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CURIOUS IDENTITIES AND CONGRUENCES INVOLVING BERNOULLI POLYNOMIALS

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Abstract. In this talk we first tell the story how the developments of some curious identities concerning Bernoulli polynomials finally led to the following unified symmetric relation (due to Z. W. Sun and H. Pan): If n is a positive integer, $r+s+t=n$ and $x+y+z=1$, then we have

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y).$$

Let p be a prime, and let $n>0$ and r be integers. We will also talk about some congruences involving the Felck quotient

$$F_p(n,r) := (-p)^{-\lfloor (n-1)/(p-1)\rfloor}\sum_{k\equiv r \pmod p}\binom{n}{k}(-1)^k \in \mathbb{Z}$$

and Bernoulli polynomials (initiated by Z. W. Sun and D. Wan), obtained by p-adic methods. We also mention some related work of D. M. Davis and Z. W. Sun on homotopy exponents of special unitary groups.

1. A symmetric identity on Bernoulli polynomials

Usually number theorists are more interested in congruences rather than combinatorial identities. The following Rogers-Ramanujan identities are very famous in combinatorics:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Here is G. H. Hardy's comment: "**It would be difficult to find more beautiful formulae than the Rogers-Ramanujan identities!**"

Let $\mathbb{N}=\{0,1,2,\ldots\}$ and $\mathbb{Z}^+=\{1,2,3,\ldots\}$. The well-known Bernoulli numbers B_n $(n\in\mathbb{N})$ are rational numbers defined by

$$B_0=1 \ \text{ and } \ \sum_{k=0}^{n}\binom{n+1}{k}B_k=0 \ \ (n\in\mathbb{Z}^+).$$

For $n\in\mathbb{N}$ the Bernoulli polynomial $B_n(x)$ of degree n is given by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}.$$

The exponential generating function of $\{B_n(x)\}_{n\geqslant 0}$ is

$$\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}=\frac{ze^{xz}}{e^z-1}\qquad(|z|<2\pi).$$

Raabe's multiplication formula

$$m^{n-1}\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)=B_n(x)$$

can be easily proved by the generating function method:

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{r=0}^{m-1}m^nB_n\left(\frac{x+r}{m}\right)\right)\frac{z^n}{n!}\\ =&\sum_{r=0}^{m-1}\frac{mze^{mz(x+r)/m}}{e^{mz}-1}=\frac{mze^{xz}}{e^{mz}-1}\sum_{r=0}^{m-1}(e^z)^r\\ =&\frac{mze^{xz}}{e^{mz}-1}\cdot\frac{e^{mz}-1}{e^z-1}=\frac{mze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}mB_n(x)\frac{z^n}{n!}.\end{aligned}$$

Below we give a somewhat surprising symmetric identity on Bernoulli polynomials due to me and my student H. Pan.

Theorem 1.1 [Z. W. Sun and H. Pan, Acta Arith., to appear]**.** *Provided that $x+y+z=1$ and $r+s+t=n\in\mathbb{Z}^+$, we have the symmetric identity*

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0, \tag{1.1}$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y). \tag{1.2}$$

It would be very interesting to compare the symmetric identity (1.1) with the following easy property of determinants:

$$0=\begin{vmatrix} r & s & t \\ r & s & t \\ z & x & y \end{vmatrix} = r\begin{vmatrix} s & t \\ x & y \end{vmatrix} + s\begin{vmatrix} t & r \\ y & z \end{vmatrix} + t\begin{vmatrix} r & s \\ z & x \end{vmatrix}.$$

There is also a similar identity involving both Euler polynomials and Bernoulli polynomials.

Now let us look at some special cases of the symmetric identity (1.1).

(a) [H. Miki, J. Number Theory, 1978] *Let $n\geqslant 4$ be an integer and $H_n=\sum_{k=1}^n 1/k$. Then*

$$\sum_{k=2}^{n-2}\frac{B_kB_{n-k}}{k(n-k)} - \sum_{k=2}^{n-2}\binom{n}{k}\frac{B_kB_{n-k}}{k(n-k)} = 2H_n\frac{B_n}{n}. \tag{1.3}$$

Miki's identity involves both an ordinary convolution and a binomial convolution of Bernoulli numbers, thus it is particularly curious and the usual generating function method does not work for this identity. In the original proof of his identity, Miki showed that the two sides of (1.3) are congruent modulo all sufficiently large primes. In 1982 Shiratani and Yokoyama [Mem. Fac. Sci. Kyushu Univ. Ser. A] reproved Miki's identity by p-adic analysis. In 2005 I. Gessel [J. Number Theory] derived Miki's identity by using Stirling numbers of the second kind.

