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Combinatorial Quantities and Arithmetic Means

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

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Abstract

In combinatorics there are some basic combinatorial quantities arising from enumeration problems, e.g., central trinomial coefficients T_n, Motzkin numbers M_n, Delannoy numbers D_n, little Schröder numbers s_n and large Schröder numbers S_n. Surprisingly, such combinatorial quantities have nice arithmetic properties. In this talk we introduce recent work of the speaker and his coauthor on weighted arithmetic means involving the above important combinatorial quantities. For example, Y.-P. Mu and the speaker showed that

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+5)T_k^2 \in \mathbb{Z} \quad \text{for all } n=1,2,3,\ldots$$

which was first conjectured by the speaker. The speaker proved that

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k M_k (-3)^{n-1-k} \in \mathbb{Z} \ \text{ and } \ \frac{1}{n}\sum_{k=0}^{n-1} D_k s_{k+1} \in \mathbb{Z}$$

for $n=1,2,3,\ldots$. We will also mention some open conjectures.

Part I. The period 2010–2015

Some easy facts

Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} 1 = 1 \in \mathbb{Z},$$

$$\frac{2}{n}\sum_{k=0}^{n-1} k = n - 1 \in \mathbb{Z},$$

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k + 1) = 1 \in \mathbb{Z}.$$

For $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$, their arithmetic mean is given by

$$\frac{a_0 + a_1 + \cdots + a_{n-1}}{n} = \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

Apéry numbers

In 1978 Apéry proved that $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ is irrational! Those *Apéry numbers*

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2$$

play important roles in Apéry's proof.

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1}A_k\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod{8}.\end{cases}$$

Remark. I [JNT, 2011] proved the mod p version of the conjectural congruence. The conjecture still remains open!

Arithmetic means involving Apéry numbers

Theorem. Let n be a positive integer.

(i) (Z. W. Sun [J. Number Theory 132(2012)]) We have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k \in \mathbb{Z}.$$

For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

(ii) (Conjectured by Z. W. Sun and proved by V.J.W. Guo and J. Zeng [JNT, 2012])

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \in \mathbb{Z}.$$

Franel numbers

In 1894 J. Franel introduced the Franel numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

and noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1)+2) f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

In 2008 D. Callan gave a combinatorial interpretation of the Franel numbers.

V. Strehl's Identity:

$$A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k.$$

Barrucand's Identity:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{where } g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Weighted arithmetic means for f_n and g_n

Z.-W. Sun [Adv. in Appl. Math. 51(2013), JNT 133(2013)] investigated congruences for Franel numbers. His conjecture that

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k \in \mathbb{Z} \quad \text{for all } n=1,2,3,\ldots$$

was confirmed by V.J.W. Guo [Integral Transforms. Spec. Funct. 24(2013)].

Theorem (Sun, Ramanujan J., in press)

$$\frac{1}{3n^2}\sum_{k=0}^{n-1}(4k+3)g_k = \sum_{k=0}^{n-1}\binom{n-1}{k}^2 C_k \in \mathbb{Z} \ \text{ for } n=1,2,3,\ldots.$$

Define

$$g_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}x^k \quad \text{for } n=0,1,2,\ldots.$$

Guo, Mao and Pan [arXiv:1511.04005] confirmed the speaker's conjecture that $\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x) \in \mathbb{Z}[x]$ for all $n=1,2,3\ldots$.

Central Delannoy numbers

For $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the Delannoy number

$$D_{m,n} := \sum_{k\in\mathbb{N}} \binom{m}{k}\binom{n}{k}2^k$$

in combinatorics counts lattice paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed The n-th central Delannoy number $D_n = D_{n,n}$ has another well-known expression:

$$D_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}.$$

Theorem (Z.-W. Sun [Sci. China Math. 57(2014), J. Number Theory 131(2011)]). For any odd prime p, we have

$$\sum_{k=1}^{p-1}\frac{D_k}{k} \equiv \frac{1-2^{p-1}}{p} \pmod{p}, \quad \sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p},$$

(where $(\frac{\cdot}{p})$ denotes the Legendre symbol).

