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On Arithmetic Properties of Bell Numbers, Delannoy Numbers and Schröder Numbers

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Abstract

Bell numbers, Delannoy numbers and Schröder numbers are important quantities arising from enumerative combinatorics. Surprisingly they also have nice number-theoretic properties. We will talk about congruences related to Bell numbers, Delannoy numbers and Schroder numbers.

Congruences for p-integers

Let p be a prime. A rational number is called a *p-integer* (or a rational p-adic integer) if it can be written in the form a/b with $a, b \in \mathbb{Z}$ and $(b, p) = 1$. All p-integers form a ring R_p which is a subring of the ring $\mathbb{Z}_p$ of all p-adic integers. For a p-integer a/b, an integer c and a nonnegative integer n if $a/b = c + p^n q$ for some $q \in R_p$ (equivalently, $a \equiv bc \pmod{p^n}$), then we write

$$\frac{a}{b} \equiv c \pmod{p^n}.$$

An Example for Congruences involving p-Integers:

$$1 + \frac{1}{2} \equiv 1 - 4 = -3 \pmod{3^2}.$$

Classical congruences for central binomial coefficients

A central binomial coefficient has the form

$$\binom{2k}{k}\ (k = 0, 1, 2, \ldots).$$

Wolstenholme's Congruence. For any prime $p > 3$ we have

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Remark. In 1900 Glaiser proved that for any prime $p > 3$ we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4},$$

where B_n denotes the nth Bernoulli number.

Classical congruences for central binomial coefficients

Morley's Congruence. For any prime $p > 3$ we have

$$\binom{p-1}{(p-1)/2} \equiv \left(\frac{-1}{p}\right) 4^{p-1} \pmod{p^3}.$$

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Catalan numbers

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

Recursion.

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n = 0, 1, 2, \ldots).$$

Generating Function.

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

Combinatorial Interpretations. The Catalan numbers arise in many enumeration problems. For example, C_n is the number of binary parenthesizations of a string of $n+1$ letters, and it is also the number of ways to triangulate a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

Recent results on $\sum_{k=0}^{p-1}\binom{2k}{k}$ and $\sum_{k=0}^{p-1}C_k \bmod p^2$

Let p be a prime and let $a\in\mathbb{Z}^+=\{1,2,3,\ldots\}$.

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\quad\text{for } p>3.$$

Sun & R. Tauraso [Int. JNT 2011, Adv. in Appl. Math. 2010].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p^a-1}C_k\equiv\frac{3(\frac{p^a}{3})-1}{2}\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3.$$

Determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$

Let p be an odd prime. If $p/2 < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \pmod{p},$$

where m is an integer with $p \nmid m$.

Sun [Sci. China Math. 2010]: Let p be an odd prime and let $a, m \in \mathbb{Z}$ with $a > 0$ and $p \nmid m$. Then

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p^a}\right) + \left(\frac{m^2-4m}{p^{a-1}}\right) u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $(-)$ is the Jacobi symbol and $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{ and } u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

Euler numbers and some congruences mod p^3

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}\ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453].

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) \frac{p^2}{4} E_{p-3} \pmod{p^3}.$$

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Bell numbers

For $n = 1, 2, 3, \ldots$, the nth Bell number B_n denotes the number of partitions of a set of cardinality n. In addition, $B_0 := 1$. Here are values of $B_1, \ldots, B_{10}$:

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975.$$

Recursion:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (n = 0, 1, 2, \ldots).$$

Exponential Generating Function:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Touchard's Congruence: For any prime p and $m, n = 0, 1, 2, \ldots$ we have

$$B_{p^m+n} \equiv mB_n + B_{n+1} \pmod{p}.$$

A conjecture on Bell numbers

Conjecture (Sun, July 17, 2010). For any positive integer n there is a unique integer $a(n)$ such that

$$\sum_{k=0}^{p-1}\frac{B_k}{(-n)^k}\equiv a(n) \pmod p \ \text{ for any prime } p\nmid n.$$

In particular,

$$a(2)=1,\ a(3)=2,\ a(4)=-1,\ a(5)=10,\ a(6)=-43,$$
$$a(7)=266,\ a(8)=-1853,\ a(9)=14834,\ a(10)=-133495.$$

Remark. It is easy to see that $a(1)=2$. In fact, if p is a prime then

$$\sum_{k=0}^{p-1}(-1)^kB_k\equiv\sum_{k=0}^{p-1}\binom{p-1}{k}B_k=B_p$$
$$\equiv B_0+B_1=2 \pmod p \text{ (by Touchard's congruence).}$$

Sun and Zagier's results on Bell numbers

For $n = 1, 2, 3, \ldots$ the nth derangement number D_n is the number of permutations σ of $\{1, \ldots, n\}$ with $\sigma(i) \neq i$ for all $i = 1, \ldots, n$; in addition, $D_0 := 1$. It is well known that