We can also deduce from Theorem 1.1 the following result: *If* $n \in \mathbb{Z}^+$ *and* $x + y + z = 1$*, then*

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k^2}\left(B_{n-k}(y)+(-1)^nB_{n-k}(z)\right)\\&=\sum_{k=1}^{n-1}(-1)^{n-k}\frac{B_k(y)}{k}\cdot\frac{B_{n-k}(z)}{n-k}-H_{n-1}\frac{B_n(y)+(-1)^nB_n(z)}{n}.\end{aligned}$$

In the case $x = y = 0$ and $z = 1$, this yields Miki's identity.

(b) (Y. Matiyasevich, 1997) *For* $n = 4, 5, \ldots$ *we have*

$$(n+2)\sum_{k=2}^{n-2}B_kB_{n-k}-2\sum_{l=2}^{n-2}\binom{n+2}{l}B_lB_{n-l}=n(n+1)B_n. \tag{1.4}$$

Inspired by Miki's work, Matiyasevich found the above identity by the software *Mathematica* and he was unable to give a proof.

In 2004, Dunne and Schubert [`arXiv:math.NT/0406610`] presented a new approach to Miki's and Matiyasevich's identities motivated by **quantum field theory** and **string theory**.

In the case $s = t = -1$, Theorem 1.1 yields that

$$\begin{aligned}(n+2)\sum_{k=0}^{n} B_k(x)B_{n-k}(y)\\ =\sum_{k=0}^{n}\binom{n+2}{k}((-1)^{n-k}B_k(x)+B_k(y))B_{n-k}(x-y).\end{aligned}$$

In the case $x = y = 0$, this yields Matiyasevich's identity since $B_{2l+1} = 0$ for $l = 1, 2, 3, \ldots$.

Here are the natural polynomial forms of Miki's and Matiyasevich's identities (cf. [Pan and Sun, J. Combin. Theory Ser. A, 2006]):

$$\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(x)}{k(n-k)} - 2\sum_{l=2}^{n}\binom{n-1}{l-1}\frac{B_l B_{n-l}(x)}{l^2} = 2H_{n-1}\frac{B_n(x)}{n}$$

and

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) - 2\sum_{l=2}^{n}\binom{n+1}{l+1}\frac{B_l B_{n-l}(x)}{l+2} = (n+1)B_n(x).$$

(c) [C. F. Woodcock, J. London Math. Soc., 1979] *For any* $m, n \in \mathbb{Z}^+$ *we have the symmetric relation*

$$A_{m-1,\,n} = A_{n-1,\,m}, \tag{1.5}$$

where

$$A_{m,\,n} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^k B_{m+k}B_{n-k}. \tag{1.6}$$

Here is the polynomial form of Woodcock's identity given by H. Pan and Z. W. Sun [J. Combin. Theory Ser. A, 2006]: If $m, n \in \mathbb{Z}^+$ and

$x+y+z=1$ then

$$\begin{aligned}&\frac{(-1)^m}{m}\sum_{k=0}^m\binom{m}{k}B_{m-k}(x)B_{n-1+k}(y)-\frac{B_m(z)}{m}B_{n-1}(y)\\=&\frac{(-1)^n}{n}\sum_{k=0}^n\binom{n}{k}B_{n-k}(x)B_{m-1+k}(z)-\frac{B_n(y)}{n}B_{m-1}(z).\end{aligned}\tag{1.7}$$

(d) [Z. W. Sun, European J. Combin. 2003] *If* $m,n\in\mathbb{N}$ *and* $x+y+z=1$, *then*

$$(-1)^m\sum_{k=0}^m\binom{m}{k}x^{m-k}B_{n+k}(y)=(-1)^n\sum_{k=0}^n\binom{n}{k}x^{n-k}B_{m+k}(z).\tag{1.8}$$