Catalan numbers and large Schröder numbers

The Catalan numbers are given by

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1} \in \mathbb{Z} \quad (k = 0, 1, 2, \ldots).$$

In combinatorics, the (large) Schröder numbers are given by

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k \quad (n \in \mathbb{N}).$$

Both Catalan numbers and Schröder numbers have many combinatorial interpretations. For example, $S(n)$ is the number of lattice paths from the point $(0,0)$ to (n,n) with only allowed steps $(1,0)$, $(0,1)$ and $(1,1)$ which never rise above the line $y = x$.

Little Schröder numbers

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, the n-th little Schröder number is given by

$$s_n := \sum_{k=1}^{n} N(n,k) 2^{k-1}$$

with the Narayana number $N(n,k)$ defined by

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \in \mathbb{Z}.$$

Combinatorial Interpretation: s_n is the number of ways to insert parentheses into an expression of $n+1$ terms with two or more items within a parenthesis.

Relation to the Large Schröder Numbers:

$$S_n = 2s_n \text{ for all } n = 1, 2, 3, \dots.$$

Congruences involving Schröder numbers

Theorem (i) (Sun [J. Number Theory 131(2011)]) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{S_k}{6^k} \equiv 0 \pmod{p}.$$

(ii) (Sun [Acta Arith. 156(2012)]) If $p \equiv 1 \pmod 4$ is a prime and we write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$, then

$$S_{(p-1)/2} \equiv 2\left(\frac{2}{p}\right)\left(2x - \frac{p}{x}\right) \pmod{p^2}.$$

Conjecture (Sun [JNT 131(2011)]) Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} D_k S_k \equiv -2p \sum_{k=1}^{p-1} \frac{(-1)^k + 3}{k} \pmod{p^4}.$$

This challenging conjecture was recently proved by Ji-Cai Liu [J. Number Theory, in press. arXiv:1601.03938].

Central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Theorem (i) (Z.-W. Sun [Sci. China Math. 57(2014)]) For any odd prime p, we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

(ii) (H. Q. Cao and Sun [Colloq. Math. 139(2015)]). For any prime $p > 3$, we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

Conjecture on central trinomial coefficients

Conjecture (Z.-W. Sun [Sci. China Math. 57(2014)]) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k} \equiv \frac{3+(\frac{p}{3})}{2} - p\left(1+\left(\frac{p}{3}\right)\right) \quad (\text{mod } p^2),$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$.

Mod p^2 congruences for Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Z.-W. Sun [Sci. China Math. 57(2014)]). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Generating functions

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. H. S. Wilf observed that

$$\sum_{n=0}^{\infty} T_n(b, c)x^n = \frac{1}{\sqrt{1 - 2bx + dx^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b, c) = (2n+1)bT_n(b, c) - ndT_{n-1}(b, c) \ \ (n \in \mathbb{Z}^+).$$

By the Zeilberger algorithm we have

$$(n+3)M_{n+1}(b, c) = (2n+3)bM_n(b, c) - ndM_{n-1}(b, c) \ \ (n \in \mathbb{Z}^+),$$

and hence

$$2cx^2 \sum_{n=0}^{\infty} M_n(b, c)x^n = 1 - bx - \sqrt{1 - 2bx + dx^2}.$$

Relations between $T_n(b,c)$ and Legendre polynomials

For the Legendre polynomials $P_n(t)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(\frac{t-1}{2})^k$, it is known that

$$\sum_{n=0}^{\infty}P_n(t)x^n=\frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $d=b^2-4c\neq 0$ then

$$\sum_{n=0}^{\infty}T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n=\frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}}=\sum_{n=0}^{\infty}P_n\left(\frac{b}{\sqrt{d}}\right)x^n$$

and hence

$$T_n(b,c)=(\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1,x^2+x)=P_n(2x+1)\ \text{ for all } x\in\mathbb{Z};$$

in particular, $D_n=P_n(3)=T_n(3,2)$.

Congruences modulo n

Theorem (Sun [Sci. China Math. 57(2014)]). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.

(i) For any $n \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} \equiv 0 \pmod{n},$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c).$$

(ii) If $b^2 - 4c = 1$, then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c) = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k-1}{k-1}\left(\frac{b-1}{2}\right)^{k-1} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}^+$.

Congruences modulo n^2

Theorem (Sun [Sci. China Math. 57(2014)]). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k}.$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}.$$

Corollary. For each $n = 1, 2, 3, \ldots$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

The polynomials $D_n(x)$ $(n = 0, 1, 2, \ldots)$

Define

$$D_n(x) = P_n(2x+1) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k \ (n = 0, 1, 2, \ldots).$$

Note that $D_n(1)$ is the central Delannoy number D_n, and $D_n(x) = T_n(2x+1, x^2+x)$ with $(2x+1)^2 - 4(x^2+x) = 1$.

