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Introduction to Combinatorial Number Theory (II) – Combinatorial Congruences

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Abstract

In this talk we introduce various combinatorial congruences as well as related proof techniques.

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Part I. Classical Combinatorial Congruences

Binomial coefficients

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ define

$$\binom{x}{n} = \frac{x(x-1)\ldots(x-n+1)}{n!}.$$

We also set

$$\binom{x}{0} = 1, \text{ and } \binom{x}{-n} = 0 \quad (n = 1, 2, 3, \ldots).$$

Note that

$$\binom{-1}{n} = (-1)^n \text{ and } \binom{x}{n} = \frac{x}{n}\binom{x-1}{n-1} \text{ for all } n \in \mathbb{Z}^+.$$

Also,

$$\binom{-x}{n} = (-1)^n \binom{x+n-1}{n} \quad \text{and} \quad \binom{x}{n} = \binom{x-1}{n} + \binom{x-1}{n-1}.$$

Fermat's little theorem

Fermat's Little Theorem. Let p be a prime and $a \in \mathbb{Z}$. Then $a^p \equiv a \pmod{p}$. In other words, if $p \nmid a$ then $a^{p-1} \equiv 1 \pmod{p}$.

Proof (Euler). For $k \in \{1, \ldots, p-1\}$, we have $p \mid \binom{p}{k}$ since $p \nmid k!$ but $p \mid k!\binom{p}{k} = p(p-1)\ldots(p-k+1)$.

If $n^p \equiv n \pmod{p}$ then $(-n)^p \equiv (-1)^p n \equiv -n \pmod{p}$. So it suffice to prove $n^p \equiv n \pmod{p}$ for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Clearly, $0^p \equiv 0 \pmod{p}$. If $n^p \equiv n \pmod{p}$, then

$$(n+1)^p = n^p + 1 + \sum_{k=1}^{p-1} \binom{p}{k} n^k \equiv n^p + 1 \equiv n + 1 \pmod{p}.$$

Let p be a prime. For any integer $a \not\equiv 0 \pmod{p}$ we call $q_p(a) = (a^{p-1} - 1)p \in \mathbb{Z}$ a *Fermat quotient*. If $a, b \in \mathbb{Z}$ and $p \nmid ab$, then

$$q_p(ab) = b^{p-1}\frac{a^{p-1}-1}{p} + \frac{b^{p-1}-1}{p} \equiv q_p(a) + q_p(b) \pmod{p}.$$

m-integers

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. A rational number is called an m-**integer** if its denominator is relatively prime to m. All m-integers form a subring R_m of the field $\mathbb{Q}$ of rational numbers. We may study congruences in this ring. For example,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 5^2 \times \frac{1}{12} \equiv 0 \pmod{5^2}.$$

If $a, b, c \in \mathbb{Z}$ and $(b, m) = 1$, then

$$\begin{aligned}
&\frac{a}{b} \equiv c \pmod{m} \text{ in the ring } R_m \\
\iff &\frac{a - bc}{b} = m\frac{u}{v} \text{ for some } u, v \in \mathbb{Z} \text{ with } (v, m) = 1 \\
\iff &(a - bc)v = bmu \text{ for some } u, v \in \mathbb{Z} \text{ with } (v, m) = 1 \\
\iff &a - bc = mq \text{ for some } q \in \mathbb{Z} \\
\iff &a \equiv bc \pmod{m} \text{ in the ring } \mathbb{Z}.
\end{aligned}$$

For a prime p, p-integers coincide with rational p-adic integers.

On $(2^p-2)/p \bmod p^2$

Let p be a prime. For $k \in 1,\ldots,p-1$ we have

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \equiv \frac{p}{k}\binom{-1}{k-1} = p\frac{(-1)^{k-1}}{k} \pmod{p^2}.$$

Moreover, for any $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\binom{p-1}{n} &= \prod_{k=1}^{n}\frac{p-k}{k} = (-1)^n\prod_{k=1}^{n}\left(1-\frac{p}{k}\right)\\ &\equiv (-1)^n\left(1-p\sum_{k=1}^{n}\frac{1}{k}\right) \pmod{p^2}.\end{aligned}$$

Those $H_n := \sum_{0<k\leqslant n}\frac{1}{k}$ $(n \in \mathbb{N} = \{0,1,2,\ldots\})$ are called *harmonic numbers*.

