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New divisibility results on certain sums of binomial coefficients

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Abstract

In this talk we introduce the speaker's recent divisibility results on certain sums of binomial coefficients. For example, for any positive integers a and n we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\binom{n-1}{k}^a\binom{-n-1}{k}^a\in\mathbf{Z}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^a\binom{-n-1}{k}^a}{4k^2-1}\in\mathbf{Z}.$$

Main Reference:

Z.-W. Sun, *Two new kinds of numbers and and related divisibility results*, preprint, arXiv:1408.5381.

Part I. Two new kinds of numbers and related divisibility results

Some easy facts

Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} 1 = 1 \in \mathbb{Z},$$

$$\frac{2}{n}\sum_{k=0}^{n-1} k = n - 1 \in \mathbb{Z},$$

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1) = 1 \in \mathbb{Z}.$$

For $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$, their arithmetic mean is given by

$$\frac{a_0 + a_1 + \cdots + a_{n-1}}{n} = \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

Apéry numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! Those *Apéry numbers*

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2$$

play important roles in Apéry's proof.

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Remark. I [JNT, 2011] proved the mod p version of the conjectural congruence. The conjecture still remains open!

Arithmetic means involving Apéry numbers

Theorem. Let n be a positive integer.

(i) (Z. W. Sun [J. Number Theory 132(2012)]) We have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k \in \mathbb{Z}.$$

For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

(ii) (Conjectured by Z. W. Sun and proved by V.J.W. Guo and J. Zeng [JNT, 2012])

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \in \mathbb{Z}.$$

Central Delannoy numbers

The central Delannoy numbers are given by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k} \quad (n \in \mathbb{N}).$$

Combinatorial interpretation: D_n is the number of lattice paths from the point $(0,0)$ to (n,n) with only allowed steps $(1,0)$, $(0,1)$ and $(1,1)$.

Theorem [Z.-W. Sun, arXiv1008.3887, Sci. China Math. 57(2014)].

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p} \ \text{ for any odd prime } p$$

(where $(\frac{\cdot}{p})$ denotes the Legendre symbol), and

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)D_k^2 \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

Catalan numbers and Schröder numbers

The Catalan numbers are given by

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1} \in \mathbb{Z} \quad (k = 0, 1, 2, \ldots).$$

In combinatorics, the (large) Schröder numbers are given by

$$S(n) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n}\binom{n+k}{2k}C_k \quad (n \in \mathbb{N}).$$

Both Catalan numbers and Schröder numbers have many combinatorial interpretations. For example, $S(n)$ is the number of lattice paths from the point $(0,0)$ to (n,n) with only allowed steps $(1,0)$, $(0,1)$ and $(1,1)$ which never rise above the line $y = x$.

The numbers R_n $(n = 0, 1, 2, \dots)$

We note that $(2k-1) \mid \binom{2k}{k}$ for all $k \in \mathbb{N}$. This is obvious for $k = 0$. For each $k \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, we have

$$\frac{\binom{2k}{k}}{2k-1} = \frac{2}{2k-1}\binom{2k-1}{k} = \frac{2}{k}\binom{2k-2}{k-1} = 2C_{k-1}.$$

Motivated by this and the (large) Schröder numbers, we introduce a new kind of numbers:

$$R_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{2k-1} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1} \ (n \in \mathbb{N}).$$

Below are the values of $R_0, R_1, \dots, R_{16}$ respectively:

$-1,\ 1,\ 7,\ 25,\ 87,\ 329,\ 1359,\ 6001,\ 27759,\ 132689,\ 649815,$
$3242377,\ 16421831,\ 84196761,\ 436129183,\ 2278835681,\ 11996748255.$

The recurrence of the sequence $(R_n)_{n\geqslant 0}$

Applying the Zeilberger algorithm via `Mathematica 9`, we get the following third-order recurrence for $(R_n)_{n\geqslant 0}$:

$$(n+1)R_n-(7n+15)R_{n+1}+(7n+13)R_{n+2}-(n+3)R_{n+3}=0 \quad \text{for } n\in\mathbb{N}.$$

In contrast, there is a second-order recurrence for Schröder numbers:

$$nS(n)-3(2n+3)S(n+1)+(n+3)S(n+2)=0 \ \ (n=0,1,2,\ldots).$$

So $(R_n)_{n\geqslant 0}$ looks more sophisticated than Schröder numbers.

