Introduction to Combinatorial Number Theory (I) – Bernoulli Numbers and Euler Numbers

Zhi-Wei Sun

Nanjing University
zwsun@nju.edu.cn
http://maths.nju.edu.cn/∼zwsun

May 22, 2020

Abstract

In this talk we introduce basic properties of Bernoulli numbers and Euler numbers, and related identities and congruences such as Euler's formula, von Staudt-Clausen theorem and Miki's identity and its generalizations.

Introduction to Combinatorial Number Theory (I) – Bernoulli Numbers and Euler Numbers

Zhi-Wei Sun

Nanjing University
zwsun@nju.edu.cn
http://maths.nju.edu.cn/$\sim$zwsun

May 22, 2020

Part I. Classical Results on Bernoulli Numbers and Euler Numbers

On the sum $S_k(n) = \sum_{r=0}^{n-1} r^k$

Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. By induction, one can show that

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}, \sum_{k=0}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}, \sum_{k=0}^{n-1} k^3 = \frac{n^2(n-1)^2}{4}.$$

For $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, how to find a closed formula for the sum

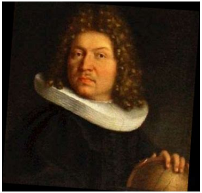
$$S_k(n) = \sum_{r=0}^{n-1} r^k.$$

Clearly $S_0(n) = n$ (we adopt $x^0 = 1$ even when $x = 0$). From the formulas for $S_1(n), S_2(n), S_3(n)$, it seems that

$S_k(n)$ **is a polynomial in** n **of degree** $k+1$.

Bernoulli numbers

Jacob Bernoulli (1654-1705) determined $S_k(n) = \sum_{r=0}^{n-1} r^k$ in his book published in 1713.

Jacob Bernoulli 

The so-called Bernoulli numbers play important roles in Bernoulli's solution. The **Bernoulli numbers** $B_0, B_1, B_2, \ldots$ are given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi).$$

In other words, the **exponential generating function** of the sequence $(B_n)_{n \geqslant 0}$ is $x/(e^x - 1)$.

Recurrence for the Bernoulli numbers

Recurrence: $B_0 = 1$, and for $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \quad \text{i.e. } B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k.$$

This is because

$$1 = \frac{x}{e^x - 1} \cdot \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \sum_{l=1}^{\infty} \frac{x^{l-1}}{l!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k}{k!} \cdot \frac{1}{(n+1-k)!} \right) x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+1}{k} B_k \frac{x^n}{(n+1)!}.$$

Using the recurrence and induction, we see that $B_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. The initial 7 Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}.$$

Bernoulli's summation formula

Bernoulli's Summation Formula. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we have

$$S_k(n) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j}.$$

Proof. Observe that

$$\sum_{k=0}^{\infty} S_k(n) \frac{x^k}{k!} = \sum_{r=0}^{n-1} \sum_{k=0}^{\infty} \frac{(rx)^k}{k!}$$

$$= \sum_{r=0}^{n-1} (e^x)^r = \frac{e^{nx}-1}{e^x - 1} = \frac{x}{e^x - 1} \cdot \frac{e^{nx}-1}{x}$$

$$= \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} \sum_{l=1}^{\infty} \frac{n^l x^{l-1}}{l!}.$$

Comparing the coefficients of x^k, we get that

$$\frac{S_k(n)}{k!} = \sum_{j=0}^{k} \frac{B_j}{j!} \cdot \frac{n^{k+1-j}}{(k+1-j)!} = \frac{1}{(k+1)!} \sum_{j=9}^{k} \binom{k+1}{j} B_j n^{k+1-j}.$$

$B_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$

Theorem. We have $B_{2k+1} = 0$ for all $k \in \mathbb{Z}^+$.

Proof. For $0 < |x| < 2\pi$, we have

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{n=2}^{\infty} B_n \frac{x^n}{n!}$$

and hence

$$1 + \sum_{n=2}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1}.$$

Note that

$$\frac{-x}{2} \cdot \frac{e^{-x} + 1}{e^{-x} - 1} = -\frac{x}{2} \cdot \frac{1 + e^x}{1 - e^x} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1}.$$

So

$$1 + \sum_{n=2}^{\infty} B_n \frac{(-x)^n}{n!} = 1 + \sum_{n=2}^{\infty} B_n \frac{x^n}{n!}.$$

If $n > 2$ is odd, then $-B_n = (-1)^n B_n = B_n$ and hence $B_n = 0$.

The Taylor expansion of $x\cot x$ and Euler's product formula

By Euler's formula $e^{ix} = \cos x + i\sin x$, we have

$$x\cot x = x\frac{\cos x}{\sin x} = x\frac{(e^{ix}+e^{-ix})/2}{(e^{ix}-e^{-ix})/(2i)}$$
$$= \frac{2ix}{2}\cdot\frac{e^{2ix}+1}{e^{2ix}-1} = 1+\sum_{n=2}^{\infty} B_n\frac{(2ix)^n}{n!}.$$

So we have

$$x\cot x = \sum_{k=0}^{\infty}(-1)^k 2^{2k} B_{2k}\frac{x^k}{k!}.$$

If $P(x)$ is a polynomial of degree n with n zeroes $x_1, \ldots, x_n$ and $P(0)=1$, then $P(x) = \prod_{k=1}^{n}(1-x/x_i)$. Euler considered $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots$ as a "*polynomial*" of infinity degree with constant 1. This led him to find that

