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A SHARP RESULT ON m-COVERS

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Abstract. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be a finite system of arithmetic sequences which forms an m-cover of $\mathbb{Z}$ (i.e., every integer belongs at least to m members of A). In this paper we show the following sharp result: For any positive integers $m_1, \ldots, m_k$ and $\theta \in [0, 1)$, if there is $I \subseteq \{1, \ldots, k\}$ such that the fractional part of $\sum_{s\in I} m_s/n_s$ is θ, then there are at least 2^m such subsets of $\{1, \ldots, k\}$. This extends an earlier result of M. Z. Zhang and an extension by Z. W. Sun. Also, we generalize the above result to m-covers of the integral ring of any algebraic number field with a power integral basis.

1. Introduction

For an integer a and a positive integer n, we simply let $a(n)$ represent the set $a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod n\}$. Following Sun [S95, S96] we call a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of such sets an *m-cover* of $\mathbb{Z}$ (where $m \in \{1, 2, 3, \ldots\}$) if every integer lies in at least m members of (1.1). We use the term *cover* (or covering system) instead of 1-cover. For problems and results in this area, the reader may consult [G04, pp. 383–390], [PS] and [S05]. P. Erdős [E97] once said: *"Perhaps my favorite problem of all concerns covering systems."*

Example 1.1. For each integer $m \geqslant 1$, there is an m-cover of $\mathbb{Z}$ which is not the union of two covers of $\mathbb{Z}$. To wit, we let $p_1, \ldots, p_r$ be distinct primes with $r \geqslant 2m - 1$, and set $N = p_1 \cdots p_r$. Clearly $A_* = \{\prod_{s\in I} p_s(N)\}_{I\subseteq\{1,\ldots,r\},\,|I|\geqslant m}$ does not cover any integer relatively prime to N. Let $a_1, \ldots, a_n$ be the list of those integers in $\{0, 1, \ldots, N-1\}$ not covered by A_* with each occurring exactly m times. If $x \in \mathbb{Z}$ is covered by A_*, then $x \in \bigcap_{s\in I} 0(p_s)$ for some $I \subseteq \{1, \ldots, r\}$ with $|I| \geqslant m$. Therefore

$$A_0 = \{0(p_1), \ldots, 0(p_r), a_1(N), \ldots, a_n(N)\}$$

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forms an m-cover of $\mathbb{Z}$. Suppose that $I_1 \cup I_2 = \{1, \dots, r\}$, $J_1 \cup J_2 = \{1, \dots, n\}$ and $I_1 \cap I_2 = J_1 \cap J_2 = \emptyset$. For $i = 1, 2$ let A_i be the system consisting of those $0(p_s)$ with $s \in I_i$ and those $a_t(N)$ with $t \in J_i$. We claim that A_1 or A_2 is not a cover of $\mathbb{Z}$. Without loss of generality, let us assume that $|I_1| \leqslant |I_2|$. Since $2|I_2| \geqslant |I_1| + |I_2| > 2(m-1)$, we have $|I_2| \geqslant m$ and hence $\prod_{s \in I_2} p_s$ is covered by A_*. Therefore $\prod_{s \in I_2} p_s \notin \bigcup_{t=1}^n a_t(N)$. Clearly $\prod_{s \in I_2} p_s$ is not covered by $\{0(p_s)\}_{s \in I_1}$ either. Thus A_1 does not form a cover of $\mathbb{Z}$.

By means of the Riemann zeta function, in 1989 M. Z. Zhang [Z89] proved that if (1.1) forms a cover of $\mathbb{Z}$ then $\sum_{s \in I} 1/n_s$ is a positive integer for some $I \subseteq \{1, \dots, k\}$.

Let $m_1, \dots, m_k$ be any positive integers. If (1.1) is a cover of $\mathbb{Z}$, then $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s=1}^k$ is also a cover of $\mathbb{Z}$ and hence Theorem 2 of [S95] indicates that for any $J \subseteq \{1, \dots, k\}$ there is an $I \subseteq \{1, \dots, k\}$ with $I \neq J$ such that $\{\sum_{s \in I} m_s/n_s\} = \{\sum_{s \in J} m_s/n_s\}$, where $\{\alpha\}$ denotes the fractional part of a real number α. When $J = \emptyset$ and $m_1 = \cdots = m_k = 1$, this yields Zhang's result. In 1999 Z. W. Sun [S99] proved further that if (1.1) forms an m-cover of $\mathbb{Z}$ then for any $J \subseteq \{1, \dots, k\}$ we have

$$\left| \left\{ I \subseteq \{1, \dots, k\} : I \neq J \text{ and } \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} = \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} \right\} \right| \geqslant m.$$

In this paper we will show the following sharp result.

