

A CHARACTERIZATION OF COVERING EQUIVALENCE

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ABSTRACT. Let $A = \{a_s \pmod{n_s}\}_{s=1}^k$ and $B = \{b_t \pmod{m_t}\}_{t=1}^l$ be two systems of residue classes. If $|\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|$ and $|\{1 \leq t \leq l : x \equiv b_t \pmod{m_t}\}|$ are equal for all $x \in \mathbb{Z}$, then A and B are said to be covering equivalent. In this paper we characterize the covering equivalence in a simple and new way. Using the characterization we partially confirm a conjecture of R. L. Graham and K. O’Byrant.

1. INTRODUCTION

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $a \in \{0, \dots, n-1\}$, we simply use $a(n)$ to denote the residue class $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \quad (0 \leq a_s < n_s) \quad (1.1)$$

of residue classes, the n_1, \dots, n_k are called its moduli and its *covering function* $w_A : \mathbb{Z} \rightarrow \mathbb{N} = \{0, 1, \dots\}$ is given by

$$w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}|. \quad (1.2)$$

(The covering function w_\emptyset of an empty system is regarded as the zero function.) The periodic function $w_A(x)$ has many surprising properties (cf. [S03a], [S04] and [S05a]).

Let m be a positive integer. If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then (1.1) is said to be an *exact m -cover* of \mathbb{Z} as in [S95] and [S96]. Recently Z. W. Sun (cf. [S04] and [S05b]) showed that (1.1) forms an exact m -cover of \mathbb{Z} if it covers $|S(n_1, \dots, n_k)|$ consecutive integers exactly m times, where

$$S(n_1, \dots, n_k) = \left\{ \frac{r}{n_s} : r = 0, \dots, n_s - 1; s = 1, \dots, k \right\}. \quad (1.3)$$

For problems and results on covers of \mathbb{Z} by residue classes, the reader is referred to [FFKPY], [G04] and [S03b].

For two finite systems $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$, Sun [S89] called A and B *covering equivalent* (in short, $A \sim B$) if they have the same

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covering function (i.e., $w_A = w_B$). Thus (1.1) is an exact m -cover of \mathbb{Z} if and only if (1.1) is covering equivalent to the system consisting of m copies of $0(1)$.

In [S01] and [S02] Sun characterized the covering equivalence by various systems of equalities. In this paper we present a simple characterization involving roots of unity. Namely, we have the following result.

Theorem 1.1. *Let $A = \{a_s(n_s)\}_{s=1}^k$ ($0 \leq a_s < n_s$) and $B = \{b_t(m_t)\}_{t=1}^l$ ($0 \leq b_t < m_t$) be two finite systems of residue classes. Let p be a prime greater than $|S(n_1, \dots, n_k, m_1, \dots, m_l)|$, and let ζ_p be a primitive p th root of unity. Then A and B are covering equivalent if and only if*

$$\sum_{s=1}^k \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = \sum_{t=1}^l \frac{\zeta_p^{b_t}}{1 - \zeta_p^{m_t}}. \quad (1.4)$$

Corollary 1.1. *(1.1) forms an exact m -cover of \mathbb{Z} if and only if*

$$\sum_{s=1}^k \frac{e^{2\pi i a_s/p}}{1 - e^{2\pi i n_s/p}} = \frac{m}{1 - e^{2\pi i/p}}, \quad (1.5)$$

where p is any fixed prime greater than $|S(n_1, \dots, n_k)|$.

Proof. Simply apply Theorem 1.1 with B consisting m copies of $0(1)$. \square

Remark 1.1. In 1975 Š. Znáám [Z75a] used the transcendence of e to prove that (1.1) is a disjoint cover (i.e., exact 1-cover) of \mathbb{Z} if and only if

$$\sum_{s=1}^k \frac{e^{a_s}}{1 - e^{n_s}} = \frac{1}{1 - e}.$$

Corollary 1.2. *Suppose that for nonempty system (1.1) we have*

$$\sum_{s=1}^k \frac{e^{2\pi i a_s/p}}{1 - e^{2\pi i n_s/p}} = 0$$

where p is a prime. Then

$$n_1 + \dots + n_k - k + 1 \geq |S(n_1, \dots, n_k)| \geq p. \quad (1.6)$$

Proof. Clearly $|S(n_1, \dots, n_k)| \leq n_1 + \dots + n_k - k + 1$. Since we don't have $A \sim \emptyset$, applying Theorem 1.1 with $B = \emptyset$ we find that $|S(n_1, \dots, n_k)|$ cannot be smaller than p . This concludes the proof. \square

Corollary 1.2 partially confirms the following conjecture arising from the study of Fraenkel's conjecture on disjoint covers of \mathbb{N} by Beatty sequences.