Theorem (Z. W. Sun, arXiv:1408.5381) Let $p\equiv 1 \pmod 4$ be a prime, and write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$. Then we can determine $x \bmod p^2$ as follows:

$$R_{(p-1)/2}-p\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)(-16)^k}\equiv -2\left(\frac{2}{p}\right)x \pmod{p^2}.$$

Results on R_n and $R_n(x)$

For convenience, we also introduce the associated polynomials

$$R_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{x^k}{2k-1} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{x^k}{2k-1} \in \mathbb{Z}[x].$$

Theorem (Z. W. Sun, arXiv:1408.5381) (i) For any odd prime p,

$$\sum_{k=0}^{p-1} R_k \equiv -p - \left(\frac{-1}{p}\right) \pmod{p^2}.$$

(ii) For any positive integer n, we have

$$R_n(-1) = -(2n+1)$$

and consequently

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{-n}{k}}{2k-1} = -2n.$$

Open Problem. Give a combinatorial proof of the last identity.

Conjectures involving R_k^2

We can deduce that $\frac{3}{n}\sum_{k=0}^{n-1}(2k+1)R_k(x) \in \mathbb{Z}[x]$ for all $n \in \mathbb{Z}^+$.

Conjecture (Z. W. Sun, arXiv:1408.5381) We have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)R_k^2 \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

Also, for any odd prime p we have

$$\sum_{k=0}^{p-1}(2k+1)R_k^2 \equiv 4p\left(\frac{-1}{p}\right) - p^2 \pmod{p^3}.$$

Remark. This conjecture has been confirmed by W.J.W. Guo and J.-C. Liu (J. Number Theory, to appear).

Open Conjecture (Z. W. Sun, arXiv:1408.5381)

$$\frac{3}{n}\sum_{k=0}^{n-1}R_k^2 \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

The numbers S_n and the polynomials $S_n(x)$

Now we introduce another kind of new numbers:

$$S_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}(2k+1) \quad (n = 0, 1, 2, \ldots).$$

We also define the associated polynomials

$$S_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}(2k+1)x^k \quad (n = 0, 1, 2, \ldots).$$

Here are the values of $S_0, S_1, \ldots, S_{12}$ respectively:

1, 7, 55, 465, 4047, 35673, 316521, 2819295, 25173855, 225157881, 2016242265, 18070920255, 162071863425.

Now we list the polynomials $S_0(x), \ldots, S_5(x)$:

$$S_0(x) = 1, \; S_1(x) = 6x + 1, \; S_2(x) = 30x^2 + 24x + 1,$$
$$S_3(x) = 140x^3 + 270x^2 + 54x + 1,$$
$$S_4(x) = 630x^4 + 2240x^3 + 1080x^2 + 96x + 1,$$
$$S_5(x) = 2772x^5 + 15750x^4 + 14000x^3 + 3000x^2 + 150x + 1.$$

Results on S_n and $S_n(x)$

By the Zeilberger algorithm, we get the following recurrence:

$$9(n+1)^2S_n-(19n^2+74n+87)S_{n+1}+(n+3)(11n+29)S_{n+2}=(n+3)^2S_{n+3},$$

which is more complicated than the recurrence for $(R_n)_{n\geqslant 0}$.

Theorem (Z. W. Sun, arXiv:1408.5381) (i) For any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}S_k=\sum_{k=0}^{n-1}\binom{n-1}{k}^2C_k\in\mathbb{Z}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1}S_k(x)\in\mathbb{Z}[x].$$

(ii) For any prime $p>3$, we have

$$\sum_{k=1}^{p-1}\frac{S_k}{k}\equiv p\sum_{k=1}^{p-1}\frac{S_k}{k^2}\equiv-\frac{p}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod{p^2}.$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

A conjecture involving kS_k

Conjecture (Z. W. Sun, arXiv:1408.5381) We have

$$\frac{4}{n^2}\sum_{k=0}^{n-1} kS_k \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

Also, for any prime p we have

$$\sum_{k=0}^{p-1} kS_k \equiv \frac{p^2}{8}\left(5 - 9\left(\frac{p}{3}\right)\right) \pmod{p^3}.$$

Remark. This has been confirmed by V.J.W. Guo and J.-C. Liu [Int. J. Number Theory, to appear].