Theorem 1.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$, and let $m_1, \dots, m_k$ be any integers. Then for any $0 \leqslant \theta < 1$ the set*

$$I_A(\theta) = \left\{ I \subseteq \{1, \dots, k\} : \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} = \theta \right\} \tag{1.2}$$

has at least 2^m elements if it is nonempty.

Remark 1.1. Clearly m copies of $0(1)$ form an m-cover of $\mathbb{Z}$. This shows that the lower bound in Theorem 1.1 is best possible.

Corollary 1.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$, and let $m_1, \dots, m_k$ be any integers. Then $|S(A)| \leqslant 2^{k-m}$ where*

$$S(A) = \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq \{1, \dots, k\} \right\}. \tag{1.3}$$

Proof. As $|I_A(\theta)| \geqslant 2^m$ for all $\theta \in S(A)$, we have

$$|S(A)| 2^m \leqslant |\{I : I \subseteq \{1, \dots, k\}\}| = 2^k$$

and hence $|S(A)| \leqslant 2^{k-m}$. $\square$

Remark 1.2. Sun [S95, S96] showed that if $m_1, \dots, m_k$ are relatively prime to $n_1, \dots, n_k$ respectively then (1.1) forms an m-cover of $\mathbb{Z}$ whenever it covers $|S(A)|$ consecutive integers at least m times.

Corollary 1.2. *Suppose that* (1.1) *forms an m-cover of $\mathbb{Z}$ but $\{a_s(n_s)\}_{s=1}^{k-1}$ does not. If the covering function $w_A(x)=|\{1\leqslant s\leqslant k:\ x\in a_s(n_s)\}|$ is periodic modulo n_k, then for any $r=0,\ldots,n_k-1$ we have*

$$\left|\left\{I\subseteq\{1,\ldots,k-1\}:\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{n_k}\right\}\right|\geqslant 2^{m-1}. \tag{1.4}$$

Proof. By Theorem 1 of Sun [S06],

$$\left|\left\{\left\lfloor\sum_{s\in I}\frac{1}{n_s}\right\rfloor:\ I\subseteq\{1,\ldots,k-1\}\text{ and }\left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{n_k}\right\}\right|\geqslant m.$$

In particular, $\{\sum_{s\in I}1/n_s\}=r/n_k$ for some $I\subseteq\{1,\ldots,k-1\}$, and hence (1.4) holds in the case $m=1$. For $A_k=\{a_s(n_s)\}_{s=1}^{k-1}$, clearly $w_{A_k}(x)\geqslant m-1$ for all $x\in\mathbb{Z}$. In the case $m>1$, we obtain (1.4) by applying Theorem 1.1 to A_k with $m_1=\cdots=m_{k-1}=1$ and $\theta=r/n_k$. □

Remark 1.3. When n_k is divisible by all the moduli $n_1,\ldots,n_k$, Corollary 1.2 was stated by the second author in [S03, Theorem 2.5]. When $w_A(x)=m$ for all $x\in\mathbb{Z}$, the following result stronger than (1.4) was proved in [S97]:

$$\left|\left\{I\subseteq\{1,\ldots,k-1\}:\ \sum_{s\in I}\frac{1}{n_s}=n+\frac{r}{n_k}\right\}\right|\geqslant\binom{m-1}{n}$$

for every $n=0,\ldots,m-1$.

For an algebraic number field K, let O_K be the ring of algebraic integers in K. For $\alpha,\beta\in O_K$, we set

$$\alpha+\beta O_K=\{\alpha+\beta\omega:\ \omega\in O_K\}$$

and call it a residue class in O_K. For a finite system

$$\mathcal{A}=\{\alpha_s+\beta_s O_K\}_{s=1}^{k} \tag{1.5}$$

of such residue classes, if $|\{1\leqslant s\leqslant k:\ x\in\alpha_s+\beta_s O_K\}|\geqslant m$ for all $x\in O_K$ (where $m\in\{1,2,3,\ldots\}$), then we call $\mathcal{A}$ an m-cover of O_K. Covers of the ring $\mathbb{Z}[\sqrt{-2}]=O_{\mathbb{Q}(\sqrt{-2})}$ were investigated by J. H. Jordan [J68].

An algebraic number field K of degree n is said to have a *power integral basis* if there is $\gamma\in O_K$ such that $1,\gamma,\ldots,\gamma^{n-1}$ form a basis of O_K over $\mathbb{Z}$. It is well known that all quadratic fields and cyclotomic fields have power integral bases.

Here is a generalization of Theorem 1.1.

Theorem 1.2. *Let K be an algebraic number field with a power integral basis. Suppose that* (1.5) *forms an m-cover of O_K, and let $\omega_1, \ldots, \omega_k \in O_K$. Then, for any $\mu \in K$, the set*

$$\left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{\omega_s}{\beta_s} \in \mu + O_K \right\}$$

is empty or it has at least 2^m elements.