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \text{for all } n \in \mathbb{N}.$$

Theorem (Sun and D. Zagier, Bull. Austral. Math. Soc., 84(2011)) (i) For every positive integer n we have

$$\sum_{k=1}^{p-1} \frac{B_k}{(-n)^k} \equiv (-1)^{n-1} D_{n-1} \pmod{p}$$

for any prime p not dividing n.
(ii) Let p be any prime. Then for all $n = 1, \ldots, p-1$ we have

$$B_n \equiv \sum_{k=1}^{p-1} (-1)^k D_{k-1} (-k)^n \pmod{p}.$$

Part (i) implies part (ii)

Part (i) implies part (ii). For $k, n \in \{1, \ldots, p-1\}$ with $k \neq n$, as $p-1 \nmid n-k$ we have

$$\sum_{m=1}^{p-1} (-m)^{n-k} \equiv 0 \pmod{p}.$$

Thus, with the help of part (i), if $n \in \{1, \ldots, p-1\}$ then

$$\begin{aligned} -B_n &\equiv \sum_{k=1}^{p-1} B_k \sum_{m=1}^{p-1} (-m)^{n-k} = \sum_{m=1}^{p-1} (-m)^n \sum_{k=1}^{p-1} \frac{B_k}{(-m)^k} \\ &\equiv \sum_{m=1}^{p-1} (-m)^n (-1)^{m-1} D_{m-1} \pmod{p}. \end{aligned}$$

A direct proof of $B_p \equiv 2 \pmod{p}$

Note that

$$B_n = \sum_{k=0}^{n} S(n,k),$$

where $S(n,k)$ (a Stirling number of the second kind) is the number of ways to partition $\{1,\ldots,n\}$ into k nonempty sets. It is well known that

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \text{ for all } n \in \mathbb{Z}^+ \text{ and } k \in \mathbb{N}.$$

Clearly, $S(n,0) = 0$ and $S(n,1) = S(n,n) = 1$ for any $n \in \mathbb{Z}^+$.

Let p be a prime. With the help of Fermat's little theorem, if $1 < k < p$ then $S(p,k) \equiv S(1,k) = 0 \pmod{p}$. Therefore

$$B_p \equiv S(p,1) + S(p,p) = 2 \pmod{p}.$$

Prove part (i) by induction

For any prime p,

$$\sum_{k=1}^{p-1}(-1)^k B_k \equiv \sum_{k=1}^{p-1}\binom{p-1}{k}B_k = B_p - B_0 \equiv 1 \pmod{p}.$$

So the desired result holds when $n = 1$.

Now fix $n \in \mathbb{Z}^+$ and suppose that

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k} \equiv (-1)^{n-1}D_{n-1} \ (\mathrm{mod}\ p)$$

for every prime $p \nmid n$.

Let p be any prime not dividing $n+1$. Recall the easy identity

$$D_n = nD_{n-1} + (-1)^n.$$

If $p \mid n$, then $D_n \equiv (-1)^n \ (\mathrm{mod}\ p)$ and hence

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n-1)^k} \equiv \sum_{k=1}^{p-1}\frac{B_k}{(-1)^k} \equiv 1 \equiv (-1)^n D_n \ (\mathrm{mod}\ p).$$

Prove part (i) by induction

Now suppose that $p \nmid n$. Observe that

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{B_k}{(-n)^k} &= \sum_{k=1}^{p-1} \frac{\sum_{l=0}^{k-1} \binom{k-1}{l} B_l}{(-n)^k} = \sum_{l=0}^{p-2} \frac{B_l}{(-n)^l} \sum_{k=l+1}^{p-1} \frac{\binom{k-1}{l}}{(-n)^{k-l}} \\
&= \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \frac{\binom{l+r-1}{r-1}}{(-n)^{r-1}} \\
&= \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \frac{\binom{-l-1}{r-1}}{n^{r-1}} \\
&\equiv \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \binom{p-1-l}{r-1} n^{-(r-1)} \\
&\equiv \sum_{l=0}^{p-1} \frac{B_l}{(-n)^{l+1}} \left(\left(1 + \frac{1}{n}\right)^{p-1-l} - \frac{1}{n^{p-1-l}} \right) \pmod{p}.
\end{aligned}$$

Prove part (i) by induction

Thus, applying Fermat's little theorem we get

$$-n\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}\equiv\sum_{l=1}^{p-1}\frac{B_l}{(-n-1)^l}-\sum_{l=1}^{p-1}\frac{B_l}{(-1)^l}\pmod{p}.$$

Therefore

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{B_l}{(-n-1)^l}&\equiv -n\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}+\sum_{l=1}^{p-1}\frac{B_l}{(-1)^l}\\&\equiv -n(-1)^{n-1}D_{n-1}+1=(-1)^nD_n\pmod{p}.\end{aligned}$$

This concludes the induction step.