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1-\frac{x}{n\pi}\right)\left(1-\frac{x}{-n\pi}\right) = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

Euler's formula for $\zeta(2m)$

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1.$$

Theorem (Euler). For each $m \in \mathbb{Z}^+$ we have

$$2\zeta(2m) = (-1)^{m-1} B_{2m} \frac{(2\pi)^{2m}}{(2m)!}.$$

Proof. Recall Euler's product formula

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right).$$

Taking logarithmic derivatives of both sides, we get

$$\frac{\cos x}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{-2x/(n\pi)^2}{1 - x^2/(n\pi)^2}.$$

Euler's formula for $\zeta(2m)$

Thus

$$\begin{aligned} x\cot x =& 1-2\sum_{n=1}^{\infty}\left(\frac{x}{n\pi}\right)^2\cdot\frac{1}{1-(x/(n\pi))^2} \\ =& 1-2\sum_{n=1}^{\infty}\frac{x^2}{(n\pi)^2}\left(1+\left(\frac{x}{n\pi}\right)^2+\left(\frac{x}{n\pi}\right)^4+\dots\right) \\ =& 1-2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{2m}}{(n\pi)^{2m}}=1-2\sum_{m=1}^{\infty}\zeta(2m)\frac{x^{2m}}{\pi^{2m}}. \end{aligned}$$

On the other hand,

$$x\cot x=1+\sum_{n=2}^{\infty}(-1)^m B_{2m}\frac{(2x)^{2m}}{(2m)!}.$$

So

$$-\frac{2}{\pi^{2m}}\zeta(2m)=(-1)^m B_{2m}\frac{2^{2m}}{(2m)!}$$

and hence the desired result follows.

Consequences

For each $m \in \mathbb{Z}^+$, we have

$$(-1)^{m-1} B_{2m} \frac{(2\pi)^{2m}}{(2m)!} = 2\zeta(2m) > 2$$

and hence

$$(-1)^{m-1} B_{2m} > \frac{2(2m)!}{(2\pi)^{2m}} > 2\left(\frac{m}{\pi e}\right)^{2m}$$

since $e^n > n^n/n!$ for all $n \in \mathbb{Z}^+$. So, $B_2, B_4, B_6, \ldots$ are nonzero, and they have alternating signs:

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ \ldots$$

By Euler's formula,

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \zeta(10) = \frac{\pi^{10}}{93555}.$$

p-adic integers

Let p be a prime. For a nonzero integer m, its p-adic order (or p-adic valuation) is

$$\nu_p(m) = \mathrm{ord}_p(m) = \max\{n \in \mathbb{N} : \ p^n \mid m\}.$$

We define $\nu_p(0)$ as $+\infty$. For a nonzero rational number $x = a/b$ with $a, b \in \mathbb{Z}$ and $b \neq 0$, $\nu_p(x)$ is defined as $\nu_p(a) - \nu_p(b)$.

For $x \in \mathbb{Q}$, its *p-adic norm* is given by $\|x\|_p = p^{-\nu_p(x)}$. The usual absolute value $|x|$ is also written as $|x|_\infty$.

If a rational number sequence converges according to the p-adic norm then its limit is called a *p-adic number*. All the p-adic numbers form the p-adic field $\mathbb{Q}_p$. Each p-adic number has a unique p-adic series representation:
$\sum_{n=m}^{\infty} a_n p^n$ with $a_n \in \{0, 1, \ldots, p-1\}$. If $m \geqslant 0$, then this p-adic number is said to be a *p-adic integer*. All the p-adic integers form a ring $\mathbb{Z}_p$. We may consider congruences in this ring.

A rational number a/b with $a, b \in \mathbb{Z}, \ b \neq 0$ and $(a, b) = 1$ is a p-adic integer if and only if $(b, p) = 1$. E.g., $\frac{2}{3} \equiv -1 \pmod 5$.

$pB_k \in \mathbb{Z}_p$ for all $k \in \mathbb{N}$

Let p be a prime. For any $k \in \mathbb{Z}^+$ we have

$$\begin{aligned} S_k(p) =& \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j p^{k+1-j} = \frac{1}{k+1}\sum_{l=0}^{k}\binom{k+1}{l+1}p^{l+1}B_{k-l} \\ =& pB_k + \sum_{l=1}^{k}\binom{k}{l}\frac{p^l}{l+1}pB_{k-l} \end{aligned}$$

As $p^l \geqslant (1+1)^l \geqslant l+1$ for $l \in \mathbb{Z}^+$, by induction $pB_k \in \mathbb{Z}_p$ for all $k \in \mathbb{N}$. Note also that

$$\begin{aligned} \frac{S_k(p) - pB_k}{k} =& \frac{1}{k}\sum_{l=1}^{k}\binom{k}{l}\frac{p^l}{l+1}pB_{k-l} = \sum_{l=1}^{k}\binom{k-1}{l-1}\frac{p^l}{l(l+1)}pB_{k-l} \\ =& \frac{p}{2}pB_{k-1} + p\sum_{1<l\le k}\binom{k-1}{l-1}\left(\frac{p^{l-1}}{l} - \frac{p^{l-1}}{l+1}\right)pB_{k-l}. \end{aligned}$$

A Lemma

For $l \in \{3, 4, \ldots\}$, we have

$$p^{l-1} \geqslant (1+1)^{l-1} = 1 + (l-1) + \binom{l-1}{2} = l + \binom{l-1}{2} \geqslant l + 1,$$

and $p^{l-1} = l + 1 \iff p = 2 \ \& \ l = 3$. Thus we have the following lemma.

