π-Series and Their p-adic Analogues

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June 5, 2020

Abstract

In this talk I'll introduce π-series (in particular Ramanujan-type series) and their p-adic analogues. I'll also tell how I found new π-series via congruences.

Part I. Ramanujan Series for $\frac{1}{\pi}$ and Zeilberger-type Series

The Gamma function

The Classical Gamma Function:

$$\Gamma(x) = \int_0^\infty \frac{t^{x-1}}{e^t} dt \ (x > 0), \quad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{Z}^+.$$

Euler's Formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

In particular,

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England
Modular equations and approximations to π,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Elliptic integrals

Complete elliptic integrals (with $0 < k < 1$):

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \quad \text{(the first kind)},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \quad \text{(the seond kind)}.$$

Legendre's Relation:

$$E(k)K(\sqrt{1-k^2}) + E(\sqrt{1-k^2})K(k) - K(k)K(\sqrt{1-k^2}) = \frac{\pi}{2}.$$

A Central Result:

$$_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \mid k^2\right) = \frac{2}{\pi} K(k) = \varphi^2(q)$$

where $q = e^{-\pi K(\sqrt{1-k^2})/K(k)}$ and

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{(theta function)}.$$

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 Jonathan Borwein and Peter Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

My first impression on Ramanujan-type series

In a year around 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\left(-\frac{27}{512}\right)^k\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 known Ramanujan-type series for $1/\pi$ with $a,b,m\in\mathbb{Z}$. I prefer their forms in terms of binomial coefficients.

What is needed for proving $\sum_{n=0}^{\infty}(6n+1)\binom{2n}{n}^3/256^n = 4/\pi$

The proofs of Ramanujan series involve lots of things such as modulo forms, elliptic integrals, theta functions, hypergeometric series, modular equations and symbolic computation.

$$P(q) := 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j} \quad \text{(Eisenstein series)},$$

$$\varphi(q) := \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{(theta function)},$$

$$X = X(q) = q\prod_{j=1}^{\infty}\frac{(1-q^j)^{24}(1-q^{4j})^{24}}{(1-q^{2j})^{48}}.$$

$$\varphi(q)^4 = \sum_{n=0}^{\infty}\binom{2n}{n}X^n, \; P(q^2) = \sqrt{1-64X}\sum_{n=0}^{\infty}(3n+1)\binom{2n}{n}^3 X^n.$$

$$X(e^{-\pi\sqrt{3}}) = \frac{1}{256} \text{ and } P(e^{-2\pi\sqrt{3}}) = \frac{\sqrt{3}}{\pi} + \frac{\sqrt{3}}{4}\varphi(e^{-\pi\sqrt{3}})^4.$$

The p-adic Gamma function

Let p be a prime and let $\mathbb{Z}_p$ be the ring of p-adic integers. Any p-adic integer x has a unique p-adic series representation

$$x = a_0 + a_1 p + a_2 p^2 + \dots \quad \text{with } a_0, a_1, a_2, \dots \in \{0, \dots, p-1\}$$

which converges according to the p-adic norm $|\cdot|_p$. Note that

$$x \equiv \sum_{k=0}^{n-1} a_k p^k \pmod{p^n} \text{ and } \left| x - \sum_{k=0}^{n-1} a_k p^k \right|_p \leqslant p^{-n} \to 0.$$

So each p-adic integer is the limit of a sequence of natural numbers which converges p-adically.

The p-adic Gamma function: For $n \in \mathbb{Z}^+$ define

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<k<n \\ p \nmid k}} k.$$

Also set $\Gamma_p(0) = 1$. For $x \in \mathbb{Z}_p$, choose a sequence of natural numbers $(x_n)_{n \geqslant 0}$ whose p-adic limit is x, and then define

$$\Gamma_p(x) = \lim_{n \to \infty} \Gamma_p(x_n).$$

van Hamme's idea

Similar to Euler's formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

for any $x \in \mathbb{Z}_p$ we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\{x\}_p},$$

where $\{x\}_p$ is the unique $r \in \{1, \dots, p\}$ with $x \equiv r \pmod{p}$. In particular,

$$\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\{1/2\}_p} = (-1)^{(p+1)/2} = -\left(\frac{-1}{p}\right).$$

Using this idea, in 1997 van Hamme posed p-adic analogues of many series for powers of π.

van Hamme's conjectures

For the two Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi} \text{ and } \sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

in 1997 van Hamme conjectured their following p-adic analogues:

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv p\left(\frac{-2}{p}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) \pmod{p^4},$$

where p is an odd prime.

