Combinatorial Congruences via Zeilberger’s Algorithm and Trees with Prime Vertices

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/~zwsun

July 14, 2013
Abstract

In this talk we introduce the speaker’s recent work on combinatorial congruences whose proofs depend heavily on Zeilberger’s algorithm.

We will also mention some new conjectures on trees whose vertices are prime numbers.
Part I. Combinatorial congruences via Zeilberger’s algorithm
Zeilberger’s Algorithm

A hypergeometric sum has the form \( S(n) = \sum_{k \geq 0} f(n, k) \) with both \( f(n, k + 1)/f(n, k) \) and \( f(n + 1, k)/f(n, k) \) rational functions in \( n \) and \( k \).

In 1990 Doron Zeilberger published the following historical paper \textit{A fast algorithm for proving terminating hypergeometric series identities}, Discrete Math. 80 (1990), 207–211.

The Zeilberger algorithm developed in this paper (see also Chapter 6 of the famous book \( A = B \) by M. Petkovšek, H. S. Wilf and D. Zeilberger) is \textbf{an algorithm which finds a linear recurrence with recurrent coefficients polynomials in \( n \) for a terminating hypergeometric sum \( S(n) \)}.

For example, if we use Mathematica 7 and input 
\[
\text{Zb}[\text{Binomial}[n,k]^3,\{k,0,n\},n,2],
\]
then we obtain the following second-order recurrence for Franel numbers \( f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \):

\[
-8(n+1)^2f_n - (7n^2 + 21n + 16)f_{n+1} + (n+2)^2f_{n+2} = 0.
\]
Zeilberger’s algorithm can be used to prove combinatorial identities. For example, I found that

\[ \sum_{k=0}^{n} \binom{2k}{n} \binom{2k}{k} \frac{(2(n-k))}{n-k} = 2^n f_n \]

since \( u_n = 2^{-n} \sum_{k=0}^{n} \binom{2k}{n} \binom{2k}{k} \left( \frac{2(n-k)}{n-k} \right) \) satisfies the recurrence relation for \( f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \). Also, I got the identity

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(2k)}{n} \binom{n+2k}{k} = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(n+k)}{k} \]

during my study of series for \( 1/\pi \).

An example of my conjectural series for \( 1/\pi \):

\[ \sum_{k=0}^{\infty} \frac{126k + 31}{(-80)^{3k}} T_k^3(22, 21^2) = \frac{880\sqrt{5}}{21\pi} \]

where \( T_k(b, c) \) is the coefficient of \( x^k \) in \( (x^2 + bx + c)^k \).
Rodriguez-Villegas’ conjecture

Let $p > 3$ be a prime. In 2003 Rodriguez-Villegas conjectured congruences on

$$
\sum_{k=0}^{p-1} \frac{(2k) \binom{2}{k} (3k) \binom{3}{k}}{108^k}, \quad \sum_{k=0}^{p-1} \frac{(2k) \binom{2}{k} (4k) \binom{4}{2k}}{256^k}, \quad \sum_{k=0}^{p-1} \frac{(2k) (3k) (6k) \binom{3}{k} \binom{6}{3k}}{12^3 3^k}
$$

modulo $p^2$. Via an advanced approach Mortenson [2005] provided a partial solution with some remaining things including

$$
\sum_{k=0}^{p-1} \frac{(2k) \binom{2}{k} (3k) \binom{3}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod{6},
$$

$$
\sum_{k=0}^{p-1} \frac{(2k) \binom{2}{k} (4k) \binom{4}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 7 \pmod{8},
$$

$$
\sum_{k=0}^{p-1} \frac{(2k) (3k) (6k) \binom{3}{k} \binom{6}{3k}}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 11 \pmod{12}.
$$
My work on the remaining parts

In 2012 I proved all the remaining parts of the three congruences conjectured by Rodriguez-Villegas.