Lemma. Let p be a prime and let $k \in \mathbb{Z}^+$. Then

$$\frac{S_k(p) - pB_k}{k} \equiv \frac{p}{2} pB_{k-1} \pmod{p}.$$

Furthermore, if $p > 2$ and $k \geqslant 2$ then

$$\frac{S_k(p) - pB_k}{k} \equiv \frac{p}{2} pB_{k-1} - (k-1)\frac{p^2}{3} pB_{k-2} \pmod{p^2}.$$

In particular, if $p > 3$ then $S_{p-1}(p) \equiv pB_{p-1} \pmod{p^2}$.

von Staudt-Clausen Theorem

von Staudt-Clausen Theorem. We have

$$B_k + \sum_{p-1|k} \frac{1}{p} \in \mathbb{Z} \ \text{ for } k = 2, 4, 6, \ldots .$$

Proof. Let $k > 0$ be an even integer. Recall that $B_{k-1} = 0$ if $k > 2$. So, by the lemma, we have

$$S_k(p) - pB_k \equiv \delta_{k,2} p^2 B_1 \equiv 0 \pmod{p}.$$

If $p - 1 \mid k$, then $S_k(p) = \sum_{r=1}^{p-1} r^k \equiv \sum_{r=1}^{p-1} 1 \equiv -1 \pmod{p}$ (by Fermat's little theorem) and hence $B_k + 1/p \in \mathbb{Z}_p$.

If $p - 1 \nmid k$, then there is a $g \in \mathbb{Z}$ such that $g^k \not\equiv 1 \pmod{p}$, as $(g^k - 1)S_k(p) = \sum_{r=1}^{p-1} (gr)^k - \sum_{r=1}^{p-1} r^k \equiv 0 \ (\text{mod } p)$ we have $p \mid S_k(p)$ and hence $B_k \in \mathbb{Z}_p$.

By the above, $B_k + \sum_{p-1|k} p^{-1} \in \mathbb{Z}_q$ for any prime q. So $B_k + \sum_{p-1|k} p^{-1} \in \mathbb{Z}$.

On $(p-1)! \bmod p^2$

Theorem (Beeger, 1913). Let $p > 3$ be a prime. Then

$$(p-1)! \equiv pB_{p-1} - p \pmod{p^2}.$$

Proof. Wilson's theorem asserts that $w_p = ((p-1)!+1)/p \in \mathbb{Z}$. For any integer $a \not\equiv 0 \pmod{p}$ let $q_p(a)$ denote the Fermat quotient $(a^{p-1}-1)/p$. Then

$$(pw_p-1)^{p-1} = \prod_{r=1}^{p-1} r^{p-1} = \prod_{r=1}^{p-1}(1+pq_p(r)) \equiv 1+p\sum_{r=1}^{p-1} q_p(r) \pmod{p^2}$$

and hence

$$\begin{aligned} 1-(p-1)pw_p \equiv& (pw_p-1)^{p-1} \equiv 1 + \sum_{r=1}^{p-1}(r^{p-1}-1) \\ =& S_{p-1}(p) - p + 2 \equiv pB_{p-1} - p + 2 \pmod{p^2}. \end{aligned}$$

So $(p-1)! = pw_p - 1 \equiv pB_{p-1} - p \pmod{p^2}$.

Some other classical results

Theorem. Let p be a prime and $n > 0$ be an even integer.

(i) (E. Kummer) If $p-1 \nmid n$, then $B_n/n \in \mathbb{Z}_p$, moreover $B_m/m \equiv B_n/n \pmod{p}$ whenever $m \equiv n \pmod{p-1}$.

(ii) (L. Carlitz) If $p \neq 2$ and $p-1 \mid n$ then $(B_n + p^{-1} - 1)/n \in \mathbb{Z}_p$.

Theorem (Voronoi, 1889). Let $n > 0$ be even, $q \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ and $(m, q) = 1$. Then

$$(m^n - 1)\, B_n \equiv n m^{n-1} \sum_{j=1}^{q-1} j^{n-1} \left\lfloor \frac{jm}{q} \right\rfloor \pmod{q}.$$

Theorem (Z.-W. Sun [Discrete Math. 262(2003)]). Let $a \in \mathbb{Z}$, $m, n, q \in \mathbb{Z}^+$ and $(m, q) = 1$. Then

$$\frac{1}{n}\left(m^n B_n\left(\frac{x+a}{m}\right) - B_n(x)\right)$$
$$\equiv \sum_{j=0}^{q-1}\left(\left\lfloor \frac{a+jm}{q} \right\rfloor + \frac{1-m}{2}\right)(x+a+jm)^{n-1} \pmod{q}.$$

Regular primes and irregular primes

A prime $p > 3$ is called *regular* if p divides none of the numerators of $B_2, B_4, \ldots, B_{p-3}$ (i.e., $B_{2k} \not\equiv 0 \pmod{p}$ for all $k = 1, \ldots, \frac{p-3}{2}$).