All the p-adic analogue conjectures of van Hamme were proved before 2017. Following van Hamme's idea, Zudilin [JNT, 2009] proposed more p-adic analogues for Ramanujan-type series.

$\sum_{k=0}^{p-1} a_k$ modulo powers of p for the series $\sum_{k=0}^{\infty} a_k$

Recall that for each $m = 2, 3, \ldots$ we have

$$\zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m} = \lim_{n\to\infty} H_n^{(m)},$$

where $H_n^{(m)} = \sum_{0<k\leqslant n} 1/k^m$. As Euler proved, for each $m \in \mathbb{Z}^+$ we have

$$2\zeta(2m) = (-1)^{m-1} B_{2m} \frac{(2\pi)^{2m}}{(2m)!}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} = \frac{(-1)^m E_{2m} \pi^{2m+1}}{4^{m+1}(2m)!}.$$

J.W.L. Glaisher (1900): Let $p > 3$ be a prime. Then

$$H_{p-1}^{(m)} \equiv \begin{cases} \frac{pm}{m+1} B_{p-1-m} \pmod{p^2} & \text{if } m \in \{2, 4, \ldots, p-3\}, \\ -\frac{p^2 m(m+1)}{2(m+2)} B_{p-2-m} \pmod{p^3} & \text{if } m \in \{1, 3, \ldots, p-4\}, \end{cases}.$$

My Philosophy (Sun, 2010): If $(a_k)_{k\geqslant 0}$ is a sequence of rational numbers with $\sum_{k=0}^{p-1} a_k$ related to the zeta function or powers of π, then for large primes p, the partial sum $\sum_{k=0}^{p-1} a_k$ modulo powers of p is related to Bernoulli numbers or Bernoulli polynomials.

Zeilberger-type series

In 1993, D. Zeilberger used the Wilf-Zeilberger method to obtain the new identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

Define

$$F(n,k) = \frac{1}{\binom{2n}{n}(n+1)^2\binom{2n+k+1}{n+1}^2}$$

and

$$G(n,k) = \frac{n!^4(n+k)!^2}{2(2n+1)!(2n+k+2)!^2}P(n,k),$$

where $P(n,k)$ denotes

$$(n+1)^2(21n+13) + 2k^3 + k^2(13n+11) + k(28n^2+48n+20).$$

Then $\langle F, G\rangle$ is a **WZ pair** in the sense that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Zeilberger's proof

$$\sum_{k=0}^{N-1}(F(n+1,k)-F(n,k)) = \sum_{k=0}^{N-1}(G(n,k+1)-G(n,k)) = G(n,N)-G(n,0).$$

$$\sum_{n=0}^{N}\left(\sum_{k=0}^{N-1}F(n+1,k)-\sum_{k=0}^{N-1}F(n,k)\right) = \sum_{n=0}^{N}G(n,N)-\sum_{n=0}^{N}G(n,0).$$

$$\sum_{k=0}^{N-1}F(N+1,k)-\sum_{k=0}^{N-1}F(0,k) = \sum_{n=0}^{N}(G(n,N)-G(n,0)).$$

$$F(0,k)=\frac{1}{(k+1)^2},\ \ G(n,0)=\frac{21(n+1)-8}{(n+1)^3\binom{2n+2}{n+1}^3}.$$

$$\sum_{k=0}^{N-1}F(N+1,k)-\sum_{n=1}^{N}\frac{1}{n^2} = \sum_{n=0}^{N}G(n,N)-\sum_{n=1}^{N+1}\frac{21n-8}{n^3\binom{2n}{n}^3}$$

and hence $\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3}=\zeta(2)=\frac{\pi^2}{6}$ since $\sum_{k=0}^{N-1}F(N+1,k)\to 0$ and $\sum_{n=0}^{N}G(n,N)\to 0$.

