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Combinatorial Congruences via Zeilberger's Algorithm and Trees with Prime Vertices

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Abstract

In this talk we introduce the speaker's recent work on combinatorial congruences whose proofs depend heavily on Zeilberger's algorithm.

We will also mention some new conjectures on trees whose vertices are prime numbers.

Part I. Combinatorial congruences via Zeilberger's algorithm

Zeilberger's Algorithm

A hypergeometric sum has the form $S(n) = \sum_{k \geqslant 0} f(n,k)$ with both $f(n,k+1)/f(n,k)$ and $f(n+1,k)/f(n,k)$ rational functions in n and k.

In 1990 Doron Zeilberger published the following historical paper *A fast algorithm for proving terminating hypergeometric series identities*, Discrete Math. 80 (1990), 207–211.

The Zeilberger algorithm developed in this paper (see also Chapter 6 of the famous book $A = B$ by M. Petkovšek, H. S. Wilf and D. Zeilberger) is **an algorithm which finds a linear recurrence with recurrent coefficients polynomials in n for a terminating hypergeometric sum $S(n)$.**

For example, if we use `Mathematica 7` and input
`Zb[Binomial[n,k]^3,{k,0,n},n,2]`,
then we obtain the following second-order recurrence for Franel numbers $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$:

$$-8(n+1)^2 f_n - (7n^2 + 21n + 16) f_{n+1} + (n+2)^2 f_{n+2} = 0.$$

Prove combinatorial identities via Zeilberger's algorithm

Zeilberger's algorithm can be used to prove combinatorial identities. For example, I found that

$$\sum_{k=0}^{n}\binom{2k}{n}\binom{2k}{k}\binom{2(n-k)}{n-k}=2^n f_n$$

since $u_n=2^{-n}\sum_{k=0}^{n}\binom{2k}{n}\binom{2k}{k}\binom{2(n-k)}{n-k}$ satisfies the recurrence relation for $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$. Also, I got the identity

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}\binom{n+2k}{n}=\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$$

during my study of series for $1/\pi$.

An example of my conjectural series for $1/\pi$:

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

where $T_k(b,c)$ is the coefficient of x^k in $(x^2+bx+c)^k$.

Rodriguez-Villegas' conjecture

Let $p>3$ be a prime. In 2003 Rodriguez-Villegas conjectured congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2. Via an advanced approach Mortenson [2005] provided a partial solution with some remaining things including

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 7 \pmod 8,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 11 \pmod{12}.$$

My work on the remaining parts

In 2012 I proved all the remaining parts of the three congruences conjectured by Rodriguez-Villegas.

Theorem 1 (Z. W. Sun, Acta Arith. 2012). Let $p > 3$ be a prime. For each $d = 0, \ldots, (p-1)/2$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{4k}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod 8,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod 4.$$

Detailed proof of the first congruence in Theorem 1

For $d = 0, 1, 2, \ldots$, we define

$$f(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k}.$$

By the Zeilberger algorithm, we find the recursive relation:

$$\begin{aligned}&(3d+1)(6d+1)f(d) - (3d+2)(6d+5)f(d+1)\\ =&\frac{(3p-1)(3p-2)(2d+1)}{2^{2p-1}27^{p-1}p}\binom{2p}{p+2d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1}.\end{aligned}$$

Note that

$$\frac{1}{p}\binom{2p-2}{p-1} = C_{p-1} \in \mathbb{Z},\ (3p-2)\binom{3p-3}{p-1} = p\binom{3p-2}{p} \equiv 0 \pmod{p},$$

$$\begin{aligned}(2d+1)\binom{2p}{p+2d+1} \equiv&(p+2d+1)\binom{2p}{p+2d+1} = 2p\binom{2p-1}{p+2d}\\ \equiv&0 \pmod{p}.\end{aligned}$$