E. Kummer showed that a prime $p > 3$ is regular if and only if p does not divide the class number of the cyclotomic field $\mathbb{Q}(e^{2\pi i/p})$.

Theorem (Kummer, 1847) Let $p > 3$ be a regular prime. Then $x^p + y^p = z^p$ has no integer solutions with $p \nmid xyz$.

The first ten irregular primes are

$$37,\ 59,\ 67,\ 101,\ 103,\ 131,\ 149,\ 157,\ 233,\ 257.$$

Numerical computation suggests that 60% of the primes are regular primes.

Infinitely many irregular primes

In 1915 K. L. Jensen showed that there are infinitely many irregular primes $p \equiv 3 \pmod 4$.

Theorem. There are infinitely many irregular primes.

Carlitz's Proof in 1954. Suppose that there are only finitely many irregular primes, they are $p_1, \ldots, p_r$. Choose sufficiently large $k \in \mathbb{Z}^+$ such that for $n = k(p_1 - 1) \ldots (p_r - 1)$ we have $|B_n/n| > 1$. Let p be a prime divisor of the numerator of B_n/n. By the von Staudt-Clausen theorem, $p - 1 \nmid n$ and thus $p \neq 2$ and $p \neq p_1, \ldots, p_r$.

Let m be the least nonnegative residue of n mod $p - 1$. Then $m \neq 0$ and $2 \mid m$. So $2 \leqslant m \leqslant p - 3$. By the Kummer congruence, $B_n/n \equiv B_m/m \pmod p$. Since $B_n/n \equiv 0 \pmod p$, we also have $B_m/m \equiv 0 \pmod p$ and $B_m \equiv 0 \pmod p$. So p is a new irregular prime and we get a contradiction.

Remark. Up to now, nobody is able to prove that there are infinitely many regular primes.

Bernoulli polynomials

For $n \in \mathbb{N}$ the *Bernoulli polynomial* of degree n is defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Note that $B_n(0) = B_n$, and $S_k(n) = \sum_{r=0}^{n-1} r^k = \frac{B_{k+1}(n) - B_{k+1}}{k+1}$.

Basic properties of the Bernoulli polynomials:

(i) $B_n(1-x) = (-1)^n B_n(x),\ \sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n$.

(ii) $\Delta B_n(x) = B_n(x+1) - B_n(x) = nx^{n-1},\ B_n'(x) = nB_{n-1}(x)$.

(iii) $\int_0^1 B_n(x)dx = \delta_{n,0},\ B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}$.

(iv) (Raabe's formula) $\sum_{r=0}^{n-1} B_n(\frac{x+r}{m}) = m^{1-n} B_n(x)$.

(v) $B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n$. When $2 \mid n$, we have

$$B_n\left(\frac{1}{3}\right) = (3^{1-n} - 1)\frac{B_n}{2},\ B_n\left(\frac{1}{4}\right) = 2^{-n}(2^{1-n} - 1)B_n,$$

$$B_n\left(\frac{1}{6}\right) = (2^{1-n} - 1)(3^{1-n} - 1)\frac{B_n}{2}.$$

Euler numbers

The Euler numbers are given by

$$\frac{2e^z}{e^{2z}+1} = \frac{2}{e^z + e^{-z}} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}.$$

Clearly $E_n = 0$ if $2 \nmid n$. As

$$1 = \sum_{k=0}^{\infty} E_k \frac{z^k}{k!} \sum_{l=0}^{\infty} \frac{z^l + (-z)^l}{2 \times l!} = \sum_{k=0}^{\infty} E_k \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!},$$

comparing the coefficients of z^n we get

$$\sum_{\substack{k=0 \\ 2 \mid n-k}}^{n} \binom{n}{k} E_k = n!\delta_{n,0} = \delta_{n,0}.$$

Thus $E_0 = 1$ and $E_n + \sum_{\substack{0 \leqslant k < n \\ 2 \mid n-k}} \binom{n}{k} E_k = 0$ for $n \in \mathbb{Z}^+$. So $E_n \in \mathbb{Z}$.
$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521.$

Some series due to Euler

For $|x| < \pi/2$, Euler observed that

$$\sec x = \frac{1}{\cos x} = \frac{2}{e^{ix} + e^{-ix}} = \sum_{n=0}^{\infty} E_n \frac{(ix)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k E_{2k} \frac{x^{2k}}{(2k)!}$$

and

$$\tan x = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2^{2m}(2^{2m} - 1) B_{2m}}{(2m)!} x^{2m-1}.$$

Dirichlet beta function:

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} \quad (\Re(s) > 0).$$

For any $n \in \mathbb{N}$ we have

$$\beta(2n+1) = (-1)^n E_{2n} \frac{\pi^{2n+1}}{4^{n+1}(2n)!}.$$

Other things for Euler numbers

Combinatorial Interpretation of Euler Numbers (D. André, 1879): For any $n \in \mathbb{Z}^+$, we have

$$(-1)^n E_{2n} = |\{\sigma \in S_n : \sigma(1) > \sigma(2) < \sigma(3) > \ldots < \sigma(2n-1) > \sigma(2n)\}|.$$

Conjecture (Z.-W. Sun, 2012). (i) The sequences $(\sqrt[n]{(-1)^{n-1}B_{2n}})_{n\geqslant 1}$ and $\sqrt[n]{(-1)^n E_{2n}})_{n\geqslant 1}$ are strictly increasing.

