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SOME CURIOUS RESULTS ON BERNOULLI AND EULER POLYNOMIALS

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Abstract. In this talk we tell the story how the developments of some curious identities concerning Bernoulli (and Euler) polynomials finally led to the following unified symmetric relation (of Z. W. Sun and H. Pan): If n is a positive integer, $r+s+t=n$ and $x+y+z=1$, then we have

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n} (-1)^k \binom{s}{k}\binom{t}{n-k} B_{n-k}(x)B_k(y).$$

It is interesting to compare this with the easy identity

$$0 = \begin{vmatrix} r & s & t \\ r & s & t \\ z & x & y \end{vmatrix} = r\begin{vmatrix} s & t \\ x & y \end{vmatrix} + s\begin{vmatrix} t & r \\ y & z \end{vmatrix} + t\begin{vmatrix} r & s \\ z & x \end{vmatrix}.$$

We will also talk about some congruences for Euler numbers and q-Euler numbers.

All papers of the speaker mentioned in this survey are available from his homepage http://pweb.nju.edu.cn/zwsun.

1. Von Ettingshausen's identity and its generalizations

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. The well-known Bernoulli numbers B_n $(n \in \mathbb{N})$ are rational numbers defined by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n \in \mathbb{Z}^+).$$

Similarly, Euler numbers E_n $(n \in \mathbb{N})$ are integers given by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0\\ 2\mid n-k}}^{n} \binom{n}{k} E_k = 0 \quad (n \in \mathbb{Z}^+).$$

Bernoulli numbers and Euler numbers can also be given by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} = \left(\frac{e^x - 1}{x}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\right)^{-1} \quad (|x| < 2\pi)$$

and

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x} + 1} = \left(\frac{e^x + e^{-x}}{2}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)^{-1} \quad (|x| < \pi).$$

It is well known that $B_3 = B_5 = \cdots = 0$ and $E_1 = E_3 = E_5 = \cdots = 0$.

For $n \in \mathbb{N}$ the Bernoulli polynomial $B_n(x)$ and the Euler polynomial $E_n(x)$ are as follows:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

Clearly $B_n(0) = B_n$ and $E_n(1/2) = E_n/2^n$. Here are some well-known properties of Bernoulli and Euler polynomials.

$$B_n(1-x) = (-1)^n B_n(x), \ B'_{n+1}(x) = (n+1)B_n(x);$$

$$E_n(1-x) = (-1)^n E_n(x), \ E'_{n+1}(x) = (n+1)E_n(x).$$

In a book of von Ettingshausen published in 1827, the author obtained that we can compute B_{2n} in terms of $B_n, B_{n+1}, \dots, B_{2n-1}$ by the recursion:

$$\sum_{k=0}^{n}\binom{n+1}{k}(n+k+1)B_{n+k}=0 \quad (n=1,2,\dots). \tag{1.1}$$

With the help of continued fractions, in 1995 M. Kaneko [Proc. Japan Acad. Ser. A. Math. Sci. 71(1995), 192-193] rediscovered this. (The speaker thanks Prof. T. Agoh for his informing me that Kaneko repeated von Ettingshausen's discovery.) In 2001, by employing certain integrals over $\mathbb{Z}_p$, H. Momiyama [Fibonacci Quart. 39(2001), 285-288] extended the von Ettingshausen identity in the following symmetric form:

$$\begin{aligned}&(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)B_{n+k}\\ =&-(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k}\end{aligned} \tag{1.2}$$

providing that $m,n\in\mathbb{N}$ and $m+n>0$.

In November 2001, the speaker found Momiyama's paper and asked my students to provide an induction proof of (1.2) and extend it to Bernoulli polynomials. Soon, Hao Pan, one of my students, proved (1.2) by induction Then, on Dec. 1, 2001, the speaker succeeded in giving the polynomial form of (1.2):

$$\begin{aligned}&(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)B_{n+k}(x)\\ &+(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k}(-x)\\ =&(-1)^m(m+n+1)(m+n+2)x^{m+n}.\end{aligned} \tag{1.3}$$

This result appeared in the paper [K. J. Wu, Z. W. Sun and H. Pan, Fibonacci Quart. 42(2004), 295-299].

On Dec. 7, 2001 the speaker obtained the following result more general than (1.3):

$$(-1)^m\sum_{k=0}^m\binom{m}{k}x^{m-k}B_{n+k}(y)=(-1)^n\sum_{k=0}^n\binom{n}{k}x^{n-k}B_{m+k}(z) \tag{1.4}$$

providing $x+y+z=1$.

