, \ldots, a}^{e}, \overbrace{-a, \ldots, -a}^{f}) \in \Omega_m$ *with* $e \geq f \geq 0$, *and* $a \neq 0$. *Let* d *be the order of* a. *Then*

$$f(S) = 2^{m-e-f} \begin{bmatrix} e+f \\ f \end{bmatrix}_d .$$

Proof. Set $T = (\overbrace{a, \ldots, a}^{e}, \overbrace{-a, \ldots, -a}^{f})$. Then

$$\begin{aligned} f(T) &= \sum_{d|k-l} \binom{e}{k}\binom{f}{l} \\ &= \sum_{i=-\infty}^{i=+\infty} [x^{di}](1+x)^e(1+x^{-1})^f = \sum_{i=-\infty}^{i=+\infty} [x^{di}] \frac{(1+x)^e(1+x)^f}{x^f} \\ &= \sum_{i=-\infty}^{i=+\infty} [x^{f+di}](1+x)^{e+f} \\ &= \sum_{\substack{0 \leq k \leq e+f \\ k \equiv f \pmod{d}}} \binom{e+f}{k} = \begin{bmatrix} e+f \\ f \end{bmatrix}_d . \end{aligned}$$

So we have $f(S) = 2^{m-e-f} f(T) = 2^{m-e-f} \left[\begin{smallmatrix} e+f \\ f \end{smallmatrix}\right]_d$. □

Proof of Theorem. By Lemma 1, there is an arrangement of S of the form $(0, \ldots, 0, \overbrace{a, \ldots, a}^{e}, \overbrace{-a, \ldots, -a}^{f})$ with $e \geq f \geq 0$ and $a \neq 0$. Obviously we have $e > 0$. Let d be the order of a. Then Lemma 3 implies that $f(S) = 2^{m-e-f} \left[\begin{smallmatrix} e+f \\ f \end{smallmatrix}\right]_d$, i.e.

$$2^{-m} f(S) = 2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_d .$$

By (2), $2^{-(e+f)} \left[\begin{smallmatrix} e+f \\ f \end{smallmatrix}\right]_2 = 1/2 > 3/8$. So we have $d \geq 3$. Since

$$\begin{aligned} &2^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_d = \frac{1}{2}, \qquad && 2^{-2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_d = \frac{1}{2}, \\ &2^{-2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}_d = \frac{1}{4}, \qquad && 2^{-3} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_d = \frac{3}{8}, \qquad && 2^{-4} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_d = \frac{3}{8}, \end{aligned}$$

we also have $e \geq 3$. To determine $2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_d$ we shall consider the following cases .

Case 1. $e = 3$.

$$2^{-3} \begin{bmatrix} 3 \\ 0 \end{bmatrix}_d \leq \frac{1}{4}, \qquad 2^{-4} \begin{bmatrix} 4 \\ 1 \end{bmatrix}_d = \begin{cases} \frac{5}{16} & d = 3 \\ \frac{1}{4} & d \geq 4 \end{cases},$$

$$2^{-5} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_d = \begin{cases} \frac{11}{32} & d = 3 \\ \frac{5}{16} & d \geq 4 \end{cases}, \qquad 2^{-6} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_d = \begin{cases} \frac{11}{32} & d = 3 \\ \frac{5}{16} & d \geq 4 \end{cases}.$$

Case 2. $d = 3$ and $e \geq 4$. By (3),

$$2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_3 = \begin{cases} (1 + 2^{1-(e+f)})/3 & e - f \equiv 0 \pmod 6 \\ (1 + 2^{-(e+f)})/3 & e - f \equiv \pm 1 \pmod 6 \\ (1 - 2^{-(e+f)})/3 & e - f \equiv \pm 2 \pmod 6 \\ (1 - 2^{1-(e+f)})/3 & e - f \equiv 3 \pmod 6 \end{cases}.$$

Since $e \geq 4$, we have

$$\frac{5}{16} \leq 2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_3 \leq \frac{11}{32} < \frac{3}{8}$$

and $2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_3 = 5/16$ if and only if $(e, f) \in \{(4, 0), (4, 1)\}$.