(ii) The sequences

$$\left(\sqrt[n+1]{(-1)^n B_{2n+2}}\Big/\sqrt[n]{(-1)^{n-1}B_{2n}}\right)_{n\geqslant 2}$$

and

$$\left(\sqrt[n+1]{(-1)^{n+1} E_{2n+2}}\Big/\sqrt[n]{(-1)^n E_{2n}}\right)_{n\geqslant 1}$$

are strictly decreasing.

This conjecture was confirmed by F. Luca and P. Stănică [J. Comb. Number Theory 4(2012)]. The first assertion in part (i) was also proved by W.Y.C. Chen, J.F. Guo and L.X.W. Wang [Proc. Edinb. Math. Soc. 58(2015)].

Euler polynomials

For $n \in \mathbb{N}$, the **Euler polynomial** of degree n is given by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} E_k \cdot (2x-1)^{n-k}.$$

Note that $E_n(\frac{1}{2}) = \frac{E_n}{2^n}$. Here are some basic formulas:

$$E_n(x+1) + E_n(x) = 2x^n,\ \ E_n(1-x) = (-1)^n E_n(x),$$

$$E_n'(x) = nE_{n-1}(x)\ (n > 0),\ \sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n.$$

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

Z. W. Sun (in a book published in 2006): If $k \in \mathbb{N}$, $a, m \in \mathbb{Z}$ and $2 \nmid m$ then

$$\frac{m^{k+1}}{2} E_k\left(\frac{x+a}{m}\right) - \frac{(-1)^a}{2} E_k(x) \in \mathbb{Z}[x].$$

Alternating sums

Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Then

$$\begin{aligned}\sum_{r=0}^{n-1}(-1)^r(x+r)^k =& \frac{1}{2}\sum_{r=0}^{n-1}(-1)^r\left(E_k(x+r)+E_k(x+r+1)\right)\\ =& \frac{1}{2}\sum_{r=0}^{n-1}\left((-1)^r E_k(x+r)-(-1)^{r+1}E_k(x+r+1)\right)\\ =& \frac{1}{2}\left(E_k(x)-(-1)^n E_k(x+n)\right)\end{aligned}$$

Alternating sums

On the other hand,

$$\begin{aligned}\sum_{r=0}^{n-1}(-1)^r(x+r)^k &= 2\sum_{\substack{r=0\\2|r}}^{n-1} - \sum_{r=0}^{n-1}(x+r)^k \\
=&2^{k+1}\sum_{s=0}^{\lfloor (n-1)/2\rfloor}\left(\frac{x}{2}+s\right)^k - \sum_{r=0}^{n-1}(x+r)^k \\
=&\frac{2^{k+1}}{k+1}\sum_{s=0}^{\lfloor (n-1)/2\rfloor}\Big(B_{k+1}(\frac{x}{2}+s+1) - B_{k+1}(x+s)\Big) \\
&- \frac{1}{k+1}\sum_{r=0}^{n-1}(B_{k+1}(x+r+1) - B_{k+1}(x+r)) \\
=&\frac{2^{k+1}}{k+1}\left(B_{k+1}\left(\frac{x}{2}+\left\lfloor\frac{n+1}{2}\right\rfloor\right) - B_{k+1}\left(\frac{x}{2}\right)\right) \\
&- \frac{B_{k+1}(x+n) - B_{k+1}(x)}{k+1}.\end{aligned}$$

Connections between Euler polynomials and Bernoulli polynomials

So we have

$$\frac{E_k(x)-(-1)^n E_k(x+n)}{2} = \frac{2^{k+1}}{k+1}\left(B_{k+1}\left(\frac{x}{2}+\left\lfloor\frac{n+1}{2}\right\rfloor\right)-B_{k+1}\left(\frac{x}{2}\right)\right) - \frac{B_{k+1}(x+n)-B_{k+1}(x)}{k+1}.$$

Taking $x=0$ and $n=2t, 2t-1$ $(t\in\mathbb{Z}^+)$ we get

$$\frac{E_k(0)-E_k(2t)}{2} = \frac{2^{k+1}}{k+1}\left(B_{k+1}(t)-B_{k+1}\right) - \frac{B_{k+1}(2t)-B_{k+1}}{k+1}$$

and

$$\frac{E_k(0)+E_k(2t-1)}{2} = \frac{2^{k+1}}{k+1}\left(B_{k+1}(t)-B_{k+1}\right) - \frac{B_{k+1}(2t-1)-B_{k+1}}{k+1}.$$

Euler polynomials in terms of Bernoulli polynomials

So

$$E_k(x) - \frac{2}{k+1}\left(B_{k+1}(x) - 2^{k+1}B_{k+1}\left(\frac{x}{2}\right)\right) = E_k(0) - \frac{2(1-2^{k+1})}{k+1}B_{k+1}$$

and also

$$E_k(x) - \frac{2}{k+1}\left(2^{k+1}B_{k+1}\left(\frac{x+1}{2}\right) - B_{k+1}(x)\right)$$
$$= \frac{2(2^{k+1}-1)}{k+1}B_{k+1} - E_k(0).$$

Adding these two identities we see that

$$E_k(x) = \frac{2^k}{k+1}\left(B_{k+1}\left(\frac{x+1}{2}\right) - B_{k+1}\left(\frac{x}{2}\right)\right).$$

By Raabe's formula, $\sum_{r=0}^{1} B_{k+1}\left(\frac{x+r}{2}\right) = 2^{-k}B_{k+1}(x)$. So

$$E_k(x) = \frac{2}{k+1}\left(B_{k+1}(x) - 2^{k+1}B_{k+1}\left(\frac{x}{2}\right)\right).$$

In particular, $E_k(0) = 0$ for $k = 2, 4, 6, \ldots$.