For any prime p we have

$$\frac{2^p-2}{p} = \frac{\sum_{k=1}^{p-1}\binom{p}{k}}{p} = \sum_{k=1}^{p-1}\frac{\binom{p-1}{k-1}}{k} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}(1-pH_{k-1}) \pmod{p^2}.$$

The Baker-Lerch Theorem

Euler's totient function: $\varphi(m) = |\{1 \leqslant a \leqslant m : \ (a, m) = 1\}|$.
If $m = \prod_{i=1}^{r} p_i^{a_i}$ with $p_1, \ldots, p_r$ distinct primes and $a_1, \ldots, a_r \in \mathbb{Z}^+$, then

$$\varphi(m) = \prod_{i=1}^{r} p_i^{a_i - 1}(p_i - 1) = m \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

Euler's Theorem. If $a \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ and $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 \pmod{m}$.

Baker-Lerch Theorem (1906). For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ with $(a, m) = 1$, we have

$$\frac{a^{\varphi(m)} - 1}{m} \equiv \sum_{\substack{k=1 \\ (k,m)=1}}^{m} \frac{1}{ak} \left\lfloor \frac{ak}{m} \right\rfloor \pmod{m}.$$

In particular, for any prime p and integer $a \not\equiv 0 \pmod{p}$, we have

$$\frac{a^{p-1} - 1}{p} \equiv \sum_{k=1}^{p-1} \frac{1}{ak} \left\lfloor \frac{ak}{p} \right\rfloor \pmod{p}.$$

Pan's proof of the Baker-Lerch Theorem

Proof (Hao Pan). For $k \in \mathbb{Z}^+$ let $r_k = \{ak\}_m$ denote the least nonnegative residue of ak modulo m. Then

$$\{r_k : \ 1 \leqslant k \leqslant m \ \& \ (k,m) = 1\} = \{1 \leqslant j \leqslant m : \ (j,m) = 1\}.$$

Thus

$$\begin{aligned} a^{\varphi(m)} &= \prod_{\substack{k=1 \\ (k,m)=1}}^{m} \frac{m\lfloor ak/m \rfloor + r_k}{k} = \prod_{\substack{k=1 \\ (k,m)=1}}^{m} \frac{r_k}{k}\left(1 + \frac{m}{r_k}\left\lfloor \frac{ak}{m} \right\rfloor\right) \\ &\equiv \prod_{\substack{k=1 \\ (k,m)=1}}^{m} \left(1 + \frac{m}{r_k}\left\lfloor \frac{ak}{m} \right\rfloor\right) \equiv 1 + \sum_{\substack{k=1 \\ (k,m)=1}}^{m} \frac{m}{r_k}\left\lfloor \frac{ak}{m} \right\rfloor \\ &\equiv 1 + m \sum_{\substack{k=1 \\ (k,m)=1}}^{m} \frac{1}{ak}\left\lfloor \frac{ak}{m} \right\rfloor \pmod{m^2}. \end{aligned}$$

Lucas' congruence

Lucas' Theorem (E. Lucas, 1878). Let p be a prime. For any $k, n \in \mathbb{N}$ and $s, t \in \{0, \ldots, p-1\}$ we have

$$\binom{pn+s}{pk+t} \equiv \binom{n}{k}\binom{s}{t} \pmod{p}.$$

Equivalently, if $a_i, b_i \in \{0, \ldots, p-1\}$ for all $i = 0, \ldots, k$ then

$$\binom{\sum_{i=0}^{k} a_i p^i}{\sum_{i=0}^{k} b_i p^i} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod{p}.$$

Proof. $(1+x)^p = 1 + x^p + \sum_{k=1}^{p-1} \binom{p}{k} x^k \equiv 1 + x^p \pmod{p}$. So

$$(1+x)^{pn+s} \equiv (1+x)^s (1+x^p)^n \pmod{p}.$$

Combining the coefficients of x^{pk+t} in the expansions of both sides, we get

$$\binom{pn+s}{pk+t} \equiv \binom{s}{t}\binom{n}{k} \pmod{p}.$$

Wolstenholme's theorem

Theorem (Wolstenholme, 1862). Let $p > 3$ be a prime. Then

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}, \quad H_{p-1}^{(2)} := \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p},$$

and

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

Proof. As $\{\{2k\}_p : \ k = 1, \ldots, p-1\} = \{1, \ldots, p-1\}$, we have

$$\sum_{k=1}^{p-1} \frac{1}{(2k)^2} \equiv \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p}.$$

So $3H_{p-1}^{(2)} \equiv 0 \pmod{p}$ and hence $H_{p-1}^{(2)} \equiv 0 \pmod{p}$ since $p > 3$. Observe that

$$2H_{p-1} = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \frac{1}{p-k} \right) = \sum_{k=1}^{p-1} \frac{p}{k(p-k)} \equiv -pH_{p-1}^{(2)} \equiv 0 \pmod{p^2}.$$

Wolstenhilme's congruence

$$\begin{aligned}\binom{2p-1}{p-1} &= \prod_{k=1}^{p-1} \frac{p+k}{k} = \prod_{k=1}^{p-1} \left(1+\frac{p}{k}\right) \\ &\equiv 1 + \sum_{k=1}^{p-1} \frac{p}{k} + \sum_{1\leqslant j<k\leqslant p-1} \frac{p^2}{jk} \\ &\equiv 1 + pH_{p-1} + \frac{p^2}{2}\left(H_{p-1}^2 - H_{p-1}^{(2)}\right) \equiv 1 \pmod{p^3}.\end{aligned}$$

Open Problem (J. P. Jone, 1982). Are there any composite numbers $n > 3$ such that $\binom{2n-1}{n-1} \equiv 1 \pmod{n^3}$?