Theorem 1 (Z. W. Sun, Acta Arith. 2012). Let \( p > 3 \) be a prime. For each \( d = 0, \ldots, (p - 1)/2 \), we have

\[
\sum_{k=0}^{p-1} \frac{{2k \choose k+2d} {2k \choose k} {3k \choose k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod{6},
\]

\[
\sum_{k=0}^{p-1} \frac{{2k \choose k+2d} {2k \choose k} {4k \choose 2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod{8},
\]

\[
\sum_{k=0}^{p-1} \frac{{2k \choose k+2d} {3k \choose k} {6k \choose 3k}}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod{4}.
\]
Detailed proof of the first congruence in Theorem 1

For \( d = 0, 1, 2, \ldots, \) we define

\[
f(d) = \sum_{k=0}^{p-1} \frac{(2k) (2k) (3k)}{(k+2d) (k) (k)} \cdot 108^k.
\]

By the Zeilberger algorithm, we find the recursive relation:

\[
(3d + 1)(6d + 1)f(d) - (3d + 2)(6d + 5)f(d + 1) = (3p - 1)(3p - 2)(2d + 1) \cdot \left( \frac{2p}{p + 2d + 1} \right) \left( \frac{2p - 2}{p - 1} \right) \left( \frac{3p - 3}{p - 1} \right).
\]

Note that

\[
\frac{1}{p} \left( \frac{2p - 2}{p - 1} \right) = C_{p-1} \in \mathbb{Z}, \quad (3p - 2) \left( \frac{3p - 3}{p - 1} \right) = p \left( \frac{3p - 2}{p} \right) \equiv 0 \pmod{p},
\]

\[
(2d + 1) \left( \frac{2p}{p + 2d + 1} \right) \equiv (p + 2d + 1) \left( \frac{2p}{p + 2d + 1} \right) = 2p \left( \frac{2p - 1}{p + 2d} \right) \equiv 0 \pmod{p}.
\]
Detailed proof of the first congruence in Theorem 1

Thus, for each \( d = 0, \ldots, (p - 1)/2 \) we have

\[
(3d + 1)(6d + 1)f(d) \equiv (3d + 2)(6d + 5)f(d + 1) \pmod{p^2}.
\]

Suppose that \( p \equiv 5 \pmod{6} \) and \( 0 \leq d < (p - 1)/2 \). Then \( 3d + 1, 6d + 1 \not\equiv p, 2p \pmod{p} \) and hence \( 3d + 1, 6d + 1 \not\equiv 0 \pmod{p} \). So

\[
f(d + 1) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.
\]

Therefore

\[
f \left( \frac{p - 1}{2} \right) = \sum_{k=0}^{p-1} \frac{(2k)(2k)(3k)}{(k+p-1)108^k} \equiv 0 \pmod{p^2}
\]

\[
\implies f(0) = \sum_{k=0}^{p-1} \frac{(2k)^2(3k)}{108^k} \equiv 0 \pmod{p^2}.
\]
Detailed proof of the first congruence in Theorem 1

Note that

\[
\sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1} \binom{2k}{k} \binom{3k}{k}}{108^k}
\]

\[
= \frac{(2p-2)(3p-3)}{(p-1)(p-1)}
\]

\[
= \frac{108^{p-1} \cdot p}{2p-1} \left( \binom{2p-1}{p} \right) \frac{p}{3p-2} \left( \binom{3p-2}{p} \right)
\]

\(\equiv 0 \pmod{p^2}.
\]

This concludes the proof of

\[
\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.
\]
A new result via the Zeilberger algorithm

**Theorem 2** (Z. W. Sun, Acta Arith. 2012) Let $p > 3$ be a prime. Then

$$
\sum_{k=0}^{p-1} \frac{(2k)^2 \binom{2k}{k} \binom{2k}{k+d}}{64^k} \equiv 0 \pmod{p^2}
$$

for all $d \in \{0, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod{2}$.

We prove the theorem (known for $d = 0$) via the Z-algorithm.