Graham–O’Bryant Conjecture ([GO]). *Let n_1, \dots, n_k be distinct positive integers less than and relatively prime to $q \in \mathbb{Z}^+$. If $a_1, \dots, a_k \in \mathbb{Z}$ and*

$$\sum_{s=1}^k \frac{e^{2\pi i a_s / q}}{1 - e^{2\pi i n_s / q}} = 0,$$

then we must have $\sum_{s=1}^k n_s \geq q$.

The following example shows that we cannot replace the prime p in Corollary 1.2 or Theorem 1.1 by a composite number.

Example 1.1. Let $q > 1$ be an integer and let p be a prime divisor of q . Then, for any $n = 1, \dots, q - 1$, we have

$$\sum_{s=0}^{p-1} \frac{e^{2\pi i (sq/p)/q}}{1 - e^{2\pi i n/q}} = \frac{\sum_{s=0}^{p-1} e^{2\pi i s/p}}{1 - e^{2\pi i n/q}} = 0$$

but $|S(n, \dots, n)| = n < q$. Thus the conditions $0 \leq a_s < n_s$ ($s = 1, \dots, k$) in Corollary 1.2 cannot be cancelled. If q is composite, then there are $q/p - 1 > 0$ integers in the interval $((p - 1)q/p, q - 1]$. So we cannot substitute a composite number for the prime p in Corollary 1.2.

Corollary 1.3. *Let $A = \{a_s(n_s)\}_{s=1}^k$ ($0 \leq a_s < n_s$) and $B = \{b_t(m_t)\}_{t=1}^l$ ($0 \leq b_t < m_t$) both have distinct moduli. Let p be a prime greater than $|S(n_1, \dots, n_k, m_1, \dots, m_l)|$, and let ζ_p be a primitive p th root of unity. Then A and B are identical if and only if (1.4) holds.*

Proof. By a result of Znám [S75b], A and B are identical if they have the same covering function. Combining this with Theorem 1.1 we immediately get the desired result. \square

Observe that $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ are covering equivalent if and only if

$$\sum_{\substack{s=1 \\ x \in a_s(n_s)}}^k 1 + \sum_{\substack{t=1 \\ x \in b_t(m_t)}}^l (-1) = 0 \quad \text{for every } x \in \mathbb{Z}.$$

Thus Theorem 1.1 has the following equivalent form which will be proved in the next section.

Theorem 1.2. *Let $\mathcal{A} = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^k$ where $\lambda_s, a_s, n_s \in \mathbb{Z}$ and $0 \leq a_s < n_s$. Let $p > |S(n_1, \dots, n_k)|$ be a prime, and let ζ_p be any primitive p th root of unity. Then $\mathcal{A} \sim \emptyset$ (i.e., $w_{\mathcal{A}}(x) = \sum_{1 \leq s \leq k, x \in a_s(n_s)} \lambda_s = 0$ for all $x \in \mathbb{Z}$) if and only if*

$$\sum_{s=1}^k \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0. \tag{1.7}$$

2. PROOF OF THEOREM 1.2

Let $S = S(n_1, \dots, n_k)$. As $p > |S| \geq \max\{n_1, \dots, n_k\}$, there is a common multiple $N \in \mathbb{Z}^+$ of the moduli n_1, \dots, n_k such that $N \equiv 1 \pmod{p}$. Just as in [S05a], we have

$$\begin{aligned} \sum_{r=0}^{N-1} w_{\mathcal{A}}(r) z^r &= \sum_{r=0}^{N-1} \sum_{\substack{1 \leq s \leq k \\ n_s | a_s - r}} \lambda_s z^r = \sum_{s=1}^k \lambda_s \sum_{\substack{0 \leq r < N \\ r \in a_s(n_s)}} z^r \\ &= \sum_{s=1}^k \lambda_s z^{a_s} \sum_{0 \leq q < N/n_s} (z^{n_s})^q \\ &= N \sum_{\substack{1 \leq s \leq k \\ z^{n_s} = 1}} \frac{\lambda_s}{n_s} z^{a_s} + (1 - z^N) \sum_{\substack{1 \leq s \leq k \\ z^{n_s} \neq 1}} \lambda_s \frac{z^{a_s}}{1 - z^{n_s}}. \end{aligned}$$

Thus

$$\sum_{r=0}^{N-1} w_{\mathcal{A}}(r) \zeta_p^r = (1 - \zeta_p^N) \sum_{s=1}^k \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}}.$$

It follows that

$$\sum_{s=1}^k \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0 \iff \sum_{l=0}^{p-1} c_l \zeta_p^l = 0, \quad (2.1)$$

where

$$c_l = \sum_{\substack{x=0 \\ x \in l(p)}}^{N-1} w_{\mathcal{A}}(x) \in \mathbb{Z}.$$

If $w_{\mathcal{A}}(x) = 0$ for all $x \in \mathbb{Z}$, then (1.7) holds by the above.