Now let me explain how (1.4) was found originally. Let $m,n\in\mathbb{N}$. Then, for any $h\in\mathbb{Z}^+$ we have

$$\begin{aligned}&(-1)^m\sum_{k=0}^m\binom{m}{k}\sum_{r=0}^{h-1}(x+r)^{n+k}a^{m-k}\\ =&(-1)^m\sum_{r=0}^{h-1}(x+r)^n(x+r+a)^m\\ =&(-1)^m\sum_{s=0}^{h-1}(x+h-1-s)^n(x+h-1-s+a)^m\\ =&(-1)^n\sum_{k=0}^n\binom{n}{k}a^{n-k}\sum_{s=0}^{h-1}(-x-h+1-a+s)^{m+k}.\end{aligned}$$

As

$$\sum_{r=0}^{h-1}(x+r)^{n+k}=\frac{B_{n+k+1}(x+h)-B_{n+k+1}(x)}{n+k+1},$$

we have

$$\begin{aligned}&(-1)^m\sum_{k=0}^m\binom{m}{k}a^{m-k}\frac{B_{n+k+1}(x+h)-B_{n+k+1}(x)}{n+k+1}\\ =&(-1)^n\sum_{k=0}^n\binom{n}{k}a^{n-k}\frac{B_{m+k+1}(-x-a+1)-B_{m+k+1}(-x-a+1-h)}{m+k+1},\end{aligned}$$

i.e., $f(a,x+h)$ does not depend on $h\in\mathbb{Z}^+$ where

$$\begin{aligned}f(a,x)=&(-1)^m\sum_{k=0}^m\binom{m}{k}a^{m-k}\frac{B_{n+k+1}(x)}{n+k+1}\\&+(-1)^n\sum_{k=0}^n\binom{n}{k}a^{n-k}\frac{B_{m+k+1}(1-a-x)}{m+k+1}.\end{aligned}$$

Therefore $f(a,x)=f(a,0)$ and hence $\frac{\partial}{\partial x}f(a,x)=0$ which gives the identity

$$(-1)^m\sum_{k=0}^m\binom{m}{k}a^{m-k}B_{n+k}(x)=(-1)^n\sum_{k=0}^n\binom{n}{k}a^{n-k}B_{m+k}(1-a-x).$$

On Dec. 10, 2002, with help of the beta function

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

the speaker got the following result: *If $m,n\in\mathbb{N}$ and $x+y+z=1$, then*

$$\begin{aligned}&(-1)^m\sum_{k=0}^m\binom{m}{k}x^{m-k}\frac{B_{n+k+1}(y)}{n+k+1}\\&+(-1)^n\sum_{k=0}^n\binom{n}{k}x^{n-k}\frac{B_{m+k+1}(z)}{m+k+1}\\=&\frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}},\end{aligned}\tag{1.5}$$

we can also replace Bernoulli polynomials in (1.5) *by corresponding Euler polynomials.* If we take partial derivative of (1.5) with respect to y and view $z=1-x-y$ as a function of y, we then obtain (1.4).

For a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers, its dual sequence $\{a_n\}_{n\in\mathbb{N}}$ are given by $a_n^*=\sum_{k=0}^n\binom{n}{k}(-1)^k a_k$ $(n\in\mathbb{N})$. It is well known that $a_n^{**}=a_n$. The sequences $\{(-1)^nB_n\}_{n\in\mathbb{N}}$ and $\{(-1)^nE_n(0)\}_{n\in\mathbb{N}}$ are both self-dual sequences.

In December 2001, the speaker obtained the following general result.

Theorem 1.1 [Z. W. Sun, European J. Combin. 24(2003), 709-718]. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. For $n\in\mathbb{N}$ let*

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k x^{n-k}$$

and

$$A_n^*(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k^* x^{n-k}.$$

If $m,n\in\mathbb{N}$ and $x+y+z=1$, then we have the identity

$$\begin{aligned}(-1)^m&\sum_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{A_{n+k+1}(y)}{n+k+1}\\&+(-1)^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\frac{A^*_{m+k+1}(z)}{m+k+1}\\=&a_0\frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}},\end{aligned}\tag{1.6}$$

consequently

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}x^{m-k}A_{n+k}(y)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}A^*_{m+k}(z)\tag{1.7}$$

and

$$\begin{aligned}(-1)^m&\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)x^{n-k+1}A_{n+k}(y)\\&+(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}x^{n-k+1}(m+k+1)A^*_{m+k}(z)\\=&(m+n+2)\left((-1)^{m+1}A_{m+n+1}(y)+(-1)^{n+1}A^*_{m+n+1}(z)\right).\end{aligned}\tag{1.8}$$

Quite recently R. Chapman [Integers 5(2005)] subsumes these three identities of Sun into an infinite family of identities, and J. X. Hou and J. Zeng [European J. Combin., in press, `arXiv:math.CO/0501186`] got the q-analogue of the above theorem.