Conjecture (Sun [Sci. China Math. 57(2014)]). Let x be any integer. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3 \equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4 \equiv p \pmod{p^2}.$$

This was recently proved by Victor J. W. Guo [Integral Transforms Spec. Funct. 26(2015)].

Part II. Our work in 2016

Mu and Sun's telescoping approach

To study some challenging conjectures of Sun on congruences for

$$S_n = \sum_{k=0}^{n-1} \sum_{l=0}^{k} F(k,l) \quad (n \in \mathbb{Z}^+),$$

where $F(k,l)$ is a bivariate hypergeometric term of k and l, Yan-Ping Mu and Z.-W. Sun [arXiv:1601.03954] search for two hypergeometric terms $G_1(k,l)$ and $G_2(k,l)$ such that

$$F(k,l) = \Delta_k(G_1(k,l)) + \Delta_l(G_2(k,l)),$$

where

$$G_1(k,l) = R_1(k,l)F(k,l) \quad \text{and} \quad G_2(k,l) = R_2(k,l)F(k,l)$$

with $R_1(k,l)$ and $R_2(k,l)$ rational functions, and

$$\Delta_k(G_1(k,l)) = G_1(k+1,l) - G_1(k,l),$$
$$\Delta_l(G_2(k,l)) = G_2(k,l+1) - G_2(k,l).$$

The resulting functions $G_1(k,l)$ and $G_2(k,l)$ we obtain are essentially well defined for $0 \leqslant l \leqslant k \leqslant n-1$.

Mu and Sun's telescoping approach

Once we have $G_1(k,l)$ and $G_2(k,l)$ in hand, the sum S_n can be transformed to a single sum

$$S_n = \sum_{l=0}^{n-1} \big(G_1(n,l) - G_1(l,l)\big) + \sum_{k=0}^{n-1} \big(G_2(k,k+1) - G_2(k,0)\big)$$

We can use the `Maple` package *DoubleSum* given by Chen-Hou-Mu [J. Comput. Appl. Math. 196(2006)] or the `Mathematica` package *HolonomicFunctions* given by C. Koutschan [Math. Comput. Sci. 4(2010)], together with the package *DoubleSum* or the package *MultiSum* [Appl. Algebra Engrg. Comm. Comput. 13(2002)], to compute a suitable pair $(G_1(k,l), G_2(k,l))$.

Once we get a single sum for S_n, it would be be convenient to deduce Sun's conjectural congruences for S_n. Using this powerful method, Mu and Sun confirm several sophisticated open conjectures of Sun.

Congruences involving f_k

Theorem (Conjectured by Sun in 2011 and confirmed by Mu and Sun in 2016) For any integer $n > 1$, we have

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{n^2(n-1)}.$$

Moreover, for any odd prime p we have

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1) - 16p^3 q_p(2) \pmod{p^4},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

In the proof, we make use of Strehl's identity

$$f_k = \sum_{l=0}^{k}\binom{k}{l}^2\binom{2l}{k}.$$

Sketch of the proof

Consider

$$S_n := \sum_{k=0}^{n-1} \sum_{l=0}^{k} F(k,l),$$

where

$$F(k,l) = (9k^2+5k)(-1)^k \binom{k}{l}^2 \binom{2l}{k}.$$

.

Via certain math. package, we find that

$$F(k,l) = \Delta_k(G_1(k,l)) + \Delta_l(G_2(k,l)),$$

where

$$G_1(k,l) = (-1)^{k-1}k^2(3k+4l-3)\frac{1}{l+1}\binom{2l}{l}\binom{k-1}{l}\binom{l}{k-1-l},$$

$$G_2(k,l) = (-1)^{k-1}(2l-k)(3k-l-2)\binom{k}{l}^2\binom{2l}{k}.$$

Sketch of the proof

Noting that $G_1(l,l)=G_2(k,k+1)=G_2(k,0)=0$, we get

$$\begin{aligned}
S_n =& \sum_{l=0}^{n-1} G_1(n,l) \\
=&(-1)^{n-1}n^2\sum_{l=0}^{n-1}(3n+4l-3)\frac{1}{l+1}\binom{2l}{l}\binom{n-1}{l}\binom{l}{n-1-l} \\
=&(-1)^{n-1}3n^2(n-1)\sum_{l=0}^{n-1}\frac{1}{l+1}\binom{2l}{l}\binom{n-1}{l}\binom{l}{n-1-l} \\
&+(-1)^{n-1}4n^2(n-1)\sum_{l=1}^{n-1}\frac{1}{l+1}\binom{2l}{l}\binom{n-2}{l-1}\binom{l}{n-1-l}.
\end{aligned}$$