For $d = 0, 1, 2, \ldots$ set

$$
u_d = \sum_{k=0}^{p-1} \frac{(2k)^2 \binom{2k}{k} \binom{2k}{k+d}}{64^k} = \sum_{d \leq k < p} \frac{(2k)^2 \binom{2k}{k+d}}{64^k}.
$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2 \nu_d - (2d+3)^2 \nu_{d+2} = \frac{(2p-1)^2(d+1)}{64^{p-1}p} \binom{2p}{p+d+1} \binom{2p-2}{p-1}^2.
$$

Note that

$$
\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod{p} \text{ and } p|(d+1)\binom{2p}{p+d+1}.
$$
A new result via the Zeilberger algorithm

So, if $0 \leq d < p - 2$ then

$$(2d + 1)^2 u_d \equiv (2d + 3)^2 u_{d+2} \pmod{p^2}.$$ 

For $d \in \{0, \ldots, p - 3\}$ with $d \equiv (p + 1)/2 \pmod{2}$, clearly $p \not= 2d + 1 < 2p$ and hence

$$u_{d+2} \equiv 0 \pmod{p^2} \implies u_d \equiv 0 \pmod{p^2}.$$ 

If $p \equiv 3 \pmod{4}$ then $p - 1 \equiv (p + 1)/2 \pmod{2}$; if $p \equiv 1 \pmod{4}$ then $p - 2 \equiv (p + 1)/2 \pmod{2}$. Thus, if $d \in \{p - 1, p - 2\}$ and $d \equiv (p + 1)/2 \pmod{2}$, then $d \not\geq (p + 1)/2$ and hence $u_d \equiv 0 \pmod{p^2}$ since $p \mid \binom{2k}{k}$ for $d \leq k \leq p - 1$.

It follows that $u_d \equiv 0 \pmod{p^2}$ for all $d \in \{0, \ldots, p - 1\}$ with $d \equiv (p + 1)/2 \pmod{2}$. 

Another theorem via the Zeilberger algorithm

**Theorem 3** (Z. W. Sun, J. Number Theory, 2012). Let $p$ be an odd prime and let $x$ be any $p$-adic integer.

(i) If $x \equiv 2k \pmod{p}$ with $k \in \{0, \ldots, (p - 1)/2\}$, then we have

$$
\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.
$$

(ii) If $x \equiv k \pmod{p}$ with $k \in \{0, \ldots, p - 1\}$, then

$$
\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}.
$$

(iii) If $p > 3$ and $x \equiv -2k \pmod{p}$ for some $k \in \{1, \ldots, \lfloor p/3 \rfloor \}$. Then we have

$$
\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^3 \equiv 0 \pmod{p^2}.
$$
Comments

It is interesting to compare Theorem 3 (i)-(ii) with the known identities

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \]

and

\[ \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \]

Here is a consequence of Theorem 3(i)-(ii).

**Corollary** (conjectured by Rodriguez-Villegas and proved by Mortenson). Let \( p \) be an odd prime. Then

\[ \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \equiv \sum_{k=0}^{p-1} \left(-\frac{1}{2}\right)^2 \equiv (-1)^{(p-1)/2} \pmod{p^2}. \]

**Conjecture** (Z. W. Sun, J. Number Theory 2012). We may replace 3 in Theorem 3(iii) by any odd integer greater than one.
Proof of Theorem 3(i)

Define

\[ f_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{2k + py}{r}^2 \quad \text{for } k \in \mathbb{N}. \]

We want to prove that

\[ f_k(y) \equiv (-1)^k \binom{2k + py}{k} \pmod{p^2} \]

for any \( p \)-adic integer \( y \) and \( k \in \{0, 1, \ldots, (p - 1)/2\} \).