2. Miki's and Matiyasevich's identities and their polynomial forms

In 1978 H. Miki [J. Number Theory 10(1978), 297-302] discovered the following curious identity which involves both an ordinary convolution and a binomial convolution of Bernoulli numbers:

$$\sum_{k=2}^{n-2}\frac{B_kB_{n-k}}{k(n-k)}-\sum_{k=2}^{n-2}\binom{n}{k}\frac{B_kB_{n-k}}{k(n-k)}=2H_n\frac{B_n}{n} \tag{2.1}$$

for every $n=4,5,\ldots$, where

$$H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}.$$

In the original proof of this identity, Miki showed that the two sides of (2.1) are congruent modulo all sufficiently large primes. In 1982 Shiratani and Yokoyama [Mem. Fac. Sci. Kyushu Univ. Ser. A 36(1982), 73-83] gave another proof of (2.1) by p-adic analysis.

Inspired by Miki's work, Matiyasevich found the following two identities of the same nature by the software *Mathematica*.

$$\sum_{k=2}^{n-2}\frac{B_k}{k}B_{n-k}-\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l}{l}B_{n-l}=H_nB_n$$

and

$$(n+2)\sum_{k=2}^{n-2}B_kB_{n-k}-2\sum_{l=2}^{n-2}\binom{n+2}{l}B_lB_{n-l}=n(n+1)B_n \tag{2.2}$$

for each $n=4,5,\ldots$. Clearly the first one is actually equivalent to Miki's identity (2.1) since

$$\begin{aligned}&\sum_{k=2}^{n-2}\frac{B_kB_{n-k}}{k(n-k)}-\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_lB_{n-l}}{l(n-l)}\\=&\frac{1}{n}\sum_{k=2}^{n-2}\left(\frac{1}{k}+\frac{1}{n-k}\right)B_kB_{n-k}-\frac{1}{n}\sum_{l=2}^{n-2}\binom{n}{l}\left(\frac{1}{l}+\frac{1}{n-l}\right)B_lB_{n-l}\\=&\frac{2}{n}\sum_{k=2}^{n-2}\frac{B_k}{k}B_{n-k}-\frac{2}{n}\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l}{l}B_{n-l}.\end{aligned}$$

In June 2004, Dunne and Schubert [arXiv:math.NT/0406610] presented a new approach to (2.1) and (2.2) motivated by quantum field theory and string theory.

Since all previous proofs of Miki's identity are non-natural and complicated, in May 2004 H. Pan and Z. W. Sun developed a new method which only involves differences and derivatives of polynomials.

Define the operators Δ and Δ^* by $\Delta(f(x)) = f(x+1) - f(x)$ and $\Delta^*(f(x)) = f(x+1) + f(x)$. It is well known that

$$\Delta(B_n(x)) = nx^{n-1} \text{ and } \Delta^*(E_n(x)) = 2x^n \quad \text{for } n = 0, 1, 2, \dots .$$

Lemma 2.1 [H. Pan and Z. W. Sun, J. Combin. Theory Ser A 113(2006)]. *Let $P(x), Q(x) \in \mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers.*

(i) *If $\Delta(P(x)) = \Delta(Q(x))$ then $P'(x) = Q'(x)$.*

(ii) *If $\Delta^*(P(x)) = \Delta^*(Q(x))$ then $P(x) = Q(x)$.*

To illustrate the power of Lemma 2.1, let us give a simple proof of Raabe's multiplication formula:

$$\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right) = m^{1-n}B_n(x) \text{ for } m \in \mathbb{Z}^+ \text{ and } n \in \mathbb{N}.$$

Clearly

$$\begin{aligned}&\Delta\Bigg(\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right)\Bigg)\\ =&\sum_{r=0}^{m-1}\left(B_n\left(\frac{x+r+1}{m}\right)-B_n\left(\frac{x+r}{m}\right)\right)\\ =&B_n\left(\frac{x}{m}+1\right)-B_n\left(\frac{x}{m}\right)=n\left(\frac{x}{m}\right)^{n-1}=\Delta(m^{1-n}B_n(x))\end{aligned}$$

and hence

$$\begin{aligned}\sum_{r=0}^{m-1}\frac{n}{m}B_{n-1}\left(\frac{x+r}{m}\right)=&\frac{d}{dx}\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)\\ =&\frac{d}{dx}(m^{1-n}B_n(x))=m^{1-n}nB_{n-1}(x)\end{aligned}$$

for $n=1,2,3,\ldots$, this proves Raabe's formula.

With help of Lemma 2.1, Pan and Sun were able to extend Miki's identity (2.1) and Matiyasevich's identity (2.2) to Bernoulli polynomials.