Since $C_l=\binom{2l}{l}/(l+1)\in\mathbb{Z}$, we derive that $n^2(n-1)\mid S_n$.

With more efforts, we can show for any odd prime p that

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1)-16p^3q_p(2) \pmod{p^4}.$$

On $F_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k$

Theorem (Conjectured by Sun in 2011, and proved by Mu and Sun in 2016). For $k = 0, 1, 2, \ldots$ define $F_k := \sum_{l=0}^{k} \binom{k}{l}^3 (-8)^l$. Then, for any positive integer n, the number

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+5)(-1)^k F_k$$

is always an odd integer.

A key step of the proof is the identity

$$\sum_{k=0}^{n-1} (6k+5)(-1)^k F_k$$
$$= -\frac{4}{3} \sum_{l=0}^{n-1} (12n^2 - 30nl + 21l^2 - 32n + 46l + 25) \binom{n}{l+1}^3 (-8)^l (-1)^n$$
$$- \frac{1}{3} \sum_{k=0}^{n-1} (12k^2 + 10k + 5)(-1)^k.$$

Arithmetic means involving $T_k(b, c^2)^2$

Theorem (Conjectured by Sun in 2010 and confirmed by Mu and Sun in 2016) Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

Corollary (Conjectured by Sun in 2010 and confirmed by Mu and Sun in 2016).

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n} \quad \text{for all } n \in \mathbb{Z}^+,$$

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2} \text{ for any prime } p.$$

Sketch of the proof

As observed by Sun, for any $k \in \mathbb{N}$ we have

$$T_k(b, c^2)^2 = \sum_{l=0}^{k} \binom{k+l}{2l} \binom{2l}{l}^2 c^{2l}(b^2 - 4c^2)^{k-l}.$$

Let

$$F(k, l) = (8ck+4c+b)\binom{k+l}{2l}\binom{2l}{l}^2 c^{2l}(b^2-4c^2)^{k-l}(b-2c)^{2(n-1-k)}$$

be the summand and

$$S_n = \sum_{k=0}^{n-1}\sum_{l=0}^{k} F(k, l) = \sum_{k=0}^{n-1}(8ck+4c+b)T_k(b, c^2)^2(b-2c)^{2(n-1-k)}$$

for any $n \in \mathbb{Z}^+$. When $b + 2c \neq 0$, we find the rational functions

$$R_1(k, l) = -\frac{(k-l)(-b^2 + 8c^2 + lb^2 + 8lc^2 - 2kbc + 2b^2l^2 - 4kbcl)}{(b+2c)(8ck+4c+b)(l+1)}$$

and

$$R_2(k, l) = -\frac{4bl^2}{8ck + 4c + b}.$$

Sketch of the proof

It follows that

$$S_n = \sum_{l=0}^{n-2} R_1(n,l)F(n,l) + R_1(n,n-1)F(n,n-1)$$

$$= -n\sum_{l=0}^{n-2}(-b^2+8c^2+lb^2+8lc^2-2nbc+2b^2l^2-4nbcl)$$

$$\times c^{2l}(b+2c)^{n-l-1}(b-2c)^{n-l-2}\binom{n-1}{l}\binom{n+l}{l}C_l$$

$$-n^2(2nb-3b-4c)c^{2n-2}\binom{2n-1}{n}C_{n-1}$$

This implies that $n \mid S_n$ since $C_l = \binom{2l}{l}/(l+1) \in \mathbb{Z}$ for all $l \in \mathbb{N}$.

