Int. J. Mod. Math. 2(2007), no. 2, 247–251.

An Extension of a Curious Binomial Identity

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Abstract

In 2002 Z. W. Sun published a curious identity involving binomial coefficients. In this paper we obtain the following generalization of the identity:

$$
\begin{aligned}
&(x+(m+1)z)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\
=&z\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}+(x-m)\binom{x}{m}.
\end{aligned}
$$

Keywords: Binomial coefficient, combinatorial identity.

2000 Mathematics Subject Classification: 05A19, 11B65.

1 Introduction

In 2002 Z. W. Sun [9] established the following combinatorial identity:

$$
\begin{aligned}
&(x+m+1)\sum_{n=0}^{m}(-1)^n\binom{x+y+n}{m-n}\binom{y+2n}{n}\\
&=\sum_{n=0}^{m}\binom{x+n}{m-n}(-4)^n+(x-m)\binom{x}{m},
\end{aligned}
\tag{1.1}
$$

where $m\in\mathbb{N}=\{0,1,2,\ldots\}$. Later A. Panholzer and H. Prodinger [8] gave a new proof using generating functions, D. Merlini and R. Sprugnoli [7] obtained another proof by means of Riordan arrays, S. B. Ekhad and M. Mohammed [4] presented a WZ proof of the identity, and W. Chu and L.V.D. Claudio [3] re-proved the identity by using Jensen's formula.

In this paper we aim to extend the curious identity as follows.

Theorem 1.1. *For any $m=0,1,2,\ldots$, we have*

$$
\begin{aligned}
&(x+(m+1)z)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\
=&z\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}+(x-m)\binom{x}{m};
\end{aligned}
\tag{1.2}
$$

equivalently,

$$\begin{aligned}&(x+(m+1)z+1)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\=&(z+1)\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l+1}{m-n}(1+z)^{n+l}(1-z)^{n-l}+(x-m)\binom{x}{m}.\end{aligned}\tag{1.3}$$

Remark 1.1. Soon after the initial version of this paper was posted as a preprint (`arXiv:math.CO/0401057`), D. Callan [C] found a nice combinatorial interpretation of the identity (1.1) (which was called "Sun's identity" by him) and also a slightly more complicated combinatorial proof of our generalization (1.2).

Clearly, (1.2) in the case $z=1$ gives Sun's identity (1.1), and (1.3) in the case $z=1$ yields the following equivalent form of (1.1).

Corollary 1.2. *Let* $m\in\mathbb{N}$. *Then*

$$\begin{aligned}&(x+m+2)\sum_{n=0}^{m}(-1)^n\binom{x+y+n}{m-n}\binom{y+2n}{n}\\&=2\sum_{n=0}^{m}\binom{x+n+1}{m-n}(-4)^n+(x-m)\binom{x}{m}.\end{aligned}\tag{1.4}$$

Remark 1.2. (1.4) in the special case $x-m\in\mathbb{N}$ and $y=1$, was ever conjectured by Z. H. Sun.

Corollary 1.3. *For any* $m\in\mathbb{N}$ *we have*

$$\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{l+(m+1)z}{m-n}(1+z)^{n-l}(1-z)^{n+l}=(m+1)\binom{(m+1)z-1}{m}.$$

Proof. Just take $x=-(m+1)z$ in (1.2) and then replace z by $-z$.

2 Proof of Theorem 1.1

The starting point of our proof of Theorem 1.1 is the following known identity:

$$\sum_{n=0}^{\infty}\binom{\alpha+n\beta}{n}\left(\frac{x-1}{x^{\beta}}\right)^n=\frac{x^{\alpha+1}}{(1-\beta)x+\beta}.\tag{2.1}$$

It appeared as (9) of H. W. Gould [5], and dates back to an identity of Lambert (cf. (E.3.1) of [1]). Both (2.1) and Lambert's identity can be proved by Lagrange's inversion formula (see pp. 631–632 of [1]). In 2005 V. J. W. Guo and J. Zeng [6] applied (2.1) to deduce some combinatorial identities originally motivated by the enumeration of convex polyominoes.

Let $\mathbb{C}$ be the complex field. For a formal power series $f(t) \in \mathbb{C}[\![t]\!]$, the coefficient of t^n in $f(t)$ will be denoted by $[t^n]f(t)$.

Proof of Theorem 1.1. In the case $m = 0$, both (1.2) and (1.3) are trivial. Below we assume that m is a positive integer.