Applying the Zeilberger algorithm via Mathematica 7, we find

\[ (py + 2k + 2)f_{k+1}(y) + 4(py + 2k + 1)f_k(y) = \frac{(p(y - 1) + 2k + 3)^2 F_k(y)}{(py + 2k + 1)(py + 2k + 2)^2} \binom{py + 2k + 2}{p-1}^2, \]

where

\[ F_k(y) = 14 + 34k + 20k^2 - 10p - 12kp + 2p^2 + 17py + 20kpy - 6p^2y + 5p^2y^2. \]
Proof of Theorem 3(i) (continued)

It follows that

\[ f_k(y) \equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} f_{k+1}(y) \pmod{p^2} \quad \text{for } k = 0, \ldots, \frac{p-3}{2}. \]

If \( 0 \leq k < (p-1)/2 \) and

\[ f_{k+1}(y) \equiv (-1)^{k+1} \binom{2(k + 1) + py}{k + 1} \pmod{p^2}, \]

then

\[ f_k(y) \equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} (-1)^{k+1} \binom{2(k + 1) + py}{k + 1} \]

\[ = \frac{(-1)^k (py + 2k + 2)^2}{4(k + 1)(py + k + 1)} \binom{2k + py}{k} \equiv (-1)^k \binom{2k + py}{k} \pmod{p^2}. \]
Proof of Theorem 3(ii)

Define

\[ g_k(y) := \sum_{r=0}^{p-1} \left( \begin{array}{c} k + py \\ r \end{array} \right)^2 \quad \text{for } k \in \mathbb{N}. \]

By the Zeilberger algorithm, we have the recursion

\[ (py + k + 1)g_{k+1}(y) - 2(2py + 2k + 1)g_k(y) = -\frac{(p(y - 1) + k + 2)^2(3py - 2p + 3k + 3)}{(py + k + 1)^2} \left( \begin{array}{c} py + k + 1 \\ p - 1 \end{array} \right)^2. \]

It follows that if \( k \in \{0, \ldots, p - 2\} \) and \( y \) is a \( p \)-adic integer then

\[ g_{k+1}(y) \equiv \left( \begin{array}{c} 2(k + 1) + 2py \\ k + 1 \end{array} \right) \pmod{p^2} \]

\[ \implies g_k(y) \equiv \left( \begin{array}{c} 2k + 2py \\ k \end{array} \right) \pmod{p^2}. \]
Proof of Theorem 3(iii)

Define

\[ w_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{py - 2k}{r}^3 \quad \text{for} \quad k \in \mathbb{N}. \]

We want to show that \( w_k(y) \equiv 0 \pmod{p^2} \) for any \( p \)-adic integer \( y \) and \( k \in \{1, \ldots, [(p - 1)/3]\} \).

By the Zeilberger algorithm, for \( k = 0, 1, 2, \ldots \) we have

\[
(py - 2k)^2 w_k(y) + 3(3py - 2(3k + 1))(3py - 2(3k + 2))w_{k+1}(y)
= \frac{P(k, p, y)(p(1 - y) + 2k - 1)^3}{(py - 2k)^3(py - 2k - 1)^3} \binom{py - 2k}{p - 1}^3
\]

where \( P(k, p, y) \) is a suitable polynomial in \( k, p, y \) with integer coefficients such that \( P(0, p, y) \equiv 0 \pmod{p^2} \). (Here we omit the explicit expression of \( P(k, p, y) \) since it is complicated.)
Proof of Theorem 3(iii) (continued)

Note that
\[ w_1(0) = \sum_{r=0}^{p-1} (-1)^r \binom{-2}{r}^3 = \sum_{r=0}^{p-1} (r+1)^3 = \frac{p^2(p+1)^2}{4} \equiv 0 \pmod{p^2}. \]

Fix a \( p \)-adic integer \( y \). If \( y \neq 0 \), then the recursion with \( k = 0 \) yields
\[
3(3py - 2)(3py - 4)w_1(y) \\
\equiv \frac{P(0, p, y)(p(1-y) - 1)^3}{(py)^3(py - 1)^3} \left( \frac{py}{p-1} \left( \frac{p(y-1) + p - 1}{p-2} \right) \right)^3 \\
\equiv 0 \pmod{p^2}
\]
and hence \( w_1(y) \equiv 0 \pmod{p^2} \). If \( 1 < k + 1 \leq \lfloor (p - 1)/3 \rfloor \), then
\[
(py - 2k)^2w_k(y) + 3(3py - 2(3k + 1))(3py - 2(3k + 2))w_{k+1}(y) \\
\equiv 0 \pmod{p^3}
\]
by the recursion since \( \left( \frac{py-2k}{p-1} \right) = \frac{p}{py-2k+1} \left( \frac{py-2k+1}{p} \right) \equiv 0 \pmod{p} \).
Proof of Theorem 3(iii) (continued)