Theorem 2.1 [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]**.** *Let $n>1$ be an integer. Then*

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(y)}{k(n-k)}-\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{B_l(x-y)B_{n-l}(y)+B_l(y-x)B_{n-l}(x)}{l^2}\\ &\quad=H_{n-1}\frac{B_n(x)+B_n(y)}{n}+\frac{B_n(x)-B_n(y)}{n(x-y)}\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&\sum_{k=0}^{n}B_k(x)B_{n-k}(y)-\sum_{l=0}^{n}\binom{n+1}{l+1}\frac{B_l(x-y)B_{n-l}(y)+B_l(y-x)B_{n-l}(x)}{l+2}\\ &\quad=\frac{B_{n+1}(x)+B_{n+1}(y)}{(x-y)^2}-\frac{2}{n+2}\cdot\frac{B_{n+2}(x)-B_{n+2}(y)}{(x-y)^3}.\end{aligned}\tag{2.4}$$

Letting y tend to x, (2.3) and (2.4) turn out to be

$$\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(x)}{k(n-k)}-2\sum_{l=2}^{n}\binom{n-1}{l-1}\frac{B_lB_{n-l}(x)}{l^2}=2H_{n-1}\frac{B_n(x)}{n} \tag{2.5}$$

and

$$\sum_{k=0}^{n}B_k(x)B_{n-k}(x)-2\sum_{l=2}^{n}\binom{n+1}{l+1}\frac{B_lB_{n-l}(x)}{l+2}=(n+1)B_n(x). \tag{2.6}$$

respectively.

Similar to Theorem 2.1 Pan and Sun proved the following identities involving Euler polynomials.

Theorem 2.2 [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]**.** *Let n be a positive integer. Then*

$$\begin{aligned}&\sum_{k=0}^{n}E_k(x)E_{n-k}(y)-\frac{4}{n+2}\cdot\frac{B_{n+2}(x)-B_{n+2}(y)}{x-y}\\ =&-2\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{E_l(x-y)B_{n+1-l}(y)+E_l(y-x)B_{n+1-l}(x)}{l+1}.\end{aligned} \tag{2.7}$$

Also,

$$\begin{aligned}&\sum_{k=1}^{n}\frac{B_k(x)}{k}E_{n-k}(y)-H_nE_n(y)-\frac{E_n(x)-E_n(y)}{x-y}\\ =&\sum_{l=1}^{n}\binom{n}{l}\left(\frac{B_l(x-y)}{l}E_{n-l}(y)-\frac{E_{l-1}(y-x)}{2}E_{n-l}(x)\right),\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\sum_{k=0}^{n}B_k(x)E_{n-k}(y)\\ =&\sum_{l=1}^{n}\binom{n+1}{l+1}\left(B_l(x-y)E_{n-l}(y)-\frac{E_{l-1}(y-x)}{2}E_{n-l}(x)\right)\\ &+(n+1)\left(\frac{E_n(x)}{x-y}+E_n(y)\right)-\frac{E_{n+1}(x)-E_{n+1}(y)}{(x-y)^2}.\end{aligned} \tag{2.9}$$

Letting y tend to x and noting that $E_l(0) = 2(1 - 2^{l+1})B_{l+1}/(l+1)$, we obtain from (2.7)–(2.9) the following identities.

$$(n+2)\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = 8\sum_{l=2}^{n+2}\binom{n+2}{l}(2^l-1)\frac{B_l}{l}B_{n+2-l}(x), \tag{2.10}$$

$$\sum_{k=1}^{n}\frac{B_k(x)}{k}E_{n-k}(x) - \sum_{l=2}^{n}\binom{n}{l}2^l\frac{B_l}{l}E_{n-l}(x) = H_nE_n(x), \tag{2.11}$$

$$\sum_{k=0}^{n}B_k(x)E_{n-k}(x) - \sum_{l=2}^{n}\binom{n+1}{l+1}(2^l+l-1)\frac{B_l}{l}E_{n-l}(x) = (n+1)E_n(x). \tag{2.12}$$

3. Woodcock's identity and its generalizations

In 1979 C. F. Woodcock [J. London Math. Soc. 20(1979), 101-108] discovered that

$$A_{m-1,\,n} = A_{n-1,\,m} \qquad \text{for } m,n \in \mathbb{Z}^+ \tag{3.1}$$

where

$$A_{m,\,n} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^kB_{m+k}B_{n-k}. \tag{3.2}$$

Thus

$$\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}B_kB_{n-k} + B_{n-1} = A_{1-1,n} = A_{n-1,1} = -B_n$$

as noted by L. Euler.

Using Lemma 2.1 H. Pan and Z. W. Sun proved in August 2004 the following theorem which implies the Woodcock identity.