Detailed proof of the first congruence in Theorem 1

Thus, for each $d = 0, \ldots, (p-1)/2$ we have

$$(3d+1)(6d+1)f(d) \equiv (3d+2)(6d+5)f(d+1) \pmod{p^2}.$$

Suppose that $p \equiv 5 \pmod 6$ and $0 \leqslant d < (p-1)/2$. Then $3d+1, 6d+1 \neq p, 2p$ and hence $3d+1, 6d+1 \not\equiv 0 \pmod p$. So

$$f(d+1) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.$$

Therefore

$$f\left(\frac{p-1}{2}\right) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}$$

$$\implies f(0) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

Detailed proof of the first congruence in Theorem 1

Note that

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \\
=&\frac{\binom{2p-2}{p-1}\binom{3p-3}{p-1}}{108^{p-1}} \\
=&108^{1-p}\frac{p}{2p-1}\binom{2p-1}{p}\frac{p}{3p-2}\binom{3p-2}{p} \\
\equiv&0 \pmod{p^2}.
\end{aligned}$$

This concludes the proof of

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

A new result via the Zeilberger algorithm

Theorem 2 (Z. W. Sun, Acta Arith. 2012) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k} \equiv 0 \pmod{p^2}$$

for all $d \in \{0, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$.

We prove the theorem (known for $d = 0$) via the Z-algorithm. For $d = 0, 1, 2, \ldots$ set

$$u_d = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k} = \sum_{d \leqslant k < p} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2 u_d - (2d+3)^2 u_{d+2} = \frac{(2p-1)^2(d+1)}{64^{p-1} p} \binom{2p}{p+d+1} \binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod p \text{ and } p \Big| (d+1) \binom{2p}{p+d+1}.$$

A new result via the Zeilberger algorithm

So, if $0 \leqslant d < p-2$ then

$$(2d+1)^2 u_d \equiv (2d+3)^2 u_{d+2} \pmod{p^2}.$$

For $d \in \{0, \ldots, p-3\}$ with $d \equiv (p+1)/2 \pmod 2$, clearly $p \neq 2d+1 < 2p$ and hence

$$u_{d+2} \equiv 0 \pmod{p^2} \implies u_d \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$ then $p-1 \equiv (p+1)/2 \pmod 2$;
if $p \equiv 1 \pmod 4$ then $p-2 \equiv (p+1)/2 \pmod 2$.
Thus, if $d \in \{p-1, p-2\}$ and $d \equiv (p+1)/2 \pmod 2$, then $d \geqslant (p+1)/2$ and hence $u_d \equiv 0 \pmod{p^2}$ since $p \mid \binom{2k}{k}$ for $d \leqslant k \leqslant p-1$.

It follows that $u_d \equiv 0 \pmod{p^2}$ for all $d \in \{0, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$.

Another theorem via the Zeilberger algorithm

Theorem 3 (Z. W. Sun, J. Number Theory, 2012). Let p be an odd prime and let x be any p-adic integer.

(i) If $x \equiv 2k \pmod p$ with $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.$$

(ii) If $x \equiv k \pmod p$ with $k \in \{0, \ldots, p-1\}$, then

$$\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}.$$

(iii) If $p > 3$ and $x \equiv -2k \pmod p$ for some $k \in \{1, \ldots, \lfloor p/3 \rfloor\}$. Then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^3 \equiv 0 \pmod{p^2}.$$

Comments

It is interesting to compare Theorem 3 (i)-(ii) with the known identities

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2=(-1)^n\binom{2n}{n}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}.$$

Here is a consequence of Theorem 3(i)-(ii).

Corollary (conjectured by Rodriguez-Villegas and proved by Mortenson). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}=\sum_{k=0}^{p-1}\binom{-1/2}{k}^2\equiv(-1)^{(p-1)/2}\pmod{p^2}.$$

Conjecture (Z. W. Sun, J. Number Theory 2012). We may replace 3 in Theorem 3(iii) by any odd integer greater than one.

Proof of Theorem 3(i)

Define

$$f_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{2k+py}{r}^2 \qquad \text{for } k \in \mathbb{N}.$$

We want to prove that

$$f_k(y) \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2}$$

for any p-adic integer y and $k \in \{0, 1, \ldots, (p-1)/2\}$.