Below we assume (1.7). Then $\sum_{l=0}^{p-1} c_l \zeta_p^l = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r) \zeta_p^r = 0$. In the case $N = 1$, it follows that $w_{\mathcal{A}}(x) = w_{\mathcal{A}}(0) = 0$ for all $x \in \mathbb{Z}$.

Now suppose $N > 1$. Clearly $N > p$ as $N \equiv 1 \pmod{p}$. Since $1 + x + \dots + x^{p-1} = (x^p - 1)/(x - 1)$ is the minimal polynomial of ζ_p over the field of rational numbers, we must have $c_0 = c_1 = \dots = c_{p-1}$. (See also M. Newman [N71].) Observe that if $x \in \mathbb{Z}$ then

$$w_{\mathcal{A}}(x) = \sum_{s=1}^k \frac{\lambda_s}{n_s} \sum_{r=0}^{n_s-1} e^{2\pi i \frac{a_s - x}{n_s} r} = \sum_{\alpha \in S} e^{-2\pi i \alpha x} \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s}. \quad (2.2)$$

(This trick appeared in [S91] and [S04].) Since $|S| < p$, for each $l = 0, \dots, |S|$ we have

$$\begin{aligned} c_l &= \sum_{\substack{x=0 \\ x \in l(p)}}^{N-1} w_{\mathcal{A}}(x) = \sum_{\alpha \in S} \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \sum_{\substack{x=0 \\ x \in l(p)}}^{N-1} e^{-2\pi i \alpha x} \\ &= \sum_{\alpha \in S} e^{-2\pi i \alpha l} \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \sum_{j=0}^{\lfloor (N-1-l)/p \rfloor} e^{-2\pi i \alpha p j}, \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. If $l \in \{1, \dots, |S|\}$ then

$$\left\lfloor \frac{N-1-l}{p} \right\rfloor = \frac{N-1}{p} + \left\lfloor \frac{-l}{p} \right\rfloor = \frac{N-1}{p} - 1;$$

if $\alpha \in S \setminus \{0\}$ then

$$C(\alpha) := \sum_{j=0}^{(N-1)/p-1} e^{-2\pi i \alpha p j} = \frac{1 - (e^{-2\pi i \alpha p})^{(N-1)/p}}{1 - e^{-2\pi i \alpha p}} = \frac{1 - e^{2\pi i \alpha}}{1 - e^{-2\pi i \alpha p}} \neq 0.$$

Let $c = c_0 = \dots = c_{p-1}$. By the above,

$$\sum_{\alpha \in S} e^{-2\pi i \alpha j} f(\alpha) = c$$

for every $j = 0, \dots, |S| - 1$, where

$$f(0) = \frac{N-1}{p} \sum_{s=1}^k \frac{\lambda_s}{n_s}$$

and

$$f(\alpha) = e^{-2\pi i \alpha} C(\alpha) \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \quad \text{for } \alpha \in S \setminus \{0\}.$$

Let $\alpha_0 = 0, \alpha_1, \dots, \alpha_{|S|-1}$ be all the distinct elements of S . Now that

$$\sum_{t=0}^{|S|-1} e^{-2\pi i \alpha_t j} f(\alpha_t) = c \quad \text{for each } j = 0, \dots, |S| - 1,$$

by Cramer's rule $D_t = Df(\alpha_t)$ vanishes for every $t = 1, \dots, |S| - 1$, where $D = \det((e^{-2\pi i \alpha_t})^j)_{0 \leq j, t < |S|}$ is of Vandermonde's type and hence nonzero. Therefore

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \quad \text{for all } \alpha \in S \setminus \{0\}$$

and hence $w_{\mathcal{A}}(x) = \sum_{s=1}^k \lambda_s/n_s$ for all $x \in \mathbb{Z}$ by (2.2). It follows that

$$0 = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r) \zeta_p^r = \sum_{s=1}^k \frac{\lambda_s}{n_s} (1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{N-1}) = \sum_{s=1}^k \frac{\lambda_s}{n_s} \cdot \frac{1 - \zeta_p^N}{1 - \zeta_p}.$$

So $\sum_{s=1}^k \lambda_s/n_s = 0$ and hence $\mathcal{A} \sim \emptyset$. We are done.

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