Theorem 3.1 [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]**.** *Let $m,n\in\mathbb{N}$ and $x+y+z=1$. Then*

$$\begin{aligned}&(-1)^m\sum_{k=0}^m\binom{m}{k}\frac{B_{m-k+1}(x)}{m-k+1}\cdot\frac{B_{n+k+1}(y)}{n+k+1}\\&+(-1)^n\sum_{k=0}^n\binom{n}{k}\frac{B_{n-k+1}(x)}{n-k+1}\cdot\frac{B_{m+k+1}(z)}{m+k+1}\\=&\frac{(-1)^{m+n+1}}{(m+n+1)\binom{m+n}{n}}\cdot\frac{B_{m+n+2}(x)}{m+n+2}-\frac{B_{m+1}(z)}{m+1}\cdot\frac{B_{n+1}(y)}{n+1}\\&+\frac{(-1)^{m+1}}{m+1}\cdot\frac{B_{m+n+2}(y)}{m+n+2}+\frac{(-1)^{n+1}}{n+1}\cdot\frac{B_{m+n+2}(z)}{m+n+2}.\end{aligned}\tag{3.3}$$

Also,

$$\begin{aligned}&(-1)^m\sum_{k=0}^m\binom{m}{k}E_{m-k}(x)\frac{B_{n+k+1}(y)}{n+k+1}\\&+(-1)^n\sum_{k=0}^n\binom{n}{k}E_{n-k}(x)\frac{B_{m+k+1}(z)}{m+k+1}\\=&\frac{(-1)^{m+n+1}E_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}}-\frac{E_m(z)E_n(y)}{2}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\frac{(-1)^m}{2}\sum_{k=0}^m\binom{m}{k}E_{m-k}(x)\frac{E_{n+k+1}(y)}{n+k+1}\\&-(-1)^n\sum_{k=0}^n\binom{n}{k}\frac{B_{n-k+1}(x)}{n-k+1}\cdot\frac{E_{m+k+1}(z)}{m+k+1}\\=&\frac{(-1)^{m+n}}{(m+n+1)\binom{m+n}{n}}\cdot\frac{B_{m+n+2}(x)}{m+n+2}+\frac{(-1)^n}{n+1}\cdot\frac{E_{m+n+2}(z)}{m+n+2}\\&-\frac{1}{n+1}\sum_{k=0}^m\frac{\binom{m}{k}}{\binom{n+k+1}{k}}E_{m-k}(z)\frac{B_{n+k+2}(y)}{n+k+2}.\end{aligned}\tag{3.5}$$

Fix y and replace z in (1) by $1-x-y$. Then, by taking differences of both sides of (3.3) with respect to x, we can get (1.5) again. The similar identity for Euler polynomials is also implied by Theorem 3.1.

If $m, n \in \mathbb{Z}^+$ and $x+y+z=1$, then we have the following equivalent version of (3.3):

$$\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)\frac{B_{n+k}(y)}{n+k}+\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)\frac{B_{m+k}(z)}{m+k}$$
$$=\frac{(-1)^{m+n}(m-1)!(n-1)!}{(m+n)!}B_{m+n}(x)+\frac{B_m(z)}{m}\cdot\frac{B_n(y)}{n}. \tag{3.3$'$}$$

Corollary 3.1 [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]**.** *Let $x+y+z=1$. Given $m, n \in \mathbb{Z}^+$ we have the following identities:*

$$\begin{aligned}&\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)B_{n-1+k}(y)-\frac{B_m(z)}{m}B_{n-1}(y)\\ =&\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)B_{m-1+k}(z)-\frac{B_n(y)}{n}B_{m-1}(z),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&(-1)^m\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x)B_{n+k}(y)-\frac{m}{2}E_{m-1}(z)E_n(y)\\ =&(-1)^n\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(x)B_{m+k}(z)-\frac{n}{2}E_{n-1}(y)E_m(z)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\frac{(-1)^m}{2}\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x)E_{n-1+k}(y)\\ =&\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)E_{m+k}(z)-\frac{B_n(y)}{n}E_m(z).\end{aligned} \tag{3.8}$$

(3.6) and (3.7) in the case $x=1-2t$ and $y=z=t$ yield the following identities similar to the one of Woodcock.

$$A_{m-1,\,n}(t)=A_{n-1,\,m}(t)\quad\text{and}\quad C_{m,\,n}(t)=C_{n,\,m}(t), \tag{3.9}$$

where

$$A_{m,\,n}(t)=\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^kB_{m+k}(t)B_{n-k}(2t)-B_m(t)\frac{B_n(t)}{n}\qquad(3.10)$$

and

$$C_{m,\,n}(t)=\sum_{k=0}^{n}\binom{n}{k}(-1)^kB_{m+k}(t)E_{n-k}(2t)-\frac{n}{2}E_m(t)E_{n-1}(t).\qquad(3.11)$$