Further refinements

Let $p > 3$ be a prime. Then

$$\begin{aligned}\frac{2}{p}H_{p-1} &= \sum_{k=1}^{p-1}\frac{1}{k(p-k)} = \sum_{k=1}^{p-1}\frac{p+k}{k(p^2-k^2)}\\ &\equiv -\sum_{k=1}^{p-1}\frac{p+k}{k^3} = -H_{p-1}^{(2)} - p\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^3}+\frac{1}{(p-k)^3}\right)\\ &\equiv -H_{p-1}^{(2)} \pmod{p^2}\end{aligned}$$

and so $H_{p-1} \equiv -\frac{p}{2}H_{p-1}^{(2)} \pmod{p^3}$.

J.W.L. Glaisher (1900): Let $p > 3$ be a prime. Then

$$H_{p-1}^{(m)} \equiv \begin{cases} \frac{pm}{m+1}B_{p-1-m} \pmod{p^2} & \text{if } m \in \{2,4,\ldots,p-3\},\\ -\frac{p^2m(m+1)}{2(m+2)}B_{p-2-m} \pmod{p^3} & \text{if } m \in \{1,3,\ldots,p-4\},\end{cases}$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers. Also,

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3B_{p-3} \pmod{p^4}.$$

Other famous congruences

Theorem. Let $p > 3$ be a prime.

(i) (F. Morley, 1895) $\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}$.

(ii) (Gauss) If $p \equiv 1 \pmod 4$ and $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 4$, then $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$.

(iii) (E. Lehmer, 1938) $H_{(p-1)/2} \equiv -2q_p(2) + p\, q_p(2)^2 \pmod{p^2}$.

(iv) (Ljunggren, 1952) $\binom{pm}{pn} \equiv \binom{m}{n} \pmod{p^3}$ for all $m, n \in \mathbb{N}$.

(v) (Jacobsthal, 1952) For integers $m > n > 0$, the rational number

$$\frac{\binom{pm}{pn} / \binom{m}{n} - 1}{p^3 mn(m-n)}$$

is a p-integer, i.e.,

$$\binom{pm}{pn} / \binom{m}{n} \equiv 1 \pmod{p^{3+\nu_p(mn(m-n))}},$$

where $\nu_p(x) = \max\{a \in \mathbb{N} : \ p^a \mid x\}$ is the p-adic order (or p-adic valuation) of a nonzero integer x

A Lemma

Lemma. Let $p > 3$ be a prime. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{(pn)^2} \sum_{\substack{k=1 \\ p \nmid k}}^{pn-1} \frac{1}{k} \in \mathbb{Z}_p \quad \text{and} \quad \frac{1}{pn} \sum_{\substack{k=1 \\ p \nmid k}}^{pn-1} \frac{1}{k^2} \in \mathbb{Z}_p.$$

Proof. Write $pn = p^a q$ with $a, q \in \mathbb{Z}^+$ and $p \nmid q$. Let g be a primitive root modulo p^a. Then $\{g^i : \ i = 0, 1, \ldots, \varphi(p^a) - 1\}$ is a reduced system of residues modulo p^a. Thus

$$\sum_{\substack{k=1 \\ p \nmid k}}^{p^a-1} \frac{1}{k^2} \equiv \sum_{i=0}^{\varphi(p^a)-1} g^{-2i} = \frac{(g^{-2})^{\varphi(p^a)} - 1}{g^{-2} - 1} \equiv 0 \pmod{p^a}.$$

(Note that $g \not\equiv \pm 1 \pmod{p}$ since $p > 3$.) Therefore

$$\sum_{\substack{k=1 \\ p \nmid k}}^{p^a q-1} \frac{1}{k^2} \equiv \sum_{s=0}^{q-1} \sum_{\substack{t=1 \\ p \nmid t}}^{p^a-1} \frac{1}{(p^a s + t)^2} \equiv q \sum_{\substack{t=1 \\ p \nmid t}}^{p^a-1} \frac{1}{t^2} \equiv 0 \pmod{p^a}.$$

This proves the second assertion in the Lemma.