Case 3. $d = 4$ and $e \geq 4$. By (4),

$$2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_4 = \begin{cases} (1 + 2^{1-(e+f)/2})/4 & e - f \equiv 0 \pmod 8 \\ (1 + 2^{(1-e-f)/2})/4 & e - f \equiv \pm 1 \pmod 8 \\ 1/4 & e - f \equiv \pm 2 \pmod 8 \\ (1 - 2^{(1-e-f)/2})/4 & e - f \equiv \pm 3 \pmod 8 \\ (1 - 2^{1-(e+f)/2})/4 & e - f \equiv 4 \pmod 8 \end{cases}.$$

Therefore $2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_4 > 1/4$ if and only if $e - f \equiv 0, \pm 1 \pmod 8$. As $e \geq 4$, we have

$$2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_4 \leq \frac{9}{32} < \frac{5}{16}$$

with equality if and only if $(e, f) \in \{(4, 3), (4, 4), (7, 0), (8, 0)\}$. Furthermore, if $2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_4 < \frac{9}{32}$ then

$$2^{-(e+f)} \begin{bmatrix} e+f \\ f \end{bmatrix}_4 \leq \frac{17}{64} < \frac{35}{128}$$

and $e+f>8$.

Case 4. $d\geq 5$ and $e\geq 4$. If $e+f\leq 8$ then one can check $\begin{bmatrix} e+f \\ f \end{bmatrix}_d$ directly.

$$\begin{array}{lll}
\begin{bmatrix} 4 \\ 0 \end{bmatrix}_d = 1 < 2^{4-2}, & \begin{bmatrix} 5 \\ 0 \end{bmatrix}_d \leq 2 < 2^{5-2}, & \begin{bmatrix} 5 \\ 1 \end{bmatrix}_d = 5 < 2^{5-2}, \\
\begin{bmatrix} 6 \\ 0 \end{bmatrix}_d \leq 7 < 2^{6-2}, & \begin{bmatrix} 6 \\ 1 \end{bmatrix}_d \leq 7 < 2^{6-2}, & \begin{bmatrix} 6 \\ 2 \end{bmatrix}_d = 15 < 2^{6-2}, \\
\begin{bmatrix} 7 \\ 0 \end{bmatrix}_d \leq 22 < 2^{7-2}, & \begin{bmatrix} 7 \\ 1 \end{bmatrix}_d \leq 14 < 2^{7-2}, & \begin{bmatrix} 7 \\ 2 \end{bmatrix}_d \leq 22 < 2^{7-2}, \\
\begin{bmatrix} 7 \\ 3 \end{bmatrix}_d = 35 = \dfrac{35}{128} \times 2^7, & \begin{bmatrix} 8 \\ 0 \end{bmatrix}_d \leq 57 < 2^{8-2}, & \begin{bmatrix} 8 \\ 1 \end{bmatrix}_d \leq 36 < 2^{8-2}, \\
\begin{bmatrix} 8 \\ 2 \end{bmatrix}_d \leq 36 < 2^{8-2}, & \begin{bmatrix} 8 \\ 3 \end{bmatrix}_d \leq 57 < 2^{8-2}, & \begin{bmatrix} 8 \\ 4 \end{bmatrix}_d = 70 = \dfrac{35}{128} \times 2^8.
\end{array}$$

Since

$$\begin{bmatrix} 9 \\ 0 \end{bmatrix}_d \leq 127 < 2^{9-2}, \quad \begin{bmatrix} 9 \\ r \end{bmatrix}_d = \begin{bmatrix} 8 \\ r \end{bmatrix}_d + \begin{bmatrix} 8 \\ r-1 \end{bmatrix}_d \leq 127 < 2^{9-2}$$

for $1\leq r\leq 4$, by Lemma 2, we have $\begin{bmatrix} e+f \\ f \end{bmatrix}_d < 2^{e+f-2}$ for $e+f\geq 9$. As a result, in this case, $2^{-(e+f)}\begin{bmatrix} e+f \\ f \end{bmatrix}_d < 1/4$ except that

$$2^{-7}\begin{bmatrix} 7 \\ 3 \end{bmatrix}_d = \frac{35}{128} = 2^{-8}\begin{bmatrix} 8 \\ 4 \end{bmatrix}_d.$$

In view of the above discussion, the proof is now complete. $\square$

References

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