Applying the Zeilberger algorithm via `Mathematica 7`, we find

$$(py+2k+2)f_{k+1}(y) + 4(py+2k+1)f_k(y)$$
$$= \frac{(p(y-1)+2k+3)^2 F_k(y)}{(py+2k+1)(py+2k+2)^2} \binom{py+2k+2}{p-1}^2,$$

where

$$F_k(y) = 14+34k+20k^2-10p-12kp+2p^2+17py+20kpy-6p^2y+5p^2y^2.$$

Proof of Theorem 3(i) (continued)

It follows that

$$f_k(y) \equiv -\frac{py+2k+2}{4(py+2k+1)} f_{k+1}(y) \pmod{p^2} \quad \text{for } k = 0, \ldots, \frac{p-3}{2}.$$

If $0 \leqslant k < (p-1)/2$ and

$$f_{k+1}(y) \equiv (-1)^{k+1} \binom{2(k+1)+py}{k+1} \pmod{p^2},$$

then

$$\begin{aligned} f_k(y) \equiv & -\frac{py+2k+2}{4(py+2k+1)} (-1)^{k+1} \binom{2(k+1)+py}{k+1} \\ = & \frac{(-1)^k (py+2k+2)^2}{4(k+1)(py+k+1)} \binom{2k+py}{k} \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2}. \end{aligned}$$

Proof of Theorem 3(ii)

Define

$$g_k(y) := \sum_{r=0}^{p-1} \binom{k+py}{r}^2 \quad \text{for } k \in \mathbb{N}.$$

By the Zeilberger algorithm, we have the recursion

$$\begin{aligned}&(py+k+1)g_{k+1}(y) - 2(2py+2k+1)g_k(y)\\ =& -\frac{(p(y-1)+k+2)^2(3py-2p+3k+3)}{(py+k+1)^2}\binom{py+k+1}{p-1}^2.\end{aligned}$$

It follows that if $k \in \{0, \ldots, p-2\}$ and y is a p-adic integer then

$$\begin{aligned} g_{k+1}(y) &\equiv \binom{2(k+1)+2py}{k+1} \pmod{p^2}\\ \Longrightarrow g_k(y) &\equiv \binom{2k+2py}{k} \pmod{p^2}.\end{aligned}$$

Proof of Theorem 3(iii)

Define
$$w_k(y) := \sum_{r=0}^{p-1}(-1)^r\binom{py-2k}{r}^3 \quad \text{for } k\in\mathbb{N}.$$
We want to show that $w_k(y)\equiv 0 \pmod{p^2}$ for any p-adic integer y and $k\in\{1,\ldots,\lfloor(p-1)/3\rfloor\}$.
By the Zeilberger algorithm, for $k=0,1,2,\ldots$ we have

$$\begin{aligned}&(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\\&\qquad=\frac{P(k,p,y)(p(1-y)+2k-1)^3}{(py-2k)^3(py-2k-1)^3}\binom{py-2k}{p-1}^3\end{aligned}$$

where $P(k,p,y)$ is a suitable polynomial in k,p,y with integer coefficients such that $P(0,p,y)\equiv 0 \pmod{p^2}$. (Here we omit the explicit expression of $P(k,p,y)$ since it is complicated.)

Proof of Theorem 3(iii) (continued)

Note that

$$w_1(0)=\sum_{r=0}^{p-1}(-1)^r\binom{-2}{r}^3=\sum_{r=0}^{p-1}(r+1)^3=\frac{p^2(p+1)^2}{4}\equiv 0 \pmod{p^2}.$$

Fix a p-adic integer y. If $y\neq 0$, then the recursion with $k=0$ yields

$$\begin{aligned}&3(3py-2)(3py-4)w_1(y)\\ \equiv&\frac{P(0,p,y)(p(1-y)-1)^3}{(py)^3(py-1)^3}\left(\frac{py}{p-1}\binom{p(y-1)+p-1}{p-2}\right)^3\\ \equiv&0 \pmod{p^2}\end{aligned}$$

and hence $w_1(y)\equiv 0 \pmod{p^2}$. If $1<k+1\leqslant\lfloor(p-1)/3\rfloor$, then

$$\begin{aligned}&(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\\ \equiv&0 \pmod{p^3}\end{aligned}$$

by the recursion since $\binom{py-2k}{p-1}=\frac{p}{py-2k+1}\binom{py-2k+1}{p}\equiv 0 \pmod{p}$.

Proof of Theorem 3(iii) (continued)

Thus, when $1 < k+1 \leqslant \lfloor (p-1)/3 \rfloor$ we have

$$w_k(y) \equiv 0 \pmod{p^2}$$
$$\Longrightarrow w_{k+1}(y) \equiv 0 \pmod{p^2}.$$

So, by induction, we have $w_k(y) \equiv 0 \pmod{p^2}$ for all $k = 1, \ldots, \lfloor (p-1)/3 \rfloor$.