(e) [Z. W. Sun and H. Pan, Acta Arith., to appear] *For any* $n\in\mathbb{Z}^+$ *we have*

$$\begin{aligned}&\sum_{k=0}^n\binom{n}{k}^2B_k(x)B_{n-k}(y)\\=&\sum_{k=0}^n\binom{n}{k}\binom{n+k-1}{k}\left((-1)^{n-k}B_k(x)+B_k(y)\right)B_{n-k}(x-y).\end{aligned}\tag{1.9}$$

In particular,

$$\sum_{k=0}^n\binom{n}{k}^2B_k(x)B_{n-k}(x)=2\sum_{\substack{k=0\\k\neq n-1}}^n\binom{n}{k}\binom{n+k-1}{k}B_k(x)B_{n-k}.\tag{1.10}$$

This follows from Theorem 1.1 since $-n+n+n=n$ and $(1-x)+y+(x-y)=1$.

An important clue to Theorem 1.1 is my following observation during the study of Miki's and Matiyasevich's idenities:

$$\sum_{k=0}^nB_k(x)B_{n-k}(y)=\sum_{k=0}^n(-1)^k\binom{-1}{k}B_k(x)B_{n-k}(y)$$

and

$$-\sum_{k=1}^{n}\frac{B_k(x)}{k}B_{n-k}(y)=\sum_{k=1}^{n}(-1)^k\binom{-1}{k-1}\frac{B_k(x)}{k}B_{n-k}(y)$$
$$=\lim_{t\to 0}\frac{1}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}B_k(x)B_{n-k}(y).$$

Our way to deduce Theorem 1.1 is based on a new method developed by Pan and Sun [J. Combin. Theory Ser. A, 2006] which involves differences and derivatives of polynomials. It is well known that

$$\Delta(B_n(x)) := B_n(x+1)-B_n(x)=nx^{n-1} \text{ and } B_n'(x)=nB_{n-1}(x).$$

Lemma 1.1 [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A, 2006]**.** *Let $P(x), Q(x) \in \mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers. Then $P'(x)=Q'(x)$ if $\Delta(P(x))=\Delta(Q(x))$. Also,*

$$\Delta(P(x)Q(x)) = P(x)\Delta(Q(x))+Q(x)\Delta(P(x))+\Delta(P(x))\Delta(Q(x)).$$

To illustrate the power of Lemma 2.1, let us give a simple proof of Raabe's multiplication formula. Clearly

$$\Delta\left(\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)\right)$$
$$=\sum_{r=0}^{m-1}\left(B_n\left(\frac{x+r+1}{m}\right)-B_n\left(\frac{x+r}{m}\right)\right)$$
$$=B_n\left(\frac{x}{m}+1\right)-B_n\left(\frac{x}{m}\right)=n\left(\frac{x}{m}\right)^{n-1}=\Delta(m^{1-n}B_n(x))$$

and hence

$$\sum_{r=0}^{m-1}\frac{n}{m}B_{n-1}\left(\frac{x+r}{m}\right)=\frac{d}{dx}\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)$$
$$=\frac{d}{dx}(m^{1-n}B_n(x))=m^{1-n}nB_{n-1}(x)$$

for $n = 1, 2, 3, \ldots$. This proves Raabe's formula.

2. Fleck's congruence and related extensions

For a prime p and a p-adic number α, we let $\mathrm{ord}_p(\alpha) = \sup\{a \in \mathbb{Z} : \alpha/p^a \in \mathbb{Z}_p\}$ where $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Theorem 2.1. *Let p be any prime, and let $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$.*

(i) (Fleck, 1913) *We have*

$$\sum_{k\equiv r \pmod{p}} \binom{n}{k}(-1)^k \equiv 0 \ \left(\mathrm{mod}\ p^{\lfloor (n-1)/(p-1)\rfloor}\right); \tag{2.1}$$

(ii) [C. S. Weisman, Michigan J. Math., 1977] *For $a \in \mathbb{Z}^+$ we have*

$$\mathrm{ord}_p\Bigg(\sum_{k\equiv r \pmod{p^a}} \binom{n}{k}(-1)^k\Bigg) \geqslant \left\lfloor \frac{n-p^{a-1}}{\varphi(p^a)} \right\rfloor, \tag{2.2}$$

where φ denotes Euler's totient function.