A further extension

The Touchard polynomial $T_n(x)$ of degree n is given by

$$T_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

Note that $T_n(1) = B_n$. Similar to the recursion for Bell numbers, we have the recursion

$$T_{n+1}(x) = x\sum_{k=0}^{n} \binom{n}{k} T_k(x).$$

Theorem (Sun & Zagier, Bull. Austral. Math. Soc., 84(2011)). For every positive integer m, we have

$$(-x)^m \sum_{0<n<p} \frac{T_n(x)}{(-m)^n} \equiv -x^p \sum_{k=0}^{m-1} \frac{(m-1)!}{k!}(-x)^k \pmod{p}$$

for any prime p not dividing m.

Consequences

Let p be a prime. The theorem implies the congruence

$$\sum_{0<n<p} \frac{T_n(x)}{(-m)^n} \equiv \frac{1}{(-x)^{m-1}} \sum_{l=0}^{m-1} \frac{(m-1)!}{l!}(-x)^l \pmod{p}$$

for any p-adic integer x not divisible by p, special cases being

$$\sum_{0<n<p} \frac{T_n(x)}{(-2)^n} \equiv \frac{x-1}{x} \pmod{p} \qquad \text{for } p \neq 2,$$

$$\sum_{0<n<p} \frac{T_n(x)}{(-3)^n} \equiv \frac{x^2-2x+2}{x^2} \pmod{p} \qquad \text{for } p \neq 3\,,$$

$$\sum_{0<n<p} \frac{T_n(x)}{(-4)^n} \equiv \frac{x^3-3x^2+6x-6}{x^3} \pmod{p} \qquad \text{for } p \neq 2\,.$$

On central Delannoy numbers

$$D_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}.$$

In combinatorics, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Theorem (Sun, 2010-2011). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}, \ \sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2) \pmod{p}, \ \sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2\left(\frac{-1}{p}\right) E_{p-3} \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \pmod{p},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

An auxiliary identity

In the proof of the theorem, the following new identity plays an important role.

New Identity (Sun). For $s = 1, 2$ we have

$$\sum_{k=-n}^{n} \frac{(-1)^k}{(2k+1)^s} \binom{2n}{n+k} = \frac{16^n}{(2n+1)^s \binom{2n}{n}}.$$

Via Zeilberger's algorithm we find that both sides satisfy the same recurrence relation.

On central Delannoy numbers

Theorem (Sun, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k D_k \equiv p - \frac{7}{12}p^4 B_{p-3} \pmod{p^5},$$
$$\sum_{k=0}^{p-1}(2k+1)D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4},$$

where $B_0, B_1, B_2 \ldots$ are Bernoulli numbers.

Conjecture (Sun, 2010). For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2) + p\, q_p(2)^2 \pmod{p^2}.$$

Remark. We can show the congruence modulo p.

Congruences involving Schröder numbers

The nth Schröder number is given by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \binom{n+k}{k}$$

which is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ that never rise above the line $y = x$.

Theorem (Sun, 2010). Let p be an odd prime and let m be an integer not divisible by p. Then

$$\sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2 - 6m + 1}{2m} \left(1 - \left(\frac{m^2 - 6m + 1}{p} \right) \right) \pmod{p}.$$

The theorem in the case $m = 6$ gives that

$$\sum_{k=1}^{p-1} \frac{S_k}{6^k} \equiv 0 \pmod{p} \qquad \text{for any prime } p > 3.$$

An observation of van Hammer

Let $p = 2n + 1$ be a prime. As observed by van Hammer, for $k = 0, \ldots, n$ we have

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$

$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}$$

$$=\frac{\prod_{j=0}^{k-1}(\frac{p-1}{2}-j)(\frac{-p-1}{2}-j)}{(k!)^2} = \prod_{j=0}^{k-1}\frac{(-\frac{1}{2}-j)^2-\frac{p^2}{4}}{(k!)^2}$$

$$\equiv\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}$$

Thus

$$D_{(p-1)/2} \equiv \sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

On $D_{(p-1)/2}$ and $S_{(p-1)/2}$ mod p^2

Theorem 1 (conjectured by Z. W. Sun in 2009 and proved by Z. H. Sun [Proc. AMS, 2011]) Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

Theorem 2 (Z. W. Sun, 2011) Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$S_{(p-1)/2} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k}{(-16)^k} \equiv -8 \sum_{k=0}^{(p-1)/2} \frac{k\binom{2k}{k}^2}{(-16)^k}$$
$$\equiv (-1)^{(p-1)/4}\, 2\left(2x - \frac{p}{x}\right) \pmod{p^2}.$$

A conjecture

Conjecture (Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} D_k S_k \equiv -2pH_{(p-1)/2} \pmod{p^3},$$

and

$$\sum_{k=1}^{(p-1)/2} D_k S_k \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \ \& \ p = x^2 + 4y^2, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Thank you!