Formulas of Raabe's type

It is known that

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1}.$$

Let $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. Similar to Raabe's formula, using the above formulas we can deduce

$$\sum_{r=0}^{m-1}(-1)^r B_{n+1}\left(\frac{x+r}{m}\right) = -\frac{n+1}{2m^n}E_n(x)$$

if m is even, and

$$\sum_{r=0}^{m-1}(-1)^r E_n\left(\frac{x+r}{m}\right) = \frac{E_n(x)}{m^n}$$

if m is odd.

Introduction to Combinatorial Number Theory (I) – Bernoulli Numbers and Euler Numbers

Zhi-Wei Sun

Nanjing University
zwsun@nju.edu.cn
http://maths.nju.edu.cn/∼zwsun

May 22, 2020

Part II. Modern Identities on Bernoulli Numbers and Euler Numbers

Ettingshausen's identity and its extensions

von Ettingshausen (1827):
$\sum_{k=0}^{n}\binom{n+1}{k}(n+k+1)B_{n+k}=0\ (n>0)$.

H. Momiyama [Fibonacci Quart. 39(2001)]: If $m+n>0$, then

$$(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)B_{n+k}$$
$$=-(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k}.$$

K. J. Wu, Z. W. Sun and H. Pan [Fibonacci Quart. 42(2004)].

$$(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)B_{n+k}(x)$$
$$+(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k}(-x)$$
$$=(-1)^m(m+n+1)(m+n+2)x^{m+n}.$$

Dual sequences and related polynomials

For a sequence $(a_n)_{n\geqslant 0}$ of complex numbers, its *dual sequence* $(a_n^*)_{n\geqslant 0}$ is given by $a_n^* = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k$ $(n\in\mathbb{N})$. It is well known that $a_n^{**} = a_n$. For $n\in\mathbb{N}$ let

$$A_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k x^{n-k},\ A_n^*(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k^* x^{n-k}.$$

The sequence $((-1)^n B_n)_{n\geqslant 0}$ is self-dual since

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k((-1)^k B_k) = B_n + \sum_{0\leqslant k<n}\binom{n}{k}B_k = (-1)^n B_n.$$

If $a_n = (-1)^n B_n$ for all $n\in\mathbb{N}$, then $A_n(x) = B_n(x) = A_n^*(x)$.
The sequence $((-1)^n E_n(0))_{n\geqslant 0}$ is self-dual since

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-1)^k E_k(0) = \sum_{k=0}^{n}\binom{n}{k}E_k(0)1^{n-k} = E_n(1) = (-1)^n E_n(0).$$

If $a_n = (-1)^n E_n(0)$ for all $n\in\mathbb{N}$, then $A_n(x) = E_n(x) = A_n^*(x)$.

A general theorem involving dual sequences

Z. W. Sun [European J. Combin. 24(2003)] Let $(a_n)_{n\geqslant 0}$ be a sequence of complex numbers. If $m,n\in\mathbb{N}$ and $x+y+z=1$, then we have the identities

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{A_{n+k+1}(y)}{n+k+1}+(-1)^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\frac{A^*_{m+k+1}(z)}{m+k+1}$$

$$=a_0\frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}},$$

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}x^{m-k}A_{n+k}(y)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}A^*_{m+k}(z),$$

$$(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)x^{n-k+1}A_{n+k}(y)$$

$$+(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}x^{n-k+1}(m+k+1)A^*_{m+k}(z)$$

$$=(m+n+2)\left((-1)^{m+1}A_{m+n+1}(y)+(-1)^{n+1}A^*_{m+n+1}(z)\right).$$

Miki's identity

In 1978 H. Miki [J. Number Theory 10(1978)] discovered the following curious identity which involves both an ordinary convolution and a binomial convolution of Bernoulli numbers:

$$\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n-k)} - \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n-k)} = 2H_n \frac{B_n}{n}$$

for every $n = 4, 5, \ldots$, where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

In the original proof of this identity, Miki showed that the two sides of the identity are congruent modulo all sufficiently large primes.

In 1982 Shiratani and Yokoyama [Mem. Fac. Sci. Kyushu Univ. Ser. A 36(1982)] gave another proof of Miki's identity by p-adic analysis. In 2005 I. M. Gessel [J. Number Theory 110(2005)] proved Miki's identity via Stirling numbers of the second kind.