Proof of the Lemma

$$2\sum_{\substack{i=1\\ p\nmid i}}^{pn-1}\frac{1}{i}=\sum_{\substack{k=1\\ p\nmid k}}^{pn-1}\left(\frac{1}{k}+\frac{1}{pn-k}\right)=\sum_{\substack{k=1\\ p\nmid k}}^{pn-1}\frac{pn}{k(pn-k)}$$

$$\equiv -pn\sum_{\substack{k=1\\ p\nmid k}}^{pn-1}\frac{1}{k^2}\equiv 0 \pmod{p^{2+2\nu_p(n)}}.$$

So we also have

$$\frac{1}{(pn)^2}\sum_{\substack{k=1\\ p\nmid k}}^{pn-1}\frac{1}{k}\in\mathbb{Z}_p.$$

A weaker version of Jacobsthal's congruence

Now we prove a slightly weaker version of Jacobsthal's congruence.

Theorem Let $p > 3$ be a prime and let $m, n \in \mathbb{Z}^+$ with $m > n$. Then

$$\binom{pm}{pn} \Big/ \binom{m}{n} \in 1 + p^3 n(m-n)\mathbb{Z}_p,$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

Proof. Without loss of generality, we assume that $\nu_p(n) \leqslant \nu_p(m-n)$. Observe that

$$\binom{pm}{pn} = \prod_{j=0}^{pn-1} \frac{pm-j}{pn-j} = \prod_{i=0}^{n-1} \frac{pm-pi}{pn-pi} \times \prod_{\substack{0 \leqslant j < pn \\ p \nmid j}} \left(1 + \frac{p(m-n)}{pn-j}\right).$$

Thus

$$\frac{\binom{pm}{pn}}{\binom{m}{n}} = \prod_{\substack{j=1 \\ p \nmid j}}^{pn-1} \left(1 + \frac{p(m-n)}{pn-j}\right) = \prod_{\substack{i=1 \\ p \nmid i}}^{pn-1} \left(1 + \frac{p(m-n)}{i}\right).$$

Continue the proof

For each integer $k \geqslant 3$, $(p(m-n))^k/(p^3(m-n)n) \in \mathbb{Z}_p$. Thus

$$\begin{aligned}\frac{\binom{pm}{pn}}{\binom{m}{n}} \equiv & 1 + \sum_{\substack{i=1 \\ p\nmid i}}^{pn-1} \frac{p(m-n)}{i} + \sum_{\substack{1\leqslant i<j<pn \\ p\nmid ij}} \frac{(p(m-n))^2}{ij} \\ = & 1 + p(m-n)\sum_{\substack{i=1 \\ p\nmid i}}^{pn-1} \frac{1}{i} + \frac{p^2(m-n)^2}{2}\left(\left(\sum_{\substack{i=1 \\ p\nmid i}}^{pn-1} \frac{1}{i}\right)^2 - \sum_{\substack{i=1 \\ p\nmid i}}^{pn-1} \frac{1}{i^2}\right) \\ \equiv & 1 \pmod{p^{3+\nu_p(n(m-n))}}\end{aligned}$$

by the Lemma. So we have

$$\frac{\binom{pm}{pn}}{\binom{m}{n}} \in 1 + p^3 n(m-n)\mathbb{Z}_p.$$

Part II. Modern Congruences via Combinatorial Identities

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $\left(\frac{a}{p}\right)$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

$\sum_{k=0}^{p-1} \binom{2k}{k}/m^k \bmod p$ via the Binomial Theorem

For $n \in \mathbb{N}$, it is easy to see that $\binom{-1/2}{n} = \binom{2n}{n}/(-4)^n$.

For any odd prime p and $k \in \{0, \ldots, p-1\}$, clearly

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \iff 2k > p \iff \frac{p}{2} < k < p.$$

For any odd prime p and integer $m \not\equiv 0 \pmod{p}$, clearly

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} = \sum_{k=0}^{(p-1)/2} \binom{-1/2}{k} \frac{(-4)^k}{m^k} \\ &\equiv \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \left(-\frac{4}{m}\right)^k = \left(1 - \frac{4}{m}\right)^{(p-1)/2} \\ &\equiv \left(\frac{m(m-4)}{p}\right) \pmod{p}.\end{aligned}$$

On $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$

Z.-W. Sun and R. Tauraso [Int. J. Number Theory 7(2011)]: For any odd prime p we have

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

Z. W. Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m\in\mathbb{Z}$ with $p\nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m^2-4m}{p}\right)+u_{p-(\frac{m^2-4m}{p})}\pmod{p^2},$$

where the sequence $(u_n)_{n\geqslant 0}$ is given by

$$u_0=0,\ u_1=1,\text{and } u_{n+1}=(m-2)u_n-u_{n-1}\ (n=1,2,3,\ldots).$$

Z.-W. Sun [Taiwanese J. Math. 17(2013)] determined $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k \bmod p^2$ for any odd prime p and integer $m\not\equiv 0 \pmod p$.