Supercongruences involving products of two binomial coefficients

In 2003 Rodriguez-Villegas conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv(-1)^{(p-1)/2}\pmod{p^2}$$

and this was proved by E. Mortenson in 2003.

For a sequence $(a_n)_{n\geqslant0}$, its dual sequence $(a_n^*)_{n\geqslant0}$ is defined by

$$a_n^*=\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k.$$

It is well known that $(a_n^*)^*=a_n$ for all $n=0,1,2,\ldots$.

Theorem 4 (Z. W. Sun, Finite Fields Appl., 2013). Let $p>3$ be a prime and let $(a_n)_{n\geqslant0}$ be any sequence of p-adic integers. Then

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}a_k\equiv(-1)^{(p-1)/2}\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}a_k^*\pmod{p^2}.$$

A corollary

As $\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k = (1-x)^n$, Theorem 4 has the following consequence.

Corollary. Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}x^k \equiv (-1)^{(p-1)/2}\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}(1-x)^k \pmod{p^2}.$$

In particular, if $p \equiv 3 \pmod 4$ then

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2}.$$

Proof of Theorem 4

Observe that

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} a_k^* \\
&= \sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \sum_{m=0}^{k} \binom{k}{m} (-1)^m a_m \\
&= \sum_{m=0}^{p-1} (-1)^m a_m \sum_{k=m}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \binom{k}{m} \pmod{p^2}.
\end{aligned}$$

So it suffices to prove that

$$h_p(m) := \sum_{k=m}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \binom{k}{m} \equiv (-1)^{(p-1)/2} \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2}$$

for all $m = 0, \ldots, p-1$.

Proof of Theorem 4 (continued)

For $0 \leqslant m < n$ define

$$h_n(m) := \sum_{k=m}^{n-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\binom{k}{m}.$$

By the Zeilberger algorithm, for $m, n \in \mathbb{N}$ with $m < n-1$, we have

$$\begin{aligned}&36(m+1)^2 h_n(m+1) + (6m+1)(6m+5)h_n(m)\\ =&\frac{(6n-1)(6n-5)}{432^{n-1}}\binom{n-1}{m}\binom{3n-3}{n-1}\binom{6n-6}{3n-3}.\end{aligned}$$

Clearly $\binom{3p-3}{p-1} = \frac{p}{3p-2}\binom{3p-2}{p} \equiv 0 \pmod{p}$, and

$$(6p-5)\binom{6p-6}{3p-3} = \frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)}\binom{6p-3}{3p-3} \equiv 0 \pmod{p}.$$

So, for every $m = 0, \ldots, p-2$ we have

$$36(m+1)^2 h_p(m+1) + (6m+1)(6m+5)h_p(m) \equiv 0 \pmod{p^2}.$$

Proof of Theorem 4 (continued)

For $0 \leqslant m < p-1$, since

$$-\frac{(6m+1)(6m+5)}{36(m+1)^2} \cdot \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} = \frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}},$$

we have

$$h_p(m) \equiv (-1)^{(p-1)/2} \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2}$$

$$\Longrightarrow h_p(m+1) \equiv (-1)^{(p-1)/2} \frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}} \pmod{p^2}.$$

As $h_p(0) = \sum_{k=0}^{p-1} \binom{6k}{3k}\binom{3k}{k}/432^k \equiv (-1)^{(p-1)/2} \pmod{p^2}$, this yields that

$$h_p(m) \equiv (-1)^{(p-1)/2} \frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \quad (\mathrm{mod}\ p^2)$$

for all $m = 0, \ldots, p-1$.

Some open conjectures on combinatorial congruences

I have raised over 100 open conjectures on combinatorial congruences. Here I state few of them.

Conjecture (Z. W. Sun, 2009-2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k+1}}{48^k} \equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{(2k+1)64^k} \equiv (-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^3}{64^k} \equiv 8 \pmod{p^2} \ \text{ if } p \equiv 1 \pmod 4,$$

$$\sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} \equiv -\frac{3}{p}(2^{p-1}-1)^2 \pmod{p^2}.$$

Remark. L. L. Zhao, Z. W. Sun and H. Pan [Proc. AMS 2010] proved that $\sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} \equiv 0 \pmod p$ for any prime $p > 3$.