4. Unified identities for Bernoulli and Euler polynomials

Let n be any positive integer. As usual, $\binom{z}{n}=z(z-1)\cdots(z-n+1)/n!$ (and $\binom{z}{0}=1$) even if $z\notin\mathbb{N}$. Observe that

$$\sum_{k=0}^{n}B_k(x)B_{n-k}(y)=\sum_{k=0}^{n}(-1)^k\binom{-1}{k}B_k(x)B_{n-k}(y)$$

and

$$-\sum_{k=1}^{n}\frac{B_k(x)}{k}B_{n-k}(y)=\sum_{k=1}^{n}(-1)^k\binom{-1}{k-1}\frac{B_k(x)}{k}B_{n-k}(y)$$
$$=\lim_{t\to0}\frac{1}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}B_k(x)B_{n-k}(y).$$

Inspired by my above observation, in Sept. 2004 the speaker and H. Pan investigated relations among the sums

$$\sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}P_k(x)Q_{n-k}(y)$$

with $P,Q\in\{B,E\}$.

Theorem 4.1 [Z. W. Sun and H. Pan, `arXiv:math.NT/0409035`]. *Let $n\in\mathbb{Z}^+$ and $x+y+z=1$.*

(i) *If* $r+s+t=n$, *then we have the symmetric relation*

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0 \tag{4.1}$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y). \tag{4.2}$$

(ii) *If* $r+s+t=n-1$, *then*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z)\\&-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\=&\frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).\end{aligned} \tag{4.3}$$

In the case $s=t=-1$, Theorem 4.1 yields that

$$\begin{aligned}&(n+2)\sum_{k=0}^{n}B_k(x)B_{n-k}(y)\\=&\sum_{k=0}^{n}\binom{n+2}{k}((-1)^{n-k}B_k(x)+B_k(y))B_{n-k}(x-y)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\frac{n+1}{2}\sum_{k=0}^{n-1}E_k(x)E_{n-1-k}(y)\\=&\sum_{k=0}^{n}\binom{n+1}{k}\left((-1)^{n-k}B_k(x)-B_k(y)\right)E_{n-k}(x-y).\end{aligned} \tag{4.5}$$

Note that (4.4) in the case $x=y=0$ yields Matiyasevich's identity since $B_{2l+1}=0$ for $l=1,2,3,\ldots$.

We can also deduce from Theorem 4.1 the following result: *If* $n \in \mathbb{Z}^+$ *and* $x+y+z=1$, *then*

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k^2}\left(B_{n-k}(y)+(-1)^nB_{n-k}(z)\right)\\ =&\sum_{k=1}^{n-1}(-1)^{n-k}\frac{B_k(y)}{k}\cdot\frac{B_{n-k}(z)}{n-k}-H_{n-1}\frac{B_n(y)+(-1)^nB_n(z)}{n}.\end{aligned}\tag{4.6}$$

In the case $x=y=0$ and $z=1$, this yields Miki's identity.

Let $l,m,n\in\mathbb{Z}^+$, $l\leqslant\min\{m,n\}$ and $x+y+z=1$. By Theorem 4.1(i),

$$-l\begin{bmatrix}m & n\\ x & y\end{bmatrix}_{\bar n}+m\begin{bmatrix}n & -l\\ y & z\end{bmatrix}_{\bar n}+n\begin{bmatrix}-l & m\\ z & x\end{bmatrix}_{\bar n}=0$$

where $\bar n=m+n-l\in\mathbb{Z}^+$. It follows that

$$\begin{aligned}&\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{n+k-1}{l-1}B_{n-l+k}(x)B_{m-k}(z)\\ &+(-1)^l\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{m+k-1}{l-1}B_{m-l+k}(y)B_{n-k}(z)\\ &=\frac{l}{mn}\sum_{k=0}^{l}(-1)^k\binom{m}{k}\binom{n}{l-k}B_{n-l+k}(x)B_{m-k}(y).\end{aligned}\tag{4.7}$$

In the case $x=y=0$ and $l=z=1$, this yields Woodcock's identity

$$\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}(-1)^kB_{m-k}B_{n-1+k}=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^kB_{n-k}B_{m-1+k}.$$

One can also deduce the von Ettingshausen identity from Theorem 4.1(i).

As $-n+n+n=n$ and $(1-x)+y+(x-y)=1$, Theorem 4.1(i) implies the following new identity:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}^2B_k(x)B_{n-k}(y)\\ =&\sum_{k=0}^{n}\binom{n}{k}\binom{n+k-1}{k}\left((-1)^{n-k}B_k(x)+B_k(y)\right)B_{n-k}(x-y).\end{aligned}\tag{4.8}$$

In particular,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}^2 B_k(x)B_{n-k}(x)\\ =&2\sum_{\substack{k=0\\k\neq n-1}}^{n}\binom{n}{k}\binom{n+k-1}{k}B_k(x)B_{n-k}.\end{aligned} \tag{4.9}$$