Other Zeilberger-type series

J. Guillera [Ramanujan J. 15(2008)] used the WZ method to give three new Zeilberger-type series:

$$\sum_{k=1}^{\infty}\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3} = -16G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3} = -2G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2},$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$.

Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [Proc. Amer. Math. Soc. 147(2019)] provided a q-analogue of the last identity with $|q|<1$:

$$\sum_{n=0}^{\infty} q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3} = (1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}.$$

Gosper's series for π

Gosper (1974):

$$\pi = \sum_{k=0}^{\infty} \frac{50k-6}{2^k \binom{3k}{k}}.$$

Proof. For $a > 0$ and $b > 0$, Euler found that

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Thus

$$\int_0^1 x^{2n}(1-x)^n dx = B(2n+1, n+1) = \frac{\Gamma(2n+1)\Gamma(n+1)}{\Gamma(3n+2)} = \frac{1}{(3n+1)\binom{3n}{n}}$$

and hence

$$\sum_{n=0}^{\infty} \frac{50n-6}{2^n\binom{3n}{n}} = \frac{(50n-6)(3n+1)}{2^n} \int_0^1 x^{2n}(1-x)^n dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} (3n+1)(50n-6)\left(\frac{x^2(1-x)}{2}\right)^n dx = \ldots = \pi.$$

Part II. Techniques to Find Series via Transforms of Congruences

A Useful Lemma

Lemma (Z.-W. Sun [Sci. China Math. 54(2011)]) Let p be an odd prime and let $k \in \{0, \ldots, p-1\}$. Then

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Thus,

$$\binom{2(p-k)}{p-k} \equiv \begin{cases} \frac{2p}{k\binom{2k}{k}} \pmod{p} & \text{if } k \in \{\frac{p+1}{2}, \ldots, p-1\}, \\ \frac{-2p}{k\binom{2k}{k}} \pmod{p^2} & \text{if } k \in \{1, \ldots, \frac{p-1}{2}\}. \end{cases}$$

Remark. R. Tauraso [J. Number Theory 130(2010)] realized that

$$\binom{2(p-k)}{p-k} \equiv \frac{2p}{k\binom{2k}{k}} \pmod{p} \text{ for all } k = 1, \ldots, p-1.$$

We have similar lemmas involving $\binom{3k}{k}$ or $\binom{4k}{2k}$.

Rediscover Zeilberger's series $\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}$

In 2010 I proved that for any odd prime p we have

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8p+16p^4B_{p-3} \pmod{p^5}.$$

As the series $\sum_{k=0}^{\infty}(21k+8)\binom{2k}{k}^3$ diverges, it does not provide a Ramanujan-type series for $1/\pi$. However, I observe that

$$\begin{aligned}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 =&8+\sum_{k=(p+1)/2}^{p-1}(21(p-k)+8)\binom{2(p-k)}{p-k}^3\\ \equiv&8-\sum_{k=(p+1)/2}^{p-1}(21k-8)\left(\frac{2p}{k\binom{2k}{k}}\right)^3 \pmod{p}\end{aligned}$$

and this led me to rediscover that

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6} \quad \text{(D. Zeilberger, 1993).}$$

Conjecture: $\sum_{k=1}^\infty \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2$

Conjecture (Z.-W. Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv \begin{cases} x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1, \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 3p+\frac{7}{2}p^4B_{p-3} \pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} \equiv 32\frac{2^{p-1}-1}{p}-\frac{64}{3}p^2B_{p-3} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^\infty\frac{(11k-3)64^k}{k^2\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2 \quad \text{(confirmed by J. Guillera in 2013).}$$

Conjecture: $\sum_{k=1}^\infty \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K$

Conjecture (Z.-W. Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2,8 \pmod{15} \ \&\ p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1; \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 4p\left(\frac{p}{3}\right) + \frac{4}{3}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4}.$$

Also,

$$\sum_{k=1}^\infty \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K := \sum_{k=1}^\infty \frac{(\frac{k}{3})}{k^2} \text{ (confirmed by}$$

Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012).