With more efforts, for any odd prime $p \nmid b(b-2c)$ we have

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

A conjecture involving $T_k(b,c)M_k(b,c)$

Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. Conjecture 5.5 of Sun [Sci. China Math. 57(2014)] asserts that

$$\frac{1}{n}\sum_{k=0}^{n-1}T_k(b,c)M_k(b,c)d^{n-1-k}\in\mathbb{Z}\ \text{ for all } n=1,2,3,\ldots.$$

How to prove this ? Recall that $T_k(3,2)=D_k$. In 2016 the speaker realized that $M_k(3,2)$ coincides with the little Schröder number s_{k+1}. Thus he was led to show

$$\frac{1}{n}\sum_{k=0}^{n-1}D_ks_{k+1}=\frac{1}{2n}\sum_{k=0}^{n-1}D_kS_{k+1}\in\mathbb{Z}$$

for all $n=1,2,3,\ldots$.

Recall that in 2011 the speaker made a conjecture on $\sum_{k=0}^{p-1}D_kS_k$ modulo p^4 with p an odd prime, which was proved by Ji-Cai Liu in 2016.

The polynomials $D_n(x), s_n(x)$ and $S_n(x)$

Recall that $D_n = D_n(1)$, where

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k.$$

Similarly, we define

$$S_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{x^k}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^k.$$

Motivated by the identities $D_n = \sum_{k=0}^{n} \binom{n}{k}^2 2^k$ and $s_n = \sum_{k=1}^{n} N(n,k) 2^{n-k}$, we note that

$$D_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 x^k (x+1)^{n-k} \quad \text{for } n \in \mathbb{N},$$

and introduce the polynomials

$$s_n(x) := \sum_{k=1}^{n} N(n,k) x^{k-1} (x+1)^{n-k} \ (n = 1, 2, 3, \ldots).$$

Relations among $D_n(x), s_n(x)$ and $S_n(x)$

For any $n \in \mathbb{Z}^+$, we have

$$D_{n+1}(x) - D_{n-1}(x) = 2x(2n+1)S_n(x)$$

and

$$(x+1)s_n(x) = S_n(x).$$

This can be easily proved by induction on n.

When $x = 1$, this gives

$$D_{n+1} - D_{n-1} = 2(2n+1)S_n \text{ and } 2s_n = S_n.$$

Result on $\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x)$

Theorem (Z.-W. Sun [arxiv:1602.00574]) (i) For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x) = W_n(x(x+1)),$$

where

$$W_n(x) = \sum_{k=1}^{n} w(n,k)C_{k-1}x^{k-1} \in \mathbb{Z}[x]$$

with

$$w(n,k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1} \in \mathbb{Z}.$$

(ii) For any odd prime p, we have

$$\sum_{k=0}^{p-1} D_k s_{k+1} \equiv 2p^2(1-3q_p(2)) \pmod{p^3}$$

with $q_p(2) = (2^{p-1}-1)/p$.

Two Lemmas

Lemma 1 (Sun [Acta Arith. 156(2012)]). Let $n \in \mathbb{Z}^+$. Then

$$n(n+1)S_n(x)^2 = \sum_{k=1}^{n} \binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1} x^{k-1}(x+1)^{k+1}$$

and

$$\frac{D_{n-1}(x)+D_{n+1}(x)}{2} S_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}^2 \frac{2k+1}{(k+1)^2} x^k (x+1)^{k+1}.$$

Lemma 2. For any $m, n \in \mathbb{Z}^+$ with $m \leqslant n$, we have the identity

$$\sum_{k=m}^{n} \binom{k+m}{2m}\left(2m+1-m(m+1)\frac{2k+1}{k(k+1)}\right) = \frac{(n-m)(n+m+1)}{n+1}\binom{n+m}{2m}.$$

Sketch of the proof of part (i) of the theorem

Fix $n \in \mathbb{Z}^+$. For each $k \in \mathbb{Z}^+$, by the lemmas we have

$$\begin{aligned}
& D_{k-1}(x)\frac{S_k(x)}{x+1} \\
=& \frac{D_{k-1}(x)+D_{k+1}(x)}{2}\cdot\frac{S_k(x)}{x+1} - (2k+1)\frac{x}{x+1}S_k(x)^2 \\
=& \sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}^2\frac{2j+1}{(j+1)^2}(x(x+1))^j \\
& - \frac{2k+1}{k(k+1)}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}\binom{2j}{j+1}(x(x+1))^j \\
=& \sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}^2\left(\frac{2j+1}{(j+1)^2} - \frac{2k+1}{k(k+1)}\cdot\frac{j}{j+1}\right)(x(x+1))^j \\
=& \sum_{j=0}^{k}\binom{k+j}{2j}C_j^2\left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j.
\end{aligned}$$