(iii) [D. Wan, 2005, to appear in Finite Fields Appl.] *If $a \in \mathbb{Z}^+$ and $l \in \mathbb{N}$ then*

$$\mathrm{ord}_p\Bigg(\sum_{k\equiv r \pmod{p^a}} \binom{n}{k}(-1)^k \binom{(k-r)/p^a}{l}\Bigg) \geqslant \left\lfloor \frac{n-lp^a-p^{a-1}}{\varphi(p^a)} \right\rfloor. \tag{2.3}$$

We don't know Fleck's motivation to obtain his congruence which is very fundamental nowadays. Unaware of Fleck's result, Weisman got his inequality during his study of p-adically continuous functions. D. Wan derived his extension of Fleck's and Weisman's congruences via his study of the ψ-opeator which plays an essential role in Fontaine's theory, Iwasawa theory and p-adic Langlands correspondence. A combinatorial proof of (iii) was given by Z. W. Sun and D. Wan [arXiv:math.NT/0512012].

Let p be a prime. If $n \in \mathbb{Z}^+$ then

$$\operatorname{ord}_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{n}{p^i} = \frac{n}{p-1}$$

and hence $\operatorname{ord}_p(n!) \leqslant (n-1)/(p-1)$. Thus, when $a \in \mathbb{Z}^+$, $n \geqslant p^{a-1}$ and $r \in \mathbb{Z}$, by Weisman's result we have

$$\begin{aligned}&\operatorname{ord}_p\bigg(\sum_{k\equiv r \,(\mathrm{mod}\ p^a)} \binom{n}{k}(-1)^k\bigg)\\ \geqslant&\left\lfloor \frac{n-p^{a-1}}{\varphi(p^a)} \right\rfloor = \left\lfloor \frac{\lfloor n/p^{a-1}\rfloor - 1}{p-1} \right\rfloor \geqslant \operatorname{ord}_p\left(\left\lfloor \frac{n}{p^{a-1}} \right\rfloor !\right).\end{aligned}$$

For a topological purpose, few years ago the topologist D. M. Davis at Leigh Univ. conjectured that if $n > l \geqslant 0$ are integers then

$$\operatorname{ord}_2\left(\sum_{k\in\mathbb{N}} \binom{n}{2k} k^l\right) \geq \operatorname{ord}_2\left(\left\lfloor \frac{l+1}{2} \right\rfloor !\right).$$

This and his other related conjectures are indeed very sophisticated! In July 2005 he made his conjectures public. Then I attacked this conjecture and got success finally.

Theorem 2.2 [Davis and Sun, J. Pure Appl. Algebra, in press]**.** *Let p be a prime, $a, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then, for any polynomial $f(x) \in \mathbb{Z}[x]$, we have*

$$\begin{aligned}&\operatorname{ord}_p\bigg(\sum_{k\equiv r \,(\mathrm{mod}\ p^a)} \binom{n}{k}(-1)^k f\left(\frac{k-r}{p^a}\right)\bigg)\\ &\geqslant \operatorname{ord}_p\left(\left\lfloor \frac{n}{p^a} \right\rfloor !\right) + \tau_p(\{r\}_{p^a}, \{n-r\}_{p^a}),\end{aligned} \tag{2.4}$$

where $\tau_p(s,t)=\mathrm{ord}_p\binom{s+t}{s}$ *is the number of carries occurring in the addition of* s *and* t *in base* p*, and* $\{r\}_{p^a}$ *is the least nonnegative residue of* r *modulo* p^a*.*

The *special unitary group* $\mathrm{SU}(n)$ (of degree n) is the space of all $n\times n$ unitary matrices (the conjugate transpose of such a complex matrix equals its inverse) with determinant one. It plays important roles in many areas of mathematics and physics.

Here is an application of Theorem 2.2 in algebraic topology.

Theorem 2.3 [Davis and Sun, J. Pure Appl. Algebra, in press]**.** *For any prime* p *and* $n\geqslant 2$*, some homotopy group* $\pi_i(\mathrm{SU}(n))$ *contains an element of order* $p^{n-1+\mathrm{ord}_p(\lfloor n/p\rfloor !)}$*.*

Numerical examples indicate that Theorem 2.3 is very strong.

The inequality in Theorem 2.2 can be improved when $\deg f<\lfloor n/p^a\rfloor$.