Matiyasevich's identity

Inspired by Miki's work, Matiyasevich found in 1997 the following two identities of the same nature by the software *Mathematica*.

$$\sum_{k=2}^{n-2}\frac{B_k}{k}B_{n-k}-\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l}{l}B_{n-l}=H_nB_n$$

and

$$(n+2)\sum_{k=2}^{n-2}B_kB_{n-k}-2\sum_{l=2}^{n-2}\binom{n+2}{l}B_lB_{n-l}=n(n+1)B_n$$

for each $n=4,5,\ldots$. Clearly the first one is actually equivalent to Miki's identity since $\frac{n}{k(n-k)}=\frac{1}{k}+\frac{1}{n-k}$ for $0<k<n$.

Dunne and Schubert [`arXiv:math.NT/0406610`; Commun. Number Theory Physics 7(2013)] presented a new approach to Miki's and Matiyasevich's identities motivated by quantum field theory and string theory.

New approach of Pan and Sun

Since all previous proofs of Miki's identity are non-natural and complicated, in May 2004 H. Pan and Z. W. Sun developed a new method which only involves differences and derivatives of polynomials.

Define the operators Δ and Δ^* by $\Delta(f(x)) = f(x+1) - f(x)$ and $\Delta^*(f(x)) = f(x+1) + f(x)$. It is well known that

$$\Delta(B_n(x)) = nx^{n-1} \text{ and } \Delta^*(E_n(x)) = 2x^n \quad \text{for } n = 0, 1, 2, \ldots.$$

Lemma [H. Pan and Z. W. Sun, J. Combin. Theory Ser A 113(2006)]. Let $P(x), Q(x) \in \mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers.

(i) If $\Delta(P(x)) = \Delta(Q(x))$ then $P'(x) = Q'(x)$.

(ii) If $\Delta^*(P(x)) = \Delta^*(Q(x))$ then $P(x) = Q(x)$.

Prove Raabe's formula via the difference method

To illustrate the power of the lemma, let us give a simple proof of Raabe's multiplication formula. Clearly

$$\Delta\left(\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right)\right)$$
$$=\sum_{r=0}^{m-1}\left(B_n\left(\frac{x+r+1}{m}\right)-B_n\left(\frac{x+r}{m}\right)\right)$$
$$=B_n\left(\frac{x}{m}+1\right)-B_n\left(\frac{x}{m}\right)=n\left(\frac{x}{m}\right)^{n-1}=\Delta(m^{1-n}B_n(x))$$

and hence

$$\sum_{r=0}^{m-1}\frac{n}{m}B_{n-1}\left(\frac{x+r}{m}\right)=\frac{d}{dx}\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)$$
$$=\frac{d}{dx}(m^{1-n}B_n(x))=m^{1-n}nB_{n-1}(x)$$

for $n\in\mathbb{Z}^+$, this yields Raabe's formula.

Results of Pan and Sun

Theorem [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]. Let $n > 1$ be an integer. Then

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(y)}{k(n-k)} - \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{B_l(x-y)B_{n-l}(y) + B_l(y-x)B_{n-l}(x)}{l^2}$$
$$= H_{n-1} \frac{B_n(x) + B_n(y)}{n} + \frac{B_n(x) - B_n(y)}{n(x-y)}$$

and

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(y) - \sum_{l=0}^{n} \binom{n+1}{l+1} \frac{B_l(x-y)B_{n-l}(y) + B_l(y-x)B_{n-l}(x)}{l+2}$$
$$= \frac{B_{n+1}(x) + B_{n+1}(y)}{(x-y)^2} - \frac{2}{n+2} \cdot \frac{B_{n+2}(x) - B_{n+2}(y)}{(x-y)^3}.$$

Results of Pan and Sun

Letting y tend to x, we get

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(x)}{k(n-k)} - 2\sum_{l=2}^{n} \binom{n-1}{l-1} \frac{B_l B_{n-l}(x)}{l^2} = 2H_{n-1}\frac{B_n(x)}{n},$$

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) - 2\sum_{l=2}^{n} \binom{n+1}{l+1} \frac{B_l B_{n-l}(x)}{l+2} = (n+1)B_n(x).$$

This extends Miki's and Matuyasevich's identities to Bernoulli polynomials.

Results of Pan and Sun

Theorem [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]. Let $n \in \mathbb{Z}^+$. Then

$$\sum_{k=0}^{n} E_k(x)E_{n-k}(y) - \frac{4}{n+2} \cdot \frac{B_{n+2}(x) - B_{n+2}(y)}{x-y}$$
$$= -2\sum_{l=0}^{n+1} \binom{n+1}{l} \frac{E_l(x-y)B_{n+1-l}(y) + E_l(y-x)B_{n+1-l}(x)}{l+1},$$

$$\sum_{k=1}^{n} \frac{B_k(x)}{k} E_{n-k}(y) - H_n E_n(y) - \frac{E_n(x) - E_n(y)}{x-y}$$
$$= \sum_{l=1}^{n} \binom{n}{l} \left(\frac{B_l(x-y)}{l} E_{n-l}(y) - \frac{E_{l-1}(y-x)}{2} E_{n-l}(x) \right),$$

Results of Pan and Sun

$$\sum_{k=0}^{n} B_k(x)E_{n-k}(y)$$
$$= \sum_{l=1}^{n} \binom{n+1}{l+1} \left(B_l(x-y)E_{n-l}(y) - \frac{E_{l-1}(y-x)}{2} E_{n-l}(x) \right)$$
$$+ (n+1)\left(\frac{E_n(x)}{x-y} + E_n(y) \right) - \frac{E_{n+1}(x) - E_{n+1}(y)}{(x-y)^2}.$$