5. Some congruences for Euler numbers and q-Euler numbers

Euler numbers modulo an odd integer are trivial. In fact, for any $k\in\mathbb{N}$ and $q\in\mathbb{Z}^+$ we have

$$2^kE_k\left(q+\frac{1}{2}\right)=2^k\sum_{l=0}^{k}\binom{k}{l}\frac{E_l}{2^l}q^{k-l}\equiv E_k=2^kE_k\left(\frac{1}{2}\right)\pmod q$$

and

$$\begin{aligned}&E_k\left(\frac{1}{2}\right)-(-1)^qE_k\left(q+\frac{1}{2}\right)\\ =&\sum_{j=0}^{q-1}\left((-1)^jE_k\left(j+\frac{1}{2}\right)-(-1)^{j+1}E_k\left(j+1+\frac{1}{2}\right)\right)\\ =&2\sum_{j=0}^{q-1}(-1)^j\left(j+\frac{1}{2}\right)^k,\end{aligned}$$

therefore

$$E_k\equiv\sum_{j=0}^{q-1}(-1)^j(2j+1)^k \pmod q \quad \text{providing } 2\nmid q. \tag{5.1}$$

It is natural to determine Euler numbers modulo powers of two. However, this is a difficult task since $1/2$ is not a 2-adic integer.

In a recent paper I determined Euler numbers modulo powers of two in the following explicit way.

Theorem 5.1 [Z. W. Sun, J. Number Theory 115(2005), 371–380]**.** *Let $n \in \mathbb{Z}^+$. If $k \in \mathbb{N}$ is even, then*

$$\frac{3^{k+1}+1}{4}E_k \equiv \frac{3^k}{2}\sum_{j=0}^{2^n-1}(-1)^{j-1}(2j+1)^k\left\lfloor\frac{3j+1}{2^n}\right\rfloor \pmod{2^n} \tag{5.2}$$

where $\lfloor\alpha\rfloor$ denotes the greatest integer not exceeding a real number α, moreover for any positive odd integer m we have the congruence

$$\begin{aligned}&\frac{m^{k+1}-(-1)^{(m-1)/2}}{4}E_k\\ \equiv&\frac{m^k}{2}\sum_{j=0}^{2^n-1}(-1)^{j-1}(2j+1)^k\left\lfloor\frac{jm+(m-1)/2}{2^n}\right\rfloor \pmod{2^n}.\end{aligned} \tag{5.3}$$

Note that $(3^{k+1}+1)/4$ is an odd integer if $k \in \mathbb{N}$ is even.

Let $k, l \in \mathbb{N}$ be even. If $2^n \| (k-l)$ (i.e., $2^n | (k-l)$ but $2^{n+1} \nmid (k-l)$) where $n \in \mathbb{Z}^+$, then $2^n \| (E_k - E_l)$ by Theorem 5.1. In other words, for any $n \in \mathbb{Z}^+$ we have

$$E_k \equiv E_l \pmod{2^n} \iff k \equiv l \pmod{2^n}. \tag{5.4}$$

Unfortunately this discovery of the speaker repeated earlier work. In 1875 M. A. Stern [J. Reine Angew. Math. 79(1875), 67–98] stated that

$$E_{2n+2^s} \equiv E_{2n} + 2^s \pmod{2^{s+1}} \ \text{ for any } n, s \in \mathbb{N}$$

and gave a brief sketch of a proof, then Frobenius amplified Stern's sketch in 1910. In 1979 R. Ernvall said that he could not understand Frobenius' proof and provided his own proof involving umbral calculus. In 2000 an induction proof of the result was given by S. Wagstaff.

The proof of Theorem 5.1 depends heavily on the following explicit congruences for Bernoulli and Euler polynomials.

Theorem 5.2. *Let* $a \in \mathbb{Z}$ *and* $k, m \in \mathbb{Z}^+$. *Let* $q > 1$ *be an integer relatively prime to* m.

(i) [Z. W. Sun, Discrete Math. 262(2003), 253-276] *We have*

$$\begin{aligned}&\frac{1}{k}\left(m^k B_k\left(\frac{x+a}{m}\right) - B_k(x)\right)\\ \equiv&\sum_{j=0}^{q-1}\left(\left\lfloor\frac{a+jm}{q}\right\rfloor + \frac{1-m}{2}\right)(x+a+jm)^{k-1} \pmod{q}.\end{aligned} \tag{5.5}$$

(ii) [Z. W. Sun, J. Number Theory 115(2005), 371–380] *If* $2 \mid q$, *then*

$$\begin{aligned}&\frac{m^{k+1}}{2}E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2}E_k(x)\\ \equiv&\sum_{j=0}^{q-1}(-1)^{j-1}\left(\left\lfloor\frac{a+jm}{q}\right\rfloor + \frac{1-m}{2}\right)(x+a+jm)^{k} \pmod{q}.\end{aligned} \tag{5.6}$$