Theorem 2.4 [Sun and Davis, Trans. Amer. Math. Soc., to appear]**.** *Let* p *be a prime, and let* $a,n\in\mathbb{N}$ *and* $r\in\mathbb{Z}$*. Then, for any polynomial* $f(x)\in\mathbb{Z}[x]$*, we have*

$$\begin{aligned}&\mathrm{ord}_p\bigg(\sum_{k\equiv r\ (\mathrm{mod}\ p^a)}\binom{n}{k}(-1)^k f\left(\frac{k-r}{p^a}\right)\bigg)\\ \geqslant&\mathrm{ord}_p\left(\left\lfloor\frac{n}{p^{a-1}}\right\rfloor!\right)-\deg f+\tau_p(\{r\}_{p^{a-1}},\{n-r\}_{p^{a-1}}).\end{aligned}\tag{2.5}$$

where $\{x\}_{p^{a-1}}$ *is regarded as* 0 *if* $a=0$*.*

A particular case of this theorem is the following conjecture of Davis:

For $l, n \in \mathbb{N}$ we have

$$\operatorname{ord}_2\bigg(\sum_{k\in\mathbb{N}}\binom{n}{4k+2}\binom{k}{l}\bigg) \geq \operatorname{ord}_2\left(\left\lfloor\frac{n}{2}\right\rfloor!\right) - l - \operatorname{ord}_2(l!).$$

Quite recently R. M. Wilson [Discrete Math.] used Fleck's congruence and Weisman's extension to reprove the Ax-Katz theorem on solutions of congruences modulo p and deduce various results on codewords in p-ary linear codes with weights. Z. W. Sun [`arXiv:math.NT/0608560`] has extended Wilson's work by taking into account the recent work of Wan, Davis and Sun mentioned above.

Theorem 2.2 of Davis and Sun has an application to p-adic orders of Stirling numbers of the second kind (cf. [Davis and Sun, J. Pure Appl. Algebra, in press]). Inspired by Theorems 2.1, 2.2 and 2.4, my PhD students H. Q. Cao and H. Pan [`arXiv:math.NT/0608564`] established some congruences of similar types for Stirling numbers and Eulerian numbers.

3. Relations between Fleck quotients and Bernoulli polynomials

For $m = 0, 1, 2, \ldots$, the mth order Bernoulli polynomials $B_n^{(m)}(x)$ ($n \in \mathbb{N}$) are defined by

$$\frac{z^m e^{xz}}{(e^z-1)^m} = \sum_{n=0}^{\infty} B_n^{(m)}(x)\frac{z^n}{n!},$$

and those $B_n^{(m)} = B_n^{(m)}(0)$ are called higher-order Bernoulli numbers. The usual Bernoulli polynomials and numbers are $B_n(x) = B_n^{(1)}(x)$ and $B_n = B_n(0) = B_n^{(1)}$ respectively. It can be easily checked that

$$B_n^{(m)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(m)}x^{n-k} \quad\text{and}\quad \frac{B_n^{(m)}}{n!} = \sum_{k_1+\cdots+k_m=n}\frac{B_{k_1}\cdots B_{k_m}}{k_1!\cdots k_m!}.$$

Theorem 3.1 [Z. W. Sun and D. Wan, `arXiv:math.NT/0603462`]. *Let p be a prime, and let $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. Define the Fleck quotient*

$$F_p(n,r) := (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{k \equiv r \pmod p} \binom{n}{k} (-1)^k. \tag{3.1}$$

(i) *If $m \in \mathbb{N}$ and $m \equiv -n \pmod p$, then*

$$F_p(n,r) \equiv -n_*! B_{n^*}^{(m)}(-r) \pmod p, \tag{3.2}$$

where n_* is the smallest positive residue of n modulo $p-1$, and $n^* = p-1-n_*$ is the least nonnegative residue $\{-n\}_{p-1}$ of $-n$ modulo $p-1$.

(ii) $F_p(n+p^b(p-1),r) \equiv F_p(n,r) \pmod{p^b}$ *for any $b = 1,2,3,\ldots$.*

This month I obtained the following further result by p-adic method.