$\sum_{k=1}^{p-1}\binom{2k}{k}/k \bmod p^3$ via Staver's identity

Z. W. Sun & Tauraso [Adv. in Appl. Math 45(2010)]. For any prime $p>5$ we have

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}. \qquad (*)$$

$(*)$ mod p was first deduced by H. Pan and Z.-W. Sun [Discrete Math. 306(2006)] from a complicated combinatorial identity.

Taking $n=p-1$ in Staver's identity

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}=\frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2},$$

we get

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{p}{3}\binom{2p-1}{p-1}\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\ (\mathrm{mod}\ p^2).$$

With more efforts we can prove $(*)$.

A trick on congruences modulo p^2

Let $p=2n+1$ be an odd prime. For each $k=0,\ldots,n$, we have

$$\begin{aligned}\binom{n}{k}\binom{n+k}{k} &= (-1)^k\binom{n}{k}\binom{-n-1}{k}\\ &= (-1)^k\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}\\ &\equiv (-1)^k\binom{-1/2}{k}^2 = (-1)^k\left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 \pmod{p^2}.\end{aligned}$$

By the Chu-Vandermonde identity,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k=\sum_{k=0}^{n}\binom{n}{n-k}\binom{-n-1}{k}=\binom{-1}{n}=(-1)^n.$$

Thus

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^n=\left(\frac{-1}{p}\right)\pmod{p^2},\qquad(*)$$

which was conjectured by Rodriguez-Villegas in 2001 and first proved by E. Mortenson in 2003 via the p-adic Gamma function. The above approach was given by Z.-H. Sun & R. Tauraso in 2009.

Connections to Euler numbers

Recall that the Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

Note that $E_{p-3} \equiv 0 \pmod{p}$ for the primes $p = 149,\ 241$.

Z. W. Sun [Sci. China Math. 54(2011)]: For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}.$$

Franel numbers

It is well known that $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.

In 1894 J. Franel introduced the *Franel numbers*

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

and noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1) + 2) f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

V. Strehl's Identity:

$$A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} f_k,$$

where A_n is the Apéry number $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$.

Barrucand's Identity:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{where } g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Supercongruences involving Franel numbers

Theorem (Z. W. Sun [Adv. Appl. Math., 2013]). Let $p > 3$ be a prime. For any p-adic integer r we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{k+r}{k}f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{k+r}{k}^2 \pmod{p^2}.$$

In particular,

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\ \sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k \equiv 0 \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k \equiv 0 \pmod{p}.$$

The polynomials $f_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^k$

Motivated by Strehl's identity $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} = f_n$, Z.-W. Sun introduced the Franel polynomials

$$f_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} x^k \ (n \in \mathbb{N}).$$

Theorem (Sun [Adv. Appl. Math. 51(2013)]) Let p be an odd prime and let r be any p-adic integer. Then

$$\sum_{l=0}^{p-1} (-1)^l \binom{l+r}{l} f_l(x) \equiv \sum_{k=0}^{p-1} \binom{2k}{k} x^k \binom{k+r}{k}^2 \pmod{p^2}.$$

To prove this we need an auxiliary identity

$$\sum_{l=k}^{2k} (-1)^l \binom{l}{k} \binom{k}{l-k} \binom{x+l}{l} = \binom{x+k}{x}^2.$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

$$\sum_{l=1}^{p-1}(-1)^l \frac{f_l(x)}{l} = \sum_{l=1}^{p-1} \frac{(-1)^l}{l} \sum_{k=0}^{l} \binom{l}{k}\binom{k}{l-k}\binom{2k}{k} x^k$$

$$= \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} x^k \sum_{l=k}^{p-1} (-1)^l \binom{l-1}{k-1}\binom{k}{l-k}.$$

If $1 \leqslant k \leqslant (p-1)/2$, then

$$\sum_{l=k}^{p-1}(-1)^l \binom{l-1}{k-1}\binom{k}{l-k} = \sum_{l=k}^{2k}(-1)^l \binom{l-1}{k-1}\binom{k}{l-k}$$

$$= \sum_{j=0}^{k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}$$

$$= (-1)^k \sum_{j=0}^{k}\binom{-k}{j}\binom{k}{k-j} = 0$$

by the Chu-Vandermonde identity.