Conjectures involving s_n, S_n^+ and S_n^-

Conjecture (Z. W. Sun, arXiv:1408.5381) For $n \in \mathbb{N}$ define

$$s_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \frac{1}{2k-1},$$

$$S_n^+ := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^2,$$

$$S_n^- := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)^2 (-1)^k.$$

Then, for any positive integer n, we have

$$\frac{1}{n^2} \sum_{k=0}^{n-1} s_k \in \mathbb{Z}, \quad \frac{1}{n^2} \sum_{k=0}^{n-1} S_k^+ \in \mathbb{Z} \ \text{ and } \ \frac{1}{n^2} \sum_{k=0}^{n-1} S_k^- \in \mathbb{Z}.$$

Remark. I deduced $\sum_{k=0}^{n-1} S_k^{\pm} \equiv 0 \pmod{n}$ for all $n \in \mathbb{Z}^+$. The conjecture for s_n and S_n^+ has been confirmed by V.J.W. Guo and J.-C. Liu [Int. J. Number Theory, to appear], and the conjecture for S_n^- was proved by my student G.-S. Mao.

Part II. New congruences on sums of binomial coefficients

New results on sums of binomial coefficients

Theorem (Sun, arXiv:1408.5381) Let $a_1, \ldots, a_m \in \mathbb{Z}$. Then

$$\sum_{k=0}^{n-1}(\pm 1)^k(2k+1)\prod_{i=1}^{m}\binom{a_in-1}{k} \equiv 0 \pmod{n},$$

$$\sum_{k=0}^{n-1}(\pm 1)^k(4k^3-1)\prod_{i=1}^{m}\binom{a_in-1}{k} \equiv 0 \pmod{n},$$

$$n^2 \,\Big|\, \gcd\Big(\sum_{i=1}^{m}a_i-1, 2\Big)\sum_{k=0}^{n-1}(-1)^{km}(2k+1)\prod_{i=1}^{m}\binom{a_in-1}{k},$$

$$6\sum_{k=0}^{n-1}(-1)^{km}(3k^2+3k+1)\prod_{i=1}^{m}\binom{a_in-1}{k} \equiv 0 \pmod{n^2},$$

$$\sum_{k=0}^{n-1}(-1)^k(4k^3-1)\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k} \equiv 0 \pmod{n^2},$$

$$n^3 \,\Big|\, \gcd\Big(\sum_{i=1}^{m}a_i-1, 2\Big)\sum_{k=0}^{n-1}(3k^2+3k+1)\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}.$$

Some remarks

(i) $\binom{-n-1}{k} = (-1)^k \binom{n+k}{k}$ for all $k, n \in \mathbb{N}$.

(ii) In 2012 Guo and Zeng [Int. J. Number Theory] used q-binomial coefficients to prove that for any $a, b \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (-1)^{(a+b)k} \binom{n-1}{k}^a \binom{-n-1}{k}^b \equiv 0 \pmod{n}.$$

(iii) By the theorem, for any $a, b, n \in \mathbb{Z}^+$ we have the congruence

$$n^2 \,\Big|\, \gcd(a+b-1, 2) \sum_{k=0}^{n-1} (-1)^{(a+b)k} (2k+1) \binom{n-1}{k}^a \binom{-n-1}{k}^b.$$

New congruences on sums of binomial coefficients

Theorem (Sun, arXiv:1408.5381) For any $a, b, n \in \mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^a\binom{-n-1}{k}^a}{4k^2-1}\in\mathbb{Z},\quad \frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^a\binom{-n-1}{k}^a}{\binom{k+2}{2}}\in\mathbb{Z},$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}}{4k^2-1}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z},$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b-1)k}k}{4k^2-1}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z},$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}}{\binom{k+2}{2}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z},$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}(3k+1)}{(2k+1)\binom{2k}{k}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}.$$

Remark. For any $n \in \mathbb{Z}^+$, we have $\sum_{k=1}^{n-1}\frac{\binom{n-1}{k}\binom{-n-1}{k}}{4k^2-1} = -n$.