Thus, when $1 < k + 1 \leq \lfloor (p - 1)/3 \rfloor$ we have

$$w_k(y) \equiv 0 \pmod{p^2}$$

$$\implies w_{k+1}(y) \equiv 0 \pmod{p^2}.$$ 

So, by induction, we have $w_k(y) \equiv 0 \pmod{p^2}$ for all $k = 1, \ldots, \lfloor (p - 1)/3 \rfloor$. 
Supercongruences involving products of two binomial coefficients

In 2003 Rodriguez-Villegas conjectured that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{(6k)(3k)\binom{3k}{k}}{432^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}$$

and this was proved by E. Mortenson in 2003.

For a sequence $(a_n)_{n \geq 0}$, its dual sequence $(a^*_n)_{n \geq 0}$ is defined by

$$a^*_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k.$$

It is well known that $(a^*_n)^* = a_n$ for all $n = 0, 1, 2, \ldots$.

**Theorem 4** (Z. W. Sun, Finite Fields Appl., 2013). Let $p > 3$ be a prime and let $(a_n)_{n \geq 0}$ be any sequence of $p$-adic integers. Then

$$\sum_{k=0}^{p-1} \frac{(6k)(3k)\binom{3k}{k}}{432^k} a_k \equiv (-1)^{(p-1)/2} \sum_{k=0}^{p-1} \frac{(6k)(3k)\binom{3k}{k}}{432^k} a^*_k \pmod{p^2}.$$
A corollary

As $\sum_{k=0}^{n} \binom{n}{k}(-1)^k x^k = (1 - x)^n$, Theorem 4 has the following consequence.

**Corollary.** Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{(6k)(3k)}{432^k} x^k \equiv (-1)^{(p-1)/2} \sum_{k=0}^{p-1} \frac{(6k)(3k)}{432^k} (1 - x)^k \pmod{p^2}.$$  

In particular, if $p \equiv 3 \pmod{4}$ then

$$\sum_{k=0}^{p-1} \frac{(6k)(3k)}{864^k} \equiv 0 \pmod{p^2}.$$
Proof of Theorem 4

Observe that

\[
\sum_{k=0}^{p-1} \frac{(6k)(3k)(3k)}{432^k} a_k^* \equiv \sum_{k=0}^{p-1} \frac{(6k)(3k)(3k)}{432^k} \sum_{m=0}^{k} \binom{k}{m} (-1)^m a_m \mod p^2.
\]

So it suffices to prove that

\[
h_p(m) := \sum_{k=m}^{p-1} \frac{(6k)(3k)(3k)}{432^k} \binom{k}{m} \equiv (-1)^{(p-1)/2} \frac{(6m)(3m)(3m)}{(-432)^m} \quad \text{(mod } p^2)\]

for all \(m = 0, \ldots, p - 1\).
Proof of Theorem 4 (continued)

For $0 \leq m < n$ define

$$h_n(m) := \sum_{k=m}^{n-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \binom{k}{m}.$$

By the Zeilberger algorithm, for $m, n \in \mathbb{N}$ with $m < n - 1$, we have

$$36(m + 1)^2 h_n(m + 1) + (6m + 1)(6m + 5)h_n(m)$$

$$= \frac{(6n - 1)(6n - 5)}{432^{n-1}} \binom{n - 1}{m} \binom{3n - 3}{n - 1} \binom{6n - 6}{3n - 3}.$$

Clearly $\binom{3p - 3}{p - 1} = \frac{p}{3p - 2} \binom{3p - 2}{p} \equiv 0 \pmod{p}$, and

$$(6p - 5)\binom{6p - 6}{3p - 3} = \frac{3p(3p - 1)(3p - 2)}{(6p - 3)(6p - 4)} \binom{6p - 3}{3p - 3} \equiv 0 \pmod{p}.$$