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

If $(p+1)/2 \leqslant k \leqslant p-1$, then

$$\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} = \sum_{j=0}^{p-1-k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}$$

$$=(-1)^k\sum_{j=0}^{p-1-k}\binom{-k}{j}\binom{k}{k-j}.$$

Applying Andersen's identity

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{m-k} = \frac{m-n}{m}\binom{x-1}{n}\binom{-x}{m-n}\ (0 \leqslant n \leqslant m),$$

we obtain

$$\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} = (-1)^{k-1}\binom{k}{p-k} \equiv \frac{1}{2}\binom{2(p-k)}{p-k} \pmod{p}$$

Note that $\binom{2k}{k} \equiv 0 \pmod{p}$ for $k = (p+1)/2, \ldots, p-1$.

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

By the above,

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}f_l(x) \equiv \sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{k}x^k\frac{\binom{2(p-k)}{p-k}}{2} \equiv p\sum_{k=(p+1)/2}^{p-1}\frac{x^k}{k^2} \pmod{p^2}$$

since for any $k = 0, \ldots, p-1$ we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p\rfloor - 1}2p \pmod{p^2}$$

by Sun [Sci. China Math. 54(2011), Lemma 2.1].

To prove $\sum_{l=1}^{p-1}(-1)^l f_l/l \equiv 0 \pmod{p^2}$, it suffices to note that

$$2\sum_{k=(p+1)/2}^{p-1}\frac{1}{k^2} \equiv \sum_{k=(p+1)/2}^{p-1}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right) = \sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 0 \pmod{p}.$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k^2 \equiv 0 \pmod{p}$

Now we show

$$\sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} f_k^{(r)} \equiv 0 \pmod{p}, \text{ where } f_k^{(r)} := \sum_{j=0}^{k} \binom{k}{j}^r.$$

Clearly,

$$\sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} = \sum_{k=1}^{(p-1)/2} \left(\frac{(-1)^{kr}}{k^{r-1}} + \frac{(-1)^{(p-k)r}}{(p-k)^{r-1}} \right) \equiv 0 \pmod{p}.$$

Thus

$$\begin{aligned}
\sum_{l=1}^{p-1} \frac{(-1)^{lr}}{l^{r-1}} f_l^{(r)} &\equiv \sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{l=k}^{p-1} (-1)^{lr} \binom{l-1}{k-1}^{r-1} \binom{l}{k} \\
&= \sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{j=0}^{p-1-k} (-1)^{(k+j)r} \binom{k+j-1}{j}^{r-1} \binom{k+j}{j} \\
&= \sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} \sum_{j=0}^{p-1-k} \binom{-k}{j}^{r-1} \binom{-k-1}{j} \pmod{p}.
\end{aligned}$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k^2 \equiv 0 \pmod p$

By the above,

$$\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-k-1}\binom{p-k}{j}^{r-1}\binom{p-k-1}{j} \pmod p.$$

For any positive integer n, we have

$$\begin{aligned} f_n^{(r)} =& \sum_{k=0}^{n}\left(\frac{k}{n}+\frac{n-k}{n}\right)\binom{n}{k}^r = 2\sum_{k=0}^{n}\frac{n-k}{n}\binom{n}{k}^r \\ =& 2\sum_{k=0}^{n-1}\binom{n}{k}^{r-1}\binom{n-1}{k}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} \equiv& \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\cdot\frac{f_{p-k}^{(r)}}{2} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{(p-k)r}f_k^{(r)}}{(p-k)^{r-1}} \\ \equiv& -\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)} \pmod p. \end{aligned}$$

Connection between $p = x^2 + 3y^2$ and Franel numbers

Z.W. Sun [J. Number Theory 133(2013)]: Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Conjecture (Z. W. Sun): For any prime $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$, we have

$$x \equiv \frac{1}{4}\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k} \equiv \frac{1}{2}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} \pmod{p^2}.$$

Connection between $p = x^2 + 3y^2$ and the numbers g_n

Conjecture (Z. W. Sun [J. Number Theory 2013]): Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$x \equiv \sum_{k=0}^{p-1} (k+1)\frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} (k+1)\frac{g_k}{(-3)^k} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$2\sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv -\sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Rodriguez-Villegas' conjecture

Let $p>3$ be a prime. In 2003 Rodriguez-Villegas conjectured congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2. Via an advanced approach Mortenson [2005] provided a partial solution with some remaining things including

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 7 \pmod 8,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 11 \pmod{12}.$$

My work on the remaining parts

In 2012 I proved all the remaining parts of the three congruences conjectured by Rodriguez-Villegas.