Corollary. For $n \in \mathbb{N}$ define

$$t_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \frac{1}{2k-1},$$

$$T_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (2k+1),$$

$$T_n^+ = \sum_{k=0}^{n} (2k+1)^2 \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$T_n^- = \sum_{k=0}^{n} (-1)^k (2k+1)^2 \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Then, for any positive integer n, we have

$$\frac{1}{n^3} \sum_{k=0}^{n-1} (2k+1) t_k \in \mathbb{Z}, \ \frac{1}{n^3} \sum_{k=0}^{n-1} (2k+1) T_k \in \mathbb{Z},$$

$$\frac{1}{n^4} \sum_{k=0}^{n-1} (2k+1) T_k^+ \in \mathbb{Z}, \ \frac{1}{n^3} \sum_{k=0}^{n-1} (2k+1) T_k^- \in \mathbb{Z}.$$

The first auxiliary theorem

Theorem (Sun, arXiv:1408.5381) Let $a_1, \ldots, a_m \in \mathbb{Z}$ and $b_1, \ldots, b_m \in \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function with $k \mid f(k)$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{Z}^+$ and set $d = \gcd(a_1, \ldots, a_m, b_1, \ldots, b_m, n)$. Then we have

$$\sum_{k=0}^{n-1} \bar{f}(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \equiv 0 \pmod{d},$$

where $\bar{f}(k) = f(k+1) - (-1)^m f(k)$. If $k^2 \mid f(k)$ for all $k \in \mathbb{N}$, then

$$\begin{aligned}&\sum_{k=0}^{n-1} \bar{f}(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \\ \equiv&(-1)^m \left(\sum_{i=1}^{m} a_i\right) \sum_{0<k<n} \frac{f(k)}{k} \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \pmod{d^2}.\end{aligned}$$

Some ideas in the proof

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \bar{f}(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \\
=&\sum_{k=0}^{n-1} f(k+1) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} - (-1)^m \sum_{k=0}^{n-1} f(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \\
=&\sum_{k=1}^{n} f(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k - 1} - (-1)^m \sum_{k=0}^{n-1} f(k) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k} \\
=&f(n) \prod_{i=1}^{m} \binom{a_i - 1}{b_i + n - 1} + \sum_{0<k<n} f(k) d_k,
\end{aligned}
$$

where

$$
d_k := \prod_{i=1}^{m} \left(\binom{a_i}{b_i + k} - \binom{a_i - 1}{b_i + k} \right) - (-1)^m \prod_{i=1}^{m} \binom{a_i - 1}{b_i + k}
$$

can be written as $\sum_{i=1}^{m} c_{i,k} \binom{a_i}{b_i + k}$ with $c_{i,k} \in \mathbb{Z}$.

Some ideas in the proof

Note that $k \mid f(k)$ and

$$k\binom{a_i}{b_i+k} = a_i\binom{a_i-1}{b_i+k-1} - b_i\binom{a_i}{b_i+k} \equiv 0 \pmod{d}$$

for all $k = 1, 2, 3, \ldots$.