Part II. Trees with prime vertices

Alternating sums of primes

Let p_n be the nth prime and define

$$s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1} p_1.$$

For example,

$$s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8.$$

Note that

$$s_{2n} = \sum_{k=1}^{n} (p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n} (p_{2k+1} - p_{2k}) + p_1 > 0.$$

Let $1 \leqslant k < n$. If $n - k$ is even, then

$$s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0.$$

If $n - k$ is odd, then

$$s_n - s_k = \sum_{l=k+1}^{n} (-1)^{n-l} p_l - 2 \sum_{j=1}^{k} (-1)^{k-j} p_j \equiv n - k \equiv 1 \pmod 2.$$

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.

An amazing recurrence for primes

We may compute the $(n+1)$-th prime p_{n+1} in terms of $p_1, \ldots, p_n$.

Conjecture (Z. W. Sun, J. Number Theory 2013). For any positive integer $n \neq 1, 2, 4, 9$, the $(n+1)$-th prime p_{n+1} is the least positive integer m such that

$$2s_1^2, \ldots, 2s_n^2$$

are pairwise distinct modulo m.

Remark. I have verified the conjecture for $n \leqslant 10^5$, and proved that $2s_1^2, \ldots, 2s_n^2$ **are indeed pairwise distinct modulo** p_{n+1}.

Let $1 \leqslant j < k \leqslant n$. Then

$$0 < |s_k - s_j| \leqslant \max\{s_k, s_j\} \leqslant \max\{p_k, p_j\} \leqslant p_n < p_{n+1}.$$

Also, $s_k + s_j \leqslant p_k + p_j < 2p_{n+1}$. If $2 \nmid k - j$, then

$$s_k + s_j = p_k - p_{k-1} + \ldots + p_{j+1} \leqslant p_k < p_{n+1}.$$

If $2 \mid k - j$, then $s_k \equiv s_j \pmod 2$ and hence $s_k + s_j \neq p_{n+1}$. Thus $2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}$.

Conjecture on alternating sums of consecutive primes

Conjecture (Z. W. Sun, J. Number Theory, 2013). For any positive integer m, there are consecutive primes $p_k, \ldots, p_n$ $(k < n)$ not exceeding $2m + 2.2\sqrt{m}$ such that

$$m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k.$$

(Moreover, we may even require that $m < p_n < m + 4.6\sqrt{m}$ if $2 \nmid m$ and $2m - 3.6\sqrt{m+1} < p_n < 2m + 2.2\sqrt{m}$ if $2 \mid m$.)

Examples.

$$10 = 17 - 13 + 11 - 7 + 5 - 3;$$

$$20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;$$

$$303 = p_{76} - p_{75} + \cdots + p_{52},$$

$$p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \ \ p_{52} = 239;$$

$$2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43},$$

$$p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \ p_{43} = 191.$$

The conjecture has been verified for m up to 10^7.

Prize. I would like to offer 1000 US dollars for the first proof.

A tree with prime vertices

Conjecture (Z. W. Sun, Feb. 24, 2013). For a prime p define $f(p)$ as the least prime p_n such that

$$p = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k \quad \text{for some } k < n.$$

Construct a simple (undirected) graph G as follows: The vertex set of G is the set of all primes, and for the vertices p and $q > p$ there is an edge connecting p and q if and only if $f(p) = q$. Then the graph G constructed above is connected and hence it is a tree!

For example,

$$2 \to 5 \to 7 \to 13 \to 17 \to 23 \to 31 \to 37 \to 43 \to 53 \to 59 \to 67$$
$$\to 73 \to 83 \to 89 \to 101 \to 109 \to 113 \to 131 \to 149$$

and

$$71 \to 79 \to 97 \to 107 \to 139 \to 149,$$

hence there is a unique path connecting the vertices 2 and 71.

Goldbach's conjecture and the Collatz conjecture

The following two conjectures are famous and difficult.

Goldbach's Conjecture. Any even number greater than 2 can be written as a sum of two primes.

Collatz's Conjecture. Let a_1 be any positive integer, and let

$$a_{n+1} = \begin{cases} 3a_n + 1 & \text{if } 2 \nmid n, \\ a_n/2 & \text{if } 2 \mid n. \end{cases}$$

Then $a_N = 1$ for some positive integer N.