Letting $y \to x$ we obtain the following identities.

$$(n+2)\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = 8\sum_{l=2}^{n+2} \binom{n+2}{l} (2^l - 1)\frac{B_l}{l} B_{n+2-l}(x),$$

$$\sum_{k=1}^{n} \frac{B_k(x)}{k} E_{n-k}(x) - \sum_{l=2}^{n} \binom{n}{l} 2^l \frac{B_l}{l} E_{n-l}(x) = H_n E_n(x),$$

$$\sum_{k=0}^{n} B_k(x)E_{n-k}(x) - \sum_{l=2}^{n} \binom{n+1}{l+1} (2^l + l - 1)\frac{B_l}{l} E_{n-l}(x) = (n+1)E_n(x).$$

Woodcock's identity

In 1979 C. F. Woodcock [J. London Math. Soc. 20(1979), 101-108] discovered that

$$A_{m-1,\,n} = A_{n-1,\,m} \quad \text{for } m, n \in \mathbb{Z}^+$$

where

$$A_{m,\,n} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^k B_{m+k}B_{n-k}.$$

Thus

$$\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}B_k B_{n-k} + B_{n-1} = A_{1-1,n} = A_{n-1,1} = -B_n$$

as noted by L. Euler.

Pan and Sun's extensions

Theorem [H. Pan and Z. W. Sun, J. Combin. Theory Ser. A 113(2006)]. Let $m, n \in \mathbb{N}$ and $x + y + z = 1$. Then

$$
\begin{aligned}
&(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m-k+1}(x)}{m-k+1} \cdot \frac{B_{n+k+1}(y)}{n+k+1} \\
&+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k+1}(x)}{n-k+1} \cdot \frac{B_{m+k+1}(z)}{m+k+1} \\
=& \frac{(-1)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} \cdot \frac{B_{m+n+2}(x)}{m+n+2} - \frac{B_{m+1}(z)}{m+1} \cdot \frac{B_{n+1}(y)}{n+1} \\
&+ \frac{(-1)^{m+1}}{m+1} \cdot \frac{B_{m+n+2}(y)}{m+n+2} + \frac{(-1)^{n+1}}{n+1} \cdot \frac{B_{m+n+2}(z)}{m+n+2}.
\end{aligned}
$$

Pan and Sun's extensions

Also,

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(x) \frac{B_{n+k+1}(y)}{n+k+1}$$

$$+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(x) \frac{B_{m+k+1}(z)}{m+k+1}$$

$$= \frac{(-1)^{m+n+1} E_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}} - \frac{E_m(z)E_n(y)}{2}$$

and

$$\frac{(-1)^m}{2} \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(x) \frac{E_{n+k+1}(y)}{n+k+1}$$

$$- (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k+1}(x)}{n-k+1} \cdot \frac{E_{m+k+1}(z)}{m+k+1}$$

$$= \frac{(-1)^{m+n}}{(m+n+1)\binom{m+n}{n}} \cdot \frac{B_{m+n+2}(x)}{m+n+2} + \frac{(-1)^n}{n+1} \cdot \frac{E_{m+n+2}(z)}{m+n+2}$$

A unified approach

Let $n \in \mathbb{Z}^+$. Observe that

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(y) = \sum_{k=0}^{n}(-1)^k \binom{-1}{k} B_k(x)B_{n-k}(y)$$

and

$$-\sum_{k=1}^{n} \frac{B_k(x)}{k} B_{n-k}(y) = \sum_{k=1}^{n}(-1)^k \binom{-1}{k-1} \frac{B_k(x)}{k} B_{n-k}(y)$$
$$= \lim_{t \to 0} \frac{1}{t} \sum_{k=1}^{n}(-1)^k \binom{t}{k} B_k(x)B_{n-k}(y).$$

Inspired by this observation, in Sept. 2004 the speaker and H. Pan investigated relations among the sums

$$\sum_{k=0}^{n}(-1)^k \binom{s}{k}\binom{t}{n-k} P_k(x)Q_{n-k}(y)$$

with $P, Q \in \{B, E\}$.

A general theorem

It is easy to see that

$$0 = \begin{vmatrix} r & s & t \\ r & s & t \\ z & x & y \end{vmatrix} = r \begin{vmatrix} s & t \\ x & y \end{vmatrix} + s \begin{vmatrix} t & r \\ y & z \end{vmatrix} + t \begin{vmatrix} r & s \\ z & x \end{vmatrix}.$$

Theorem [Z. W. Sun and H. Pan, Acta Arith. 125(2006)]. Let $n \in \mathbb{Z}^+$, $x + y + z = 1$ and $r + s + t = n$. Then we have the symmetric relation

$$r \begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s \begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t \begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n} (-1)^k \binom{s}{k} \binom{t}{n-k} B_{n-k}(x) B_k(y).$$

Another theorem

Theorem [Z. W. Sun and H. Pan, Acta Arith. 125(2006)]. Let $n \in \mathbb{Z}^+$, $x + y + z = 1$ and $r + s + t = n - 1$. Then

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z) \\
&-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z) \\
=&\frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).
\end{aligned}$$

Thank you!