So, for every $m = 0, \ldots, p - 2$ we have

$$36(m + 1)^2 h_p(m + 1) + (6m + 1)(6m + 5)h_p(m) \equiv 0 \pmod{p^2}.$$
Proof of Theorem 4 (continued)

For $0 \leq m < p - 1$, since

$$- \frac{(6m + 1)(6m + 5)}{36(m + 1)^2} \cdot \frac{(6m)(3m)}{(3m)} \frac{(3m)}{(m)} = \frac{(6(m+1))(3(m+1))}{(-432)^{m+1}}$$

we have

$$h_p(m) \equiv (-1)^{(p-1)/2} \frac{(6m)(3m)}{(3m)} \frac{(3m)}{(m)} \equiv \frac{(6(m+1))(3(m+1))}{(-432)^{m+1}} \pmod{p^2}$$

$$\Rightarrow h_p(m + 1) \equiv (-1)^{(p-1)/2} \frac{(6(m+1))(3(m+1))}{(3(m+1)) \frac{(m+1)}{(m+1)}} \pmod{p^2}.$$ 

As $h_p(0) = \sum_{k=0}^{p-1} \frac{(6k)(3k)}{(3k)} / 432^k \equiv (-1)^{(p-1)/2} \pmod{p^2}$, this yields that

$$h_p(m) \equiv (-1)^{(p-1)/2} \frac{(6m)(3m)}{(3m)} \frac{(3m)}{(m)} \equiv \frac{(6(m+1))(3(m+1))}{(-432)^{m+1}} \pmod{p^2}$$

for all $m = 0, \ldots, p - 1$. 
Some open conjectures on combinatorial congruences

I have raised over 100 open conjectures on combinatorial congruences. Here I state few of them.

Conjecture (Z. W. Sun, 2009-2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{(2k)(4k)}{48^k} \equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{(2k)(4k)}{(2k+1)64^k} \equiv (-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k^3}{64^k} \equiv 8 \pmod{p^2} \text{ if } p \equiv 1 \pmod{4},$$

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \frac{(3k)}{k} \equiv -\frac{3}{p}(2^{p-1} - 1)^2 \pmod{p^2}.$$  

Remark. L. L. Zhao, Z. W. Sun and H. Pan [Proc. AMS 2010] proved that $\sum_{k=1}^{p-1} \frac{2^k}{k} \frac{(3k)}{k} \equiv 0 \pmod{p}$ for any prime $p > 3.$
Part II. Trees with prime vertices
Alternating sums of primes

Let $p_n$ be the $n$th prime and define

$$s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1}p_1.$$  

For example,

$$s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8.$$  

Note that

$$s_{2n} = \sum_{k=1}^{n}(p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n}(p_{2k+1} - p_{2k}) + p_1 > 0.$$  

Let $1 \leq k < n$. If $n - k$ is even, then

$$s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0.$$  

If $n - k$ is odd, then

$$s_n - s_k = \sum_{l=k+1}^{n}(-1)^{n-l}p_l - 2\sum_{j=1}^{k}(-1)^{k-j}p_j \equiv n - k \equiv 1 \pmod{2}.$$  

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.
An amazing recurrence for primes

We may compute the \((n + 1)\)-th prime \(p_{n+1}\) in terms of \(p_1, \ldots, p_n\).

**Conjecture** (Z. W. Sun, J. Number Theory 2013). For any positive integer \(n \neq 1, 2, 4, 9\), the \((n + 1)\)-th prime \(p_{n+1}\) is the least positive integer \(m\) such that

\[
2s_1^2, \ldots, 2s_n^2
\]

are pairwise distinct modulo \(m\).

**Remark.** I have verified the conjecture for \(n \leq 10^5\), and proved that \(2s_1^2, \ldots, 2s_n^2\) are indeed pairwise distinct modulo \(p_{n+1}\).