For example,

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40$$
$$\to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

Another tree with prime vertices

Conjecture (Z. W. Sun, Feb. 24, 2013). For a prime p define $g(p)$ as the least prime $q > p$ such that $2(p+1) - q$ is prime. Construct a simple (undirected) graph G as follows: The vertex set of G is the set of all primes, and for the vertices p and $q > p$ there is an edge connecting p and q if and only if $g(p) = q$. Then the graph G constructed above is connected and hence it is a tree!

For example,

$$\begin{aligned}&2 \to 3 \to 5 \to 7 \to 11 \to 13 \to 17 \to 19 \to 23 \to 29 \to 31 \to 41,\\ &\to 43 \to 47 \to 53 \to 61 \to 71 \to 73 \to 89 \to 97 \to 107 \to 109\\ &\to 113 \to 127 \to 149 \to 151 \to 167 \to 173 \to 181 \to 191.\end{aligned}$$

A new Collatz-type conjecture

Conjecture 2 (Z. W. Sun, Feb. 28, 2013) For any positive integer n, define

$$f(n) = \begin{cases} (p+1)/2 & \text{if } 4 \mid p+1, \\ p & \text{otherwise,} \end{cases}$$

where p is the least prime greater than n with $2(n+1)-p$ prime. If $a_1 \in \{3, 4, \ldots\}$ and $a_{k+1} = f(a_k)$ for $k = 1, 2, 3, \ldots$, then $a_N = 4$ for some positive integer N.

For example,

$$45 \to 61 \to 36 \to 37 \to 24 \to 16 \to 17 \to 10 \to 6 \to 4 \to 5 \to 4.$$

Practical numbers

A positive integer n is called a *practical* number if every $m = 1, \ldots, n$ can be written as a sum of some distinct divisors of n, i.e., there are distinct divisors $d_1, \ldots, d_k$ of n such that

$$\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.$$

For example, 6 is practical since $1, 2, 3, 6$ divides 6, and also $4 = 1 + 3$ and $5 = 2 + 3$. As any positive integer has a unique representation in base 2 with digits in $\{0, 1\}$, powers of 2 are all practical. 1 is the only odd practical number.

Practical numbers below 50:

1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.

Goldbach-type results for practical numbers

Theorem (Stewart [Amer. J. Math., 76(1954)]). If $p_1 < \cdots < p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers then $m = p_1^{q_1} \cdots p_r^{a_r}$ is practical if and only if $p_1 = 2$ and

$$p_{s+1} - 1 \leqslant \sigma\big(p_1^{a_1} \cdots p_s^{a_s}\big) \quad \text{for all} \ \ 0 < s < r,$$

where $\sigma(n)$ stands for the sum of all divisors of n.

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

Theorem (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers m with $m - 2$ and $m + 2$ also practical.

Conjecture (Sun, 2013). Any integer $n > 4$ can be written as $p + q/2$, where p and q are practical numbers smaller than n.

Two conjectures involving practical numbers

Conjecture (Z. W. Sun, Feb. 2013). For a practical number p, define $h(p)$ as the least practical number $q > p$ such that $2(p+1) - q$ is practical. Construct a simple (undirected) graph H as follows: The vertex set of H is the set of all practical numbers, and for two vertices p and $q > p$ there is an edge connecting p and q if and only if $h(p) = q$. Then the graph H is connected and hence it is a tree!

Conjecture (Z. W. Sun, Feb. 2013). For any integer $n > 0$ define

$$g(n) = \begin{cases} q/2 & \text{if } 4 \mid q, \\ q & \text{if } 4 \mid q - 2, \end{cases}$$

where q is the least practical number greater than n with $2(n+1) - q$ practical. If $b_1 \in \{4, 5, \ldots\}$ and $b_{k+1} = g(b_k)$ for $k = 1, 2, 3, \ldots$, then $b_N = 4$ for some positive integer N.

Example. If we start from $b_1 = 316$ then we get the sequence

316, 330, 342, 378, 190, 110, 126, 64, 66, 78,
40, 42, 54, 28, 30, 16, 18, 10, 8, 6, 4, 6, 4, ...

Thank you!