Putting $\alpha = y$, $\beta = z + 1$ and $x = 1/(1 + t)$ in (2.1), we find that

$$\sum_{n=0}^{\infty}\binom{y+n(z+1)}{n}(-t(1+t)^z)^n = \frac{(1+t)^{-y}}{1+t(z+1)}.$$

Thus

$$\begin{aligned}[t^m]\frac{(1+t)^x}{1+t(z+1)} =& [t^m]\sum_{n=0}^{\infty}\binom{y+n(z+1)}{n}(-t)^n(1+t)^{nz+x+y}\\ =& \sum_{n=0}^{m}(-1)^n\binom{y+n(z+1)}{n}\binom{x+y+nz}{m-n}.\end{aligned}$$

(As pointed out by one of the referees, this identity can be reproved by a mixed use of Lagrange's inversion formula and the Riordan array method.) On the other hand,

$$\begin{aligned}&[t^m]\frac{(1+t)^x}{(1+t(z+1))^2} = [t^m]\frac{(1+t)^x}{1+t(z+1)(t(z+1)+2)}\\ =&[t^m](1+t)^x\sum_{n=0}^{\infty}(-t(z+1)\,(t(z+1)+2))^n\\ =&\sum_{n=0}^{m}(-1)^n(z+1)^n[t^{m-n}](1+t)^x\,(t(z+1)+2)^n\\ =&\sum_{n=0}^{m}(-1)^n(z+1)^n[t^{m-n}](1+t)^x\,((z+1)(1+t)+1-z)^n\\ =&\sum_{n=0}^{m}(-1)^n(z+1)^n[t^{m-n}](1+t)^x\sum_{l=0}^{n}\binom{n}{l}(z+1)^l(1+t)^l(1-z)^{n-l}\\ =&\sum_{n=0}^{m}(-1)^n(z+1)^n\sum_{l=0}^{n}\binom{n}{l}\binom{x+l}{m-n}(1+z)^l(1-z)^{n-l}\\ =&\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}.\end{aligned}$$

Since

$$\begin{aligned}[t^m]\frac{(1+t)^x}{1+t(z+1)} =&[t^m]\frac{(1+t)^x}{(1+t(z+1))^2}((1+t)(z+1)-z)\\ =&(z+1)[t^m]\frac{(1+t)^{x+1}}{(1+t(z+1))^2} - z[t^m]\frac{(1+t)^x}{(1+t(z+1))^2},\end{aligned}$$

by the above we have

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\
=&(z+1)\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+1+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}\\
&-z\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}.
\end{aligned}$$

From this we immediately see that (1.2) and (1.3) are equivalent.

Observe that

$$\begin{aligned}
m[t^m]\frac{(1+t)^x}{1+t(z+1)} =&[t^{m-1}]\frac{\partial}{\partial t}\left(\frac{(1+t)^x}{1+t(z+1)}\right)\\
=&[t^{m-1}]\left(\frac{-(z+1)(1+t)^x}{(1+t(z+1))^2}+\frac{x(1+t)^{x-1}}{1+t(z+1)}\right)\\
=&[t^m]\left(-\frac{t(z+1)(1+t)^x}{(1+t(z+1))^2}+\frac{xt(1+t)^{x-1}}{1+t(z+1)}\right)\\
=&[t^m]\left(\frac{(1+t)^x}{(1+t(z+1))^2}+\frac{xt(1+t)^{x-1}-(1+t)^x}{1+t(z+1)}\right)
\end{aligned}$$

and hence

$$(m+1)[t^m]\frac{(1+t)^x}{1+t(z+1)}-[t^m]\frac{(1+t)^x}{\big(1+t(z+1)\big)^2}=[t^m]\frac{xt(1+t)^{x-1}}{1+t(z+1)}.$$

It follows that

$$\begin{aligned}
&\big(x+(m+1)z\big)[t^m]\frac{(1+t)^x}{1+t(z+1)}-z[t^m]\frac{(1+t)^x}{\big(1+t(z+1)\big)^2}\\
=&x[t^m]\frac{(1+t)^x}{1+t(z+1)}+z[t^m]\frac{xt(1+t)^{x-1}}{1+t(z+1)}=x[t^m]\frac{(1+t)^{x-1}(1+t+zt)}{1+t(z+1)}\\
=&x[t^m](1+t)^{x-1}=x\binom{x-1}{m}=(x-m)\binom{x}{m}.
\end{aligned}$$

This, together with the previous arguments, yields the identity (1.2).

The proof of Theorem 1.1 is now complete.

Acknowledgments. The first author is supported by a Key Program (Grant No. 10331020) of the National Natural Science Foundation in China. Both authors thank the referees for helpful comments.

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