Theorem 3.2 [Z. W. Sun, `arXiv:math.NT/0608328`]. *Let p be a prime, and let $m,n \in \mathbb{Z}^+$, $m \neq n$ and $m \equiv n \pmod{p(p-1)}$. Then*

$$\frac{F_p(m,r)-F_p(n,r)}{m-n} \equiv \frac{(-1)^{n^*}}{n^*!} \sum_{1<k\leqslant n^*} \binom{n^*}{k} \frac{B_k}{k} B_{n^*-k}^{(\{-n\}_p)}(-r) \pmod p. \tag{3.3}$$

The key point in the proof of Theorem 3.2 is the following new theorem on roots of unity.

Theorem 3.3 [Z. W. Sun, `arXiv:math.NT/0608328`]. *Let p be a prime and $\pi = -\sum_{k=1}^{p-1}(1-\zeta_p)^k/k$, where ζ_p is a primitive pth root of unity.*

(i) *We have*

$$\pi^{p-1} \equiv -p \pmod{p^2}, \quad i.e., \ \frac{\pi^{p-1}}{p} \equiv -1 \pmod p. \tag{3.4}$$

(ii) *Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $m \equiv -n \pmod p$, then*

$$(\zeta_p^a - 1)^n \equiv \sum_{j=0}^{p-2} B_j^{(m)} \frac{(a\pi)^{n+j}}{j!} \pmod{p\pi^n}. \tag{3.5}$$

For each $b \in \mathbb{Z}^+$ we have

$$(\zeta_p^a - 1)^{p^b n} \equiv (a\pi)^{p^b n} + p^b n \sum_{1<k<p-1} \frac{B_k}{k!k} (a\pi)^{p^b n+k} \pmod{p^{b+1}\pi^{p^b n}}. \tag{3.6}$$

With help of Theorem 3.2 we can deduce the following result.

Theorem 3.4 [Z. W. Sun, `arXiv:math.NT/0608328`]. *Let p be an odd prime, and let $n \in \mathbb{Z}^+$ with $p-1 \nmid n$.*

(i) *If $n_* \neq p-2$ (i.e., $p-1 \nmid n+1$), then*

$$\frac{F_p(pn,0)}{pn} \equiv \frac{n_*!}{n_*+1} B_{p-1-n_*} \pmod p. \tag{3.7}$$

(ii) *If $r \in \mathbb{Z}$ and $p \nmid r$, then*

$$\begin{aligned} &\frac{(-r)^n F_p(pn,r) + n_*!}{n_*!n_*} + pH_{n_*} - pB_{p-1} + p - 1 \\ &\equiv \frac{pn}{n_*}\left(q_p(r) - \sum_{1<k<p-n_*} \binom{n_*+k}{n_*} \frac{B_k}{kr^k} \right) \pmod{p^2}, \end{aligned} \tag{3.8}$$

where $q_p(r)$ denotes the Fermat quotient $(r^{p-1}-1)/p$.

In 1862 J. Wolstenholme proved further that if $p > 3$ is a prime then

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

This is a fundamental congruence involving binomial coefficients.

Now we give our further extensions of Wolstenholme's congruence via Fleck quotients or extended Fleck quotients.

Theorem 3.5. *Let p be a prime and let $n \in \mathbb{Z}^+$.*

(i) [Z. W. Sun and D. Wan, `arXiv:math.NT/0603462`] *If $2 \leqslant n \leqslant p$ then*

$$\frac{1}{p^n}\sum_{k=1}^{n}\binom{pn-1}{pk-1}(-1)^{pk} \equiv -(n-1)!B_{p-n} \pmod{p}. \tag{3.9}$$

(ii) [Z. W. Sun, `arXiv:math.NT/0608328`] *If $a \in \mathbb{Z}^+$, $d, l, m, \in \mathbb{N}$, $0 < d \leqslant \max\{p^{a-2}, 1\}$, $m < p$ and $2 \leqslant n - l - m \leqslant p$, then*

$$\begin{aligned}&\frac{1}{p^{n-l}}\sum_{l<k\leqslant n}\binom{p^a n - p^{a-1}m - d}{p^a k - p^{a-1}m - d}(-1)^{pk}\binom{k-1}{l}\\ &\equiv \frac{(-1)^{l-1}n!/l!}{\prod_{k=0}^{m}(n-l-k)}B_{p-n+l+m}^{(m+1)} \pmod{p}.\end{aligned} \tag{3.10}$$