Let \(1 \leq j < k \leq n\). Then

\[
0 < |s_k - s_j| \leq \max\{s_k, s_j\} \leq \max\{p_k, p_j\} \leq p_n < p_{n+1}.
\]

Also, \(s_k + s_j \leq p_k + p_j < 2p_{n+1}\). If \(2 \nmid k - j\), then

\[
s_k + s_j = p_k - p_{k-1} + \cdots + p_{j+1} \leq p_k < p_{n+1}.
\]

If \(2 \mid k - j\), then \(s_k \equiv s_j \pmod{2}\) and hence \(s_k + s_j \neq p_{n+1}\). Thus

\[
2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}.
\]
Conjecture on alternating sums of consecutive primes

Conjecture (Z. W. Sun, J. Number Theory, 2013). For any positive integer \( m \), there are consecutive primes \( p_k, \ldots, p_n \) \((k < n)\) not exceeding \( 2m + 2.2\sqrt{m} \) such that

\[
m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k.
\]

(Moreover, we may even require that \( m < p_n < m + 4.6\sqrt{m} \) if \( 2 \nmid m \) and \( 2m - 3.6\sqrt{m} + 1 < p_n < 2m + 2.2\sqrt{m} \) if \( 2 \mid m \).)

Examples.

\[
10 = 17 - 13 + 11 - 7 + 5 - 3;
20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;
303 = p_{76} - p_{75} + \cdots + p_{52},
\]
\[
p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \quad p_{52} = 239;
2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43},
\]
\[
p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \quad p_{43} = 191.
\]

The conjecture has been verified for \( m \) up to \( 10^7 \).

Prize. I would like to offer 1000 US dollars for the first proof.
A tree with prime vertices

**Conjecture** (Z. W. Sun, Feb. 24, 2013). For a prime $p$ define $f(p)$ as the least prime $p_n$ such that

$$p = p_n - p_{n-1} + \cdots + (-1)^{n-k}p_k \quad \text{for some } k < n.$$  

Construct a simple (undirected) graph $G$ as follows: The vertex set of $G$ is the set of all primes, and for the vertices $p$ and $q > p$ there is an edge connecting $p$ and $q$ if and only if $f(p) = q$. Then the graph $G$ constructed above is connected and hence it is a tree!

For example,

$$2 \rightarrow 5 \rightarrow 7 \rightarrow 13 \rightarrow 17 \rightarrow 23 \rightarrow 31 \rightarrow 37 \rightarrow 43 \rightarrow 53 \rightarrow 59 \rightarrow 67 \rightarrow 73 \rightarrow 83 \rightarrow 89 \rightarrow 101 \rightarrow 109 \rightarrow 113 \rightarrow 131 \rightarrow 149$$

and

$$71 \rightarrow 79 \rightarrow 97 \rightarrow 107 \rightarrow 139 \rightarrow 149,$$

hence there is a unique path connecting the vertices 2 and 71.
Goldbach's conjecture and the Collatz conjecture

The following two conjectures are famous and difficult.

**Goldbach’s Conjecture.** Any even number greater than 2 can be written as a sum of two primes.

**Collatz’s Conjecture.** Let $a_1$ be any positive integer, and let

$$a_{n+1} = \begin{cases} 
3a_n + 1 & \text{if } 2 \nmid n, \\
\frac{a_n}{2} & \text{if } 2 \mid n.
\end{cases}$$

Then $a_N = 1$ for some positive integer $N$.

For example,

$$7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1.$$
Another tree with prime vertices

**Conjecture** (Z. W. Sun, Feb. 24, 2013). For a prime $p$ define $g(p)$ as the least prime $q > p$ such that $2(p + 1) - q$ is prime. Construct a simple (undirected) graph $G$ as follows: The vertex set of $G$ is the set of all primes, and for the vertices $p$ and $q > p$ there is an edge connecting $p$ and $q$ if and only if $g(p) = q$. Then the graph $G$ constructed above is connected and hence it is a tree!