Sketch of the proof of part (i) of the theorem

Thus

$$D_{k-1}(x)s_k(x) = \sum_{j=0}^{k} \binom{k+j}{2j} C_j^2 \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j$$

for any $k \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned}
&\sum_{k=1}^{n} D_{k-1}(x)s_k(x) \\
=&\sum_{k=1}^{n}\sum_{j=0}^{k} \binom{k+j}{2j} C_j^2 \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j \\
=&n+\sum_{j=1}^{n} C_j^2 (x(x+1))^j \sum_{k=j}^{n} \binom{k+j}{2j} \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right) \\
=&\sum_{j=0}^{n-1} C_j^2 (x(x+1))^j \frac{(n-j)(n+j+1)}{n+1}\binom{n+j}{2j}
\end{aligned}$$

with the help of Lemma 2.

Sketch of the proof of part (i) of the theorem

It follows that

$$\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x) = \sum_{j=0}^{n-1} C_j(x(x+1))^j \frac{n}{j+1}\binom{n-1}{j}\binom{n+j+1}{j}$$
$$= nW_n(x(x+1)).$$

For any $k \in \mathbb{Z}^+$, we have

$$w(n,k) = \frac{1}{n}\binom{n}{k}\binom{n+k}{k-1} = \frac{1}{n+1}\binom{n-1}{k-1}\binom{n+k}{k}$$

and hence $w(n,k) = (n+1)w(n,k) - nw(n,k) \in \mathbb{Z}$. So $W_n(x) \in \mathbb{Z}[x]$. This proves part (i) of the theorem.

Another theorem

Theorem (Sun, arXiv:1602.00574). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(b,c)M_k(b,c)d^{n-1-k} = \sum_{k=1}^{n} w(n,k)C_{k-1}c^{k-1}d^{n-k} \in \mathbb{Z}.$$

Moreover, for any odd prime p not dividing cd, we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)M_k(b,c)}{d^k} \equiv \frac{pb^2}{2c}\left(\left(\frac{d}{p}\right) - 1\right) \pmod{p^2}.$$

Corollary. For any positive integer n, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k M_k (-3)^{n-1-k} = \sum_{k=1}^{n} w(n,k)C_{k-1}(-3)^{n-k} \in \mathbb{Z}.$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{T_k M_k}{(-3)^k} \equiv \frac{p}{2}\left(\left(\frac{p}{3}\right) - 1\right) \pmod{p^2}.$$

A key lemma

Lemma. Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c \neq 0$. For any $n \in \mathbb{N}$, we have

$$T_n(b,c) = (\sqrt{d})^n D_n\left(\frac{b/\sqrt{d}-1}{2}\right)$$

and

$$M_n(b,c) = (\sqrt{d})^n s_{n+1}\left(\frac{b/\sqrt{d}-1}{2}\right),$$

therefore

$$\frac{T_n(b,c)M_n(b,c)}{d^n} = D_n(x)s_{n+1}(x)$$

with $x = (b/\sqrt{d}-1)/2$.

An open conjecture

Conjecture (Z.-W. Sun, arXiv:1602:00574). (i) We have

$$f_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x)R_k(x) \in \mathbb{Z}[x] \quad \text{for all } n = 1, 2, 3, \ldots,$$

where

$$R_k(x) := \sum_{l=0}^{k} \binom{k}{l}\binom{k+l}{l}\frac{x^l}{2l-1} = \sum_{l=0}^{k} \binom{k+l}{2l}\binom{2l}{l}\frac{x^l}{2l-1}.$$

Also, $f_n(1)$ is an odd integer for each $n \in \mathbb{Z}^+$.

(ii) Let p be any odd prime, and let $E_0, E_1, E_2, \ldots$ be Euler numbers. Set $R_k = R_k(1)$. Then

$$\sum_{k=0}^{p-1} D_k R_k \equiv \begin{cases} -p + 8p^2 q_p(2) - 2p^3 E_{p-3} \pmod{p^4} & \text{if } 4 \mid p-1, \\ -5p \pmod{p^3} & \text{if } 4 \mid p+1, \end{cases}$$

$$\sum_{k=1}^{p-1} \frac{D_k R_k}{k} \equiv \left(4 - \left(\frac{-1}{p}\right)\right) q_p(2) \pmod{p},$$

Thank you!