Theorem [Z. W. Sun, Acta Arith. 132(2012)]. Let $p > 3$ be a prime. For each $d = 0, \ldots, (p-1)/2$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{4k}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod 8,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod 4.$$

Detailed proof of the congruence involving 108^k

For $d = 0, 1, 2, \ldots$, we define

$$f(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k}.$$

By the Zeilberger algorithm, we find the recursive relation:

$$\begin{aligned}
&(3d+1)(6d+1)f(d) - (3d+2)(6d+5)f(d+1)\\
=&\frac{(3p-1)(3p-2)(2d+1)}{2^{2p-1}27^{p-1}p}\binom{2p}{p+2d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1}.
\end{aligned}$$

Note that

$$\frac{1}{p}\binom{2p-2}{p-1} = C_{p-1} \in \mathbb{Z},\ (3p-2)\binom{3p-3}{p-1} = p\binom{3p-2}{p} \equiv 0 \pmod{p},$$

$$\begin{aligned}
(2d+1)\binom{2p}{p+2d+1} \equiv&(p+2d+1)\binom{2p}{p+2d+1} = 2p\binom{2p-1}{p+2d}\\
\equiv&0 \pmod{p}.
\end{aligned}$$

Detailed proof of the congruence involving 108^k

Thus, for each $d = 0, \ldots, (p-1)/2$ we have

$$(3d+1)(6d+1)f(d) \equiv (3d+2)(6d+5)f(d+1) \pmod{p^2}.$$

Suppose that $p \equiv 5 \pmod 6$ and $0 \leqslant d < (p-1)/2$. Then $3d+1, 6d+1 \neq p, 2p$ and hence $3d+1, 6d+1 \not\equiv 0 \pmod p$. So

$$f(d+1) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.$$

Therefore

$$f\left(\frac{p-1}{2}\right) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}$$

$$\implies f(0) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

Detailed proof of the congruence involving 108^k

Note that

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \\
=&\frac{\binom{2p-2}{p-1}\binom{3p-3}{p-1}}{108^{p-1}} \\
=&108^{1-p}\frac{p}{2p-1}\binom{2p-1}{p}\frac{p}{3p-2}\binom{3p-2}{p} \\
\equiv&0 \pmod{p^2}.
\end{aligned}$$

This concludes the proof of

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

Using OEIS

In 2010 I conjectured that for any positive integer n the number

$$t_n := \frac{1}{4n\binom{2n}{n}} \sum_{k=0}^{n-1} (21k+8) \binom{2k}{k}^3$$

is always an integer.

After reading my message to Number Theory List, Kasper Andersen found that

$$t_n = \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2.$$

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After reading my message to Number Theory List, Kasper Andersen found that

$$t_n = \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2.$$

How did he find this? He just inputs few initial terms $t_0, t_1, \ldots, t_8$ and then searched this on N.J.A Sloane's OEIS (On-Line Encyclopedia of Integer Sequences) with the website `http://oeis.org`.

Using the Sigma package

To prove some congruences modulo higher power a prime p, we often need to simplify some sums involving harmonic numbers. For this purpose, we may employ the Sigma package to find a relative simple form of a complicated sum involving harmonic numbers.

I found the following identities via the software Sigma.

Z.-W. Sun [Sci. China Math. 54(2011)] For any $n \in \mathbb{Z}^+$ we have

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} (H_{n+k} - H_{n-k}) = \sum_{k=1}^{n} \frac{(-1)^k}{k} - \frac{H_n}{2},$$

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k (H_{n+k} - H_{n-k}) = \frac{3}{2} \sum_{k=1}^{n} \frac{\binom{2k}{k}}{k}.$$

Z.-W. Sun [J. Number Theory 134(2014)] For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(2(n+k)+1)16^k} = \frac{\binom{2k}{k}^2}{16^k} \sum_{k=0}^{2n} \frac{1}{2k+1}.$$

Reduce a double sum to a single sum

To study some challenging conjectures of Sun on congruences for

$$S_n=\sum_{k=0}^{n-1}\sum_{l=0}^{k}F(k,l)\ \ (n\in\mathbb{Z}^+),$$

where $F(k,l)$ is a bivariate hypergeometric term of k and l, Yan-Ping Mu and Z.-W. Sun [Int. J. Number Theory 14(2018)] search for two hypergeometric terms $G_1(k,l)$ and $G_2(k,l)$ such that

$$F(k,l)=\Delta_k(G_1(k,l))+\Delta_l(G_2(k,l)),$$

where

$$G_1(k,l)=R_1(k,l)F(k,l)\quad\text{and}\quad G_2(k,l)=R_2(k,l)F(k,l)$$

with $R_1(k,l)$ and $R_2(k,l)$ rational functions, and

$$\Delta_k(G_1(k,l))=G_1(k+1,l)-G_1(k,l),$$

$$\Delta_l(G_2(k,l))=G_2(k,l+1)-G_2(k,l).$$

The resulting functions $G_1(k,l)$ and $G_2(k,l)$ we obtain are essentially well defined for $0\leqslant l\leqslant k\leqslant n-1$.