Remark 1.4. We conjecture that the requirement in Theorem 1.2 that K has a power integral basis can be cancelled.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$. Let $m_1, \ldots, m_k \in \mathbb{Z}$ and define $S(A)$ as in* (1.3)*. Then, for any given $\theta \in S(A)$, there exists $t \in \{1, \ldots, k\}$ such that both θ and $\{\theta - m_t/n_t\}$ lie in $S(A_t)$, where $A_t = \{a_s(n_s)\}_{1 \leqslant s \leqslant k,\ s \neq t}$.*

Proof. Choose a maximal $J \subseteq \{1, \ldots, k\}$ such that $\{\sum_{s \in J} m_s/n_s\} = \theta$. As (1.1) is a cover of $\mathbb{Z}$, by [S95, Theorem 2] or [S99, Theorem 1(i)] there exists an $I \subseteq \{1, \ldots, k\}$ for which $I \neq J$ and $\{\sum_{s \in I} m_s/n_s\} = \theta$. Note that $J \not\subseteq I$ and hence $t \in J \setminus I$ for some $1 \leqslant t \leqslant k$. Clearly $\theta = \{\sum_{s \in I} m_s/n_s\} \in S(A_t)$ and also $\{\theta - m_t/n_t\} = \{\sum_{s \in J \setminus \{t\}} m_s/n_s\} \in S(A_t)$. This concludes the proof. □

Proof of Theorem 1.1. We use induction on m.

The $m = 1$ case, as mentioned above, has been handled in [S95, S99].

Now let $m > 1$ and assume that Theorem 1.1 holds for smaller positive integers. Let $\theta \in S(A)$. In light of Lemma 2.1, there is a $t \in \{1, \ldots, k\}$ such that both θ and $\theta' = \{\theta - m_t/n_t\}$ lie in $S(A_t)$. As A_t forms a $(m-1)$-cover of $\mathbb{Z}$, by the induction hypothesis we have $|I_{A_t}(\theta)| \geqslant 2^{m-1}$ and $|I_{A_t}(\theta')| \geqslant 2^{m-1}$. Observe that

$$I_A(\theta) = I_{A_t}(\theta) \cup \{I \cup \{t\} : I \in I_{A_t}(\theta')\}.$$

Therefore

$$|I_A(\theta)| = |I_{A_t}(\theta)| + |I_{A_t}(\theta')| \geqslant 2^{m-1} + 2^{m-1} = 2^m.$$

We are done. □

3. Proof of Theorem 1.2

At first we give a lemma on algebraic number fields with power integral bases.

Lemma 3.1. *Let K be an algebraic number field with a power integral basis $1, \gamma, \ldots, \gamma^{n-1}$. For any $\mu = \sum_{r=0}^{n-1} \mu_r \gamma^r \in K$ with $\mu_0, \ldots, \mu_{n-1} \in \mathbb{Q}$, we have*

$$\mu \in O_K \iff \psi(\mu), \psi(\mu\gamma), \ldots, \psi(\mu\gamma^{n-1}) \in \mathbb{Z},$$

where $\psi(\mu)$ denotes the last coordinate μ_{n-1}.

Proof. If $\mu \in O_K$, then $\mu, \mu\gamma, \ldots, \mu\gamma^{n-1} \in O_K$ and hence $\psi(\mu\gamma^j) \in \mathbb{Z}$ for every $j = 0, \ldots, n-1$.

Now assume that $\psi(\mu\gamma^j) \in \mathbb{Z}$ for all $j = 0, \ldots, n-1$. We want to show that $\mu \in O_K$ (i.e., $\mu_0, \ldots, \mu_{n-1} \in \mathbb{Z}$). Clearly $\mu_{n-1} = \psi(\mu\gamma^0) \in \mathbb{Z}$. If $0 \leqslant r < n-1$ and $\mu_{r+1}, \ldots, \mu_{n-1} \in \mathbb{Z}$, then

$$\begin{aligned}\mu_r =& \psi(\mu_0\gamma^{n-1-r} + \mu_1\gamma^{n-r} + \cdots + \mu_r\gamma^{n-1})\\ =& \psi(\mu\gamma^{n-1-r}) - \psi(\mu_{r+1}\gamma^n + \cdots + \mu_{n-1}\gamma^{2n-2-r})\end{aligned}$$

and hence $\mu_r \in \mathbb{Z}$ since $\mu_{r+1}\gamma^n + \cdots + \mu_{n-1}\gamma^{2n-2-r} \in O_K$. So, by induction, $\mu_r \in \mathbb{Z}$ for all $r = 0, \ldots, n-1$. We are done. □

Proof of Theorem 1.2. In the spirit of the proof of Theorem 1.1, it suffices to handle the case $m = 1$. That is, we should prove that for any $J \subseteq \{1, \ldots, k\}$ there is $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ such that $\sum_{s\in I} \omega_s/\beta_s - \sum_{s\in J} \omega_s/\beta_s \in O_K$.