For example,

\[ 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 11 \rightarrow 13 \rightarrow 17 \rightarrow 19 \rightarrow 23 \rightarrow 29 \rightarrow 31 \rightarrow 41, \]
\[ \rightarrow 43 \rightarrow 47 \rightarrow 53 \rightarrow 61 \rightarrow 71 \rightarrow 73 \rightarrow 89 \rightarrow 97 \rightarrow 107 \rightarrow 109 \]
\[ \rightarrow 113 \rightarrow 127 \rightarrow 149 \rightarrow 151 \rightarrow 167 \rightarrow 173 \rightarrow 181 \rightarrow 191. \]
Conjecture 2 (Z. W. Sun, Feb. 28, 2013) For any positive integer \( n \), define

\[
f(n) = \begin{cases} 
\frac{(p + 1)}{2} & \text{if } 4 \mid p + 1, \\
p & \text{otherwise},
\end{cases}
\]

where \( p \) is the least prime greater than \( n \) with \( 2(n + 1) - p \) prime. If \( a_1 \in \{3, 4, \ldots\} \) and \( a_{k+1} = f(a_k) \) for \( k = 1, 2, 3, \ldots \), then \( a_N = 4 \) for some positive integer \( N \).

For example,

\[
45 \rightarrow 61 \rightarrow 36 \rightarrow 37 \rightarrow 24 \rightarrow 16 \rightarrow 17 \rightarrow 10 \rightarrow 6 \rightarrow 4 \rightarrow 5 \rightarrow 4.
\]
A positive integer $n$ is called a *practical* number if every $m = 1, \ldots, n$ can be written as a sum of some distinct divisors of $n$, i.e., there are distinct divisors $d_1, \ldots, d_k$ of $n$ such that

$$\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.$$  

For example, 6 is practical since $1, 2, 3, 6$ divides 6, and also $4 = 1 + 3$ and $5 = 2 + 3$. As any positive integer has a unique representation in base 2 with digits in $\{0, 1\}$, powers of 2 are all practical. 1 is the only odd practical number.

Practical numbers below 50:
1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.
Goldbach-type results for practical numbers

**Theorem** (Stewart [Amer. J. Math., 76(1954)]). If \( p_1 < \cdots < p_r \) are distinct primes and \( a_1, \ldots, a_r \) are positive integers then \( m = p_1^{a_1} \cdots p_r^{a_r} \) is practical if and only if \( p_1 = 2 \) and

\[
p_{s+1} - 1 \leq \sigma(p_1^{a_1} \cdots p_s^{a_s}) \quad \text{for all } 0 < s < r,
\]

where \( \sigma(n) \) stands for the sum of all divisors of \( n \).

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

**Theorem** (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers \( m \) with \( m - 2 \) and \( m + 2 \) also practical.

**Conjecture** (Sun, 2013). Any integer \( n > 4 \) can be written as \( p + q/2 \), where \( p \) and \( q \) are practical numbers smaller than \( n \).
Two conjectures involving practical numbers

**Conjecture** (Z. W. Sun, Feb. 2013). For a practical number $p$, define $h(p)$ as the least practical number $q > p$ such that $2(p + 1) - q$ is practical. Construct a simple (undirected) graph $H$ as follows: The vertex set of $H$ is the set of all practical numbers, and for two vertices $p$ and $q > p$ there is an edge connecting $p$ and $q$ if and only if $h(p) = q$. Then the graph $H$ is connected and hence it is a tree!

**Conjecture** (Z. W. Sun, Feb. 2013). For any integer $n > 0$ define

$$g(n) = \begin{cases} 
q/2 & \text{if } 4 \mid q, \\
q & \text{if } 4 \mid q - 2,
\end{cases}$$

where $q$ is the least practical number greater than $n$ with $2(n + 1) - q$ practical. If $b_1 \in \{4, 5, \ldots\}$ and $b_{k+1} = g(b_k)$ for $k = 1, 2, 3, \ldots$, then $b_N = 4$ for some positive integer $N$.

**Example.** If we start from $b_1 = 316$ then we get the sequence

$$316, 330, 342, 378, 190, 110, 126, 64, 66, 78, 40, 42, 54, 28, 30, 16, 18, 10, 8, 6, 4, 6, 4, \ldots$$
Thank you!