The telescoping approach

Once we have $G_1(k,l)$ and $G_2(k,l)$ in hand, the sum S_n can be transformed to a single sum

$$S_n = \sum_{l=0}^{n-1} \big(G_1(n,l) - G_1(l,l)\big) + \sum_{k=0}^{n-1} \big(G_2(k,k+1) - G_2(k,0)\big)$$

We can use the `Maple` package *DoubleSum* given by Chen-Hou-Mu [J. Comput. Appl. Math. 196(2006)] or the `Mathematica` package *HolonomicFunctions* given by C. Koutschan [Math. Comput. Sci. 4(2010)], together with the package *DoubleSum* or the package *MultiSum* [Appl. Algebra Engrg. Comm. Comput. 13(2002)], to compute a suitable pair $(G_1(k,l), G_2(k,l))$.

Once we get a single sum for S_n, it would be be convenient to deduce Sun's conjectural congruences for S_n. Using this powerful method, Mu and Sun confirm several sophisticated open conjectures of Sun.

On $F_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k$

Theorem (Conjectured by Sun in 2011, and proved by Mu and Sun [Int. J. Number Theory 14(2018)]). For $k = 0, 1, 2, \ldots$ define $F_k := \sum_{l=0}^{k} \binom{k}{l}^3 (-8)^l$. Then, for any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+5)(-1)^k F_k$$

is always an odd integer.

A key step of the proof is the identity

$$\begin{aligned}
&\sum_{k=0}^{n-1} (6k+5)(-1)^k F_k \\
&= -\frac{4}{3} \sum_{l=0}^{n-1} (12n^2 - 30nl + 21l^2 - 32n + 46l + 25) \binom{n}{l+1}^3 (-8)^l (-1)^n \\
&\quad - \frac{1}{3} \sum_{k=0}^{n-1} (12k^2 + 10k + 5)(-1)^k.
\end{aligned}$$

Selected open conjectures of the speaker

Conjecture [Z.-W. Sun, J. Number Theory 131(2011)]: For any prime $p \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{(p-1)/2} \frac{kC_k^3}{16^k} \equiv 2p-2 \pmod{p^2},$$

where C_k denotes the Catalan number $\binom{2k}{k}/(k+1)$.

Conjecture [Z.-W. Sun, J. Number Theory 134(2014)]: For any prime $p > 5$, we have

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}^2}{k16^k} \equiv -\frac{21}{2}H_{p-1} \pmod{p^4}.$$

Remark. The speaker has proved the first congruence mod p and the second congruence mod p^3.

Selected open conjectures of the speaker

Conjecture [Z.-W. Sun, Nanjing Univ. Math. Biquarterly 36(2019)]: For any odd prime p, we have

$$\sum_{k=0}^{p-1}\frac{f_k}{8^k}\equiv\left(\frac{p}{3}\right)-\frac{p^2}{12}B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{f_{k-1}}{k8^{k-1}}\equiv-p^2B_{p-3}\pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{f_k}{k8^k}\equiv3q_p(2)-\frac{3}{2}p\,q_p(2)^2+p^2\,q_p(2)^3\pmod{p^3}.$$

Conjecture [Z.-W. Sun, Nanjing Univ. Math. Biquarterly 36(2019)]: For any prime $p>3$ we have

$$\sum_{n=1}^{p-1}\frac{\mathrm{Domb}(n)}{n}\equiv\left(\frac{p}{3}\right)\frac{2}{5}pB_{p-2}\left(\frac{1}{3}\right)\pmod{p^2},$$

where $\mathrm{Domb}(n)=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$.

Selected open conjectures of the speaker

Conjecture [Z.-W. Sun, Adv. Appl. Math. 51(2013)] For any prime $p>3$, we have

$$\sum_{n=0}^{p-1}(-1)^n\sum_{k=0}^{n}\binom{n}{k}^3(-8)^k\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

The Motzkin numbers are given by $M_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k$ $(n\in\mathbb{N})$.

Conjecture [Z.-W. Sun, Sci. China Math. 57(2014)]: For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}M_k^2\equiv(2-6p)\left(\frac{p}{3}\right)\pmod{p^2}.$$

Remark. I have proved (see arXiv:1801.08905) that

$$\sum_{k=0}^{p-1}(2k+1)M_k^2\equiv 12p\left(\frac{p}{3}\right)\pmod{p^2}.$$

For more open conjectures of mine on combinatorial congruences, one may look at

Z.-W. Sun, *Open conjectures on congruences*, Nanjing Univ. Math. Biquarterly **36** (2019), 1–99. Available from `http://maths.nju.edu.cn/~zwsun/191o.pdf`

Thank you!