Let $\{1, \gamma, \ldots, \gamma^{n-1}\}$ be a power integral basis of K, and define ψ as in Lemma 3.1.

Let $x_0, \ldots, x_{n-1} \in \mathbb{Z}$ and $x = \sum_{r=0}^{n-1} x_r\gamma^r$. Since $-x \in O_K$ is covered by $\mathcal{A} = \{\alpha_s + \beta_s O_K\}_{s=1}^k$, we have

$$\begin{aligned}0 =& \prod_{s=1}^{k}\left(1 - e^{2\pi i\psi(\omega_s(x+\alpha_s)/\beta_s)}\right) = \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} \prod_{s\in I} e^{2\pi i\psi(\omega_s(x+\alpha_s)/\beta_s)}\\ =& \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} \prod_{s\in I} e^{2\pi i(\psi(\omega_s\alpha_s/\beta_s)+\sum_{r=0}^{n-1} x_r\psi(\omega_s\gamma^r/\beta_s))}\\ =& \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} e^{2\pi i\psi(\sum_{s\in I}\omega_s\alpha_s/\beta_s)} \prod_{r=0}^{n-1} e^{2\pi i x_r\psi(\sum_{s\in I}\omega_s\gamma^r/\beta_s)}\\ =& \sum_{\theta_0\in S_0} e^{2\pi i x_0\theta_0} \sum_{\theta_1\in S_1} e^{2\pi i x_1\theta_1} \cdots \sum_{\theta_{n-1}\in S_{n-1}} e^{2\pi i x_{n-1}\theta_{n-1}} f(\theta_0, \ldots, \theta_{n-1}),\end{aligned}$$

where

$$S_r = \left\{\left\{\psi\left(\sum_{s\in I}\frac{\omega_s\gamma^r}{\beta_s}\right)\right\} : I \subseteq \{1, \ldots, k\}\right\}$$

and

$$f(\theta_0,\ldots,\theta_{n-1})=\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\psi(\sum_{s\in I}\omega_s\gamma^r/\beta_s)\}=\theta_r\\ \text{for all } r=0,\ldots,n-1}}(-1)^{|I|}e^{2\pi i\psi(\sum_{s\in I}\omega_s\alpha_s/\beta_s)}.$$

For each $r=0,\ldots,n-1$, if $\sum_{\theta_r\in S_r}e^{2\pi i x_r\theta_r}F(\theta_r)=0$ for all $x_r=0,\ldots,|S_r|-1$, then $F(\theta_r)=0$ for every $\theta_r\in S_r$, because the Vandermonde determinant $\det(e^{2\pi i x_r\theta_r})_{0\leqslant x_r<|S_r|,\,\theta_r\in S_r}$ does not vanish. So, by the above, we have $f(\theta_0,\ldots,\theta_{n-1})=0$ for all $\theta_0\in S_0,\ldots,\theta_{n-1}\in S_{n-1}$.

Now suppose that $\mu\in K$ and $\sum_{s\in J}\omega_s/\beta_s\in\mu+O_K$ for a unique subset J of $\{1,\ldots,k\}$. We want to deduce a contradiction.

Set $\theta_r=\{\psi(\mu\gamma^r)\}$ for $r=0,\ldots,n-1$. For any $I\subseteq\{1,\ldots,k\}$ we have

$$\begin{aligned}&\left\{\psi\left(\sum_{s\in I}\frac{\omega_s\gamma^r}{\beta_s}\right)\right\}=\theta_r\ \text{ for all } r=0,\ldots,n-1\\ \iff&\psi\left(\left(\sum_{s\in I}\frac{\omega_s}{\beta_s}-\mu\right)\gamma^r\right)\in\mathbb{Z}\ \text{ for all } r=0,\ldots,n-1\\ \iff&\sum_{s\in I}\frac{\omega_s}{\beta_s}\in\mu+O_K\qquad\text{(by Lemma 3.1)}\\ \iff&I=J.\end{aligned}$$

Thus the expression of $f(\theta_0,\ldots,\theta_{n-1})$ only contains one summand, and therefore

$$0=f(\theta_0,\ldots,\theta_{n-1})=(-1)^{|J|}e^{2\pi i\psi(\sum_{s\in J}\omega_s\alpha_s/\beta_s)}\neq0$$

which is a contradiction.

The proof of Theorem 1.2 is now complete. □

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