If $k^2 \mid f(k)$ for all $k \in \mathbb{N}$, then for any $0 < k < n$ and $1 \leqslant i < j \leqslant m$ we have

$$\begin{aligned} & f(k)\binom{a_i}{b_i+k}\binom{a_j}{b_j+k} \\ = & \frac{f(k)}{k^2}\left(k\binom{a_i}{b_i+k}\right)\left(k\binom{a_j}{b_j+k}\right) \\ \equiv & 0 \pmod{d^2}. \end{aligned}$$

The second auxiliary theorem

Theorem (Sun, arXiv:1408.5381) Let $a_1, \ldots, a_m \in \mathbb{Z}$, and let $f : \mathbb{N} \to \mathbb{Z}$ be a function with $k^3 \mid f(k)$ for all $k \in \mathbb{N}$. Then, for any positive integer n, we have

$$\sum_{k=0}^{n-1} \Delta f(k) \prod_{i=1}^{m} \binom{a_i n - 1}{k} \binom{-a_i n - 1}{k}$$

$$\equiv n^2(a_1^2 + \cdots + a_m^2) \sum_{0<k<n} \frac{f(k)}{k^2} \prod_{i=1}^{m} \binom{a_i n - 1}{k} \binom{-a_i n - 1}{k} \pmod{n^3},$$

where $\Delta f(k) = f(k+1) - f(k)$.

The third auxiliary theorem

Theorem (Sun, arXiv:1408.5381) Let $a_1, \ldots, a_m$ be positive integers with $\min\{a_1, \ldots, a_m\} = 1$, and let f be a function from $\mathbb{N}$ to the field $\mathbb{Q}$ of rational numbers. Let n be any positive integer.

(i) If $\binom{2k-1}{k} f(k) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, then we have

$$\sum_{k=0}^{n-1} \Delta f(k) \prod_{i=1}^{m} \binom{a_i n - 1}{k} \binom{-a_i n - 1}{k} \in \mathbb{Z}.$$

(ii) If $\binom{2k-1}{k} f(k) \in k\mathbb{Z}$ for all $k \in \mathbb{N}$, then we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \Delta f(k) \prod_{i=1}^{m} \binom{a_i n - 1}{k} \binom{-a_i n - 1}{k} \in \mathbb{Z}.$$

Lemma. For any $k, n \in \mathbb{N}$, we have

$$\frac{k}{\binom{2k-1}{k}} \binom{n}{k} \binom{-n}{k} \equiv 0 \pmod{n}.$$

The fourth auxiliary theorem

Theorem (Sun, arXiv:1408.5381) Let $a, b, n \in \mathbb{Z}^+$. For any function $f : \mathbb{N} \to \mathbb{Z}$ with $f(k)\binom{2k-1}{k} \in \mathbb{Z}$ for all $k \in \mathbb{N}$, we have

$$\sum_{k=0}^{n-1}(f(k+1) - (-1)^{a+b}f(k))\binom{n-1}{k}^a\binom{-n-1}{k}^b \in \mathbb{Z}.$$

Remark. The proof used Abel's partial summation and the fact that $\binom{2k-1}{k} \mid \binom{n}{k}\binom{-n}{k}$ for $k = 0, \ldots, n-1$.

Note that if $f(k) = k/(2k-1)$ for $k \in \mathbb{N}$, then

$$\binom{2k-1}{k}f(k) \in k\mathbb{Z}$$

and

$$\Delta f(k) = \frac{1}{1-4k^2}.$$

Part III. Some open conjectures

A conjecture on central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Conjecture (Sun, 2010). For any $n \in \mathbb{Z}^+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+5)T_k^2 \in \mathbb{Z}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}.$$

A conjecture on Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

A conjecture on $F_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k$

Conjecture (Sun, 2014). For $n \in \mathbb{N}$ define

$$F_n := \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k.$$

For any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)(-1)^k F_k$$

is always an odd integer. Also, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(-1)^k F_k \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Remark. For the Franel numbers $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ $(n \in \mathbb{N})$, I [Adv. in Appl. Math. 2013] proved that $\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2}$ for any prime $p > 3$.

A conjecture with \$48 prize

Conjecture (Z.-W. Sun, 2014) (i) We have

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}K,$$

where

$$K:=L\left(2,\left(\frac{-3}{\cdot}\right)\right)=\sum_{k=1}^{\infty}\frac{\left(\frac{k}{3}\right)}{k^2}=0.781302412896\ldots.$$

(ii) For any prime $p>3$, we have

$$\sum_{k=1}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}\equiv\frac{5}{12}p^2B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}\equiv 4\left(\frac{p}{3}\right)+4p\pmod{p^2},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

Thank you!