By the way we mention the following observation of the speaker [Number Theory: Tradition and Modernization, Springer, 2006]:

If $k \in \mathbb{N}$, $a, m \in \mathbb{Z}$ and $2 \nmid m$ then

$$\frac{m^{k+1}}{2}E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2}E_k(x) \in \mathbb{Z}[x]. \tag{5.7}$$

As usual we let $(a;q)_n = \prod_{0 \leqslant k < n}(1 - aq^k)$ for every $n \in \mathbb{N}$, where an empty product is regarded to have value 1 and hence $(a;q)_0 = 1$. For $n \in \mathbb{N}$ we set

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0 \leqslant k < n} q^k,$$

this is the usual q-analogue of n. For any $n, k \in \mathbb{N}$, if $k \leqslant n$ then we call

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{0<r\leqslant n}[r]_q}{\prod_{0<s\leqslant k}[s]_q \cdot \prod_{0<t\leqslant n-k}[t]_q} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

a *q-binomial coefficient*, if $k > n$ then we let $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$. Obviously we have $\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{for all } k, n = 1, 2, 3, \ldots .$$

By this recursion, each q-binomial coefficient is a polynomial in q with integer coefficients.

H. Pan and Z. W. Sun defined q-Euler numbers $E_n(q)$ $(n \in \mathbb{N})$ by

$$\sum_{n=0}^{\infty} E_n(q) \frac{x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}} x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

Multiplying both sides by $\sum_{n=0}^{\infty} q^{\binom{2n}{2}} x^{2n}/(q;q)_{2n}$, we obtain the recursion

$$\sum_{\substack{k=0 \\ 2|k}}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} E_{n-k}(q) = \delta_{n,0} \quad (n \in \mathbb{N})$$

which implies that $E_n(q) \in \mathbb{Z}[q]$. Note that $\lim_{q\to 1} E_n(q) = E_n$.

The usual way to define a q-analogue of Euler numbers is as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_n(q) \frac{x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

It is easy to see that $\tilde{E}_n(q) = q^{\binom{n}{2}} E_n(1/q)$.

Recently, with the help of cyclotomic polynomials, V.J.W. Guo and J. Zeng [European J. Combin., in press] proved that if $m, n, s, t \in \mathbb{N}$, $m - n = 2^s t$ and $2 \nmid t$ then

$$\tilde{E}_{2m}(q) \equiv q^{m-n} \tilde{E}_{2n}(q) \left(\mathrm{mod}\ \prod_{r=0}^{s} (1 + q^{2^r t}) \right).$$

This is a partial q-analogue of Stern's result.

Here is a complete q-analogue of the classical result of Stern.

Theorem 5.3 (H. Pan and Z. W. Sun, Acta Arith., to appear). *Let $n, s, t \in \mathbb{N}$ and $2 \nmid t$. Then*

$$E_{2n}(q) - E_{2n+2^s t}(q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}}. \tag{5.8}$$

A key tool in the proof of Theorem 5.3 is the following lemma.

Lemma 5.1. *For any $n \in \mathbb{N}$ we have*

$$E_{2n}(q) = 1 - \sum_{0<k\leqslant n} (-q;q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2(n-k)}(q). \tag{5.9}$$

The Salié numbers S_n ($n \in \mathbb{N}$) are given by

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \frac{\cosh x}{\cos x} = \frac{e^x + e^{-x}}{e^{ix} + e^{-ix}} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) \Big/ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

In 1965 Carlitz proved that $2^n \mid S_{2n}$ for any $n \in \mathbb{N}$.

H. Pan and Z. W. Sun defined q-Salié numbers by

$$\sum_{n=0}^{\infty} S_n(q) \frac{x^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^{2n}}{(q;q)_{2n}} \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} x^{2n}}{(q;q)_{2n}}.$$

Here is a q-analogue of Carlitz's result equivalent to a conjecture of Guo and Zeng and proved by Pan and Sun [Acta Arith., to appear]: *If $n \in \mathbb{N}$ then $(-q;q)_n = \prod_{0<k\leqslant n}(1+q^k)$ divides $S_{2n}(q)$ in the ring $\mathbb{Z}[q]$.*

A key tool in the proof is the following recursion similar to that in Lemma 5.1.

$$S_{2n}(q) \equiv -\sum_{0<k\leqslant n} (-1)^k (-q;q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2(n-k)}(q) \pmod{(-q;q)_n}.$$

It follows from the following deep congruence due to Pan and Sun:

$$\sum_{\substack{k\in\mathbb{Z} \\ 2k+l\geqslant 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} m \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n} \tag{5.10}$$

provided that $l \in \mathbb{Z}$, $m, n \in \mathbb{N}$ and $m \geqslant 2n$.