More such conjectural series

Conjecture (Z.-W. Sun, 2010; Sci. China Math. 54(2011))

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

The three conjectural identities were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97(2014)].

Central trinomial coefficients

For each $n = 0, 1, 2, \ldots$, the central trinomial coefficient T_n is defined as the coefficient of x^n in the expansion of $(x^2 + x + 1)^n$. It is easy to see that $T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}$. For any prime $p > 3$ and $k \in \{1, \ldots, p-1\}$ we have $T_{p-k} \equiv (\frac{p}{3}) T_{k-1}/(-3)^{k-1} \pmod{p}$.

Conjecture (Sun, Jan. 22, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2 T_k$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p}) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2 T_k \equiv p\left(20 + 24\left(\frac{p}{3}\right)(2 - 3^{p-1})\right) \pmod{p^3}.$$

Two new series for π involving central trinomial coefficients

Conjecture (Sun, 2019) For any prime $p > 3$, we have

$$p^2 \sum_{k=1}^{p-1} \frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} \equiv 11\left(\frac{p}{3}\right) + \frac{p}{2}\left(13 - 35\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$p^2 \sum_{k=1}^{p-1} \frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2 (k-1)3^{k-1}} \equiv -\frac{1}{2}\left(\frac{p}{3}\right) - \frac{p}{8}\left(7 + \left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

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Conjecture (Sun, Dec. 7, 2019). We have

$$\sum_{k=1}^{\infty} \frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} = \frac{5\pi}{\sqrt{3}} + 6\log 3,$$

$$\sum_{k=2}^{\infty} \frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2 (k-1)3^{k-1}} = \frac{21 - 2\sqrt{3}\,\pi - 9\log 3}{12}.$$

Remark. As the two series converge very fast, it is easy to check the two identities numerically. I posed the two identities to `MathOverflow` in 2019, nobody has idea to prove it.

A curious identity with \$480 prize for the solution

Conjecture (Z.-W. Sun) (i) (2009-11-29) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv 0 \pmod{p^2}.$$

(ii) (2014-07-07) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv \frac{5}{12}p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

$$p^2 \sum_{k=1}^{p-1} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} \equiv 4\left(\frac{p}{3}\right) + 4p \pmod{p^2}.$$

(iii) (2014-08-12, **\$480 prize for the solution**) We have

$$\sum_{k=1}^{\infty} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} = \frac{15}{2}\sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2}.$$

Three more conjectural series

Motivated by corresponding congruences, I made the following conjecture in 2010-2011.

Conjecture (Z.-W. Sun) (i) [Sci. China Math. 54(2011)] We have

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2}$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

(In 2004 H.H. Chan, S.H. Chan and Z. Liu [Adv. Math.] proved that $\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{8}{\sqrt{3}\pi}$.)

(ii) [Electron. J. Combin. 20(2013)] We have

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Another transform of congruences

Z.-W. Sun [Nanjing Univ. Math. Biquarterly 32(2015)]: Let $p = 2n+1$ be an odd prime. Then, for each $k = 0, \ldots, n$ we have

$$\frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{-1}{p}\right)\binom{2(n-k)}{n-k} \pmod{p}.$$

This is easy. In fact,

$$\begin{aligned}\binom{2k}{k} = \binom{-1/2}{k}(-4)^k \equiv& \binom{n}{k}(-4)^k = \binom{n}{n-k}(-4)^k \\ \equiv& \binom{-1/2}{n-k}(-4)^k = \frac{\binom{2(n-k)}{n-k}}{(-4)^{n-k}}(-4)^k \\ \equiv& (-1)^n\binom{2(n-k)}{n-k}16^k \pmod{p}.\end{aligned}$$

An Example

Let $p = 2n + 1$ be an odd prime. Then

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv \sum_{k=0}^{n}(21(n-k)+8)\left(\frac{-1}{p}\right)\left(\frac{\binom{2k}{k}}{16^k}\right)^3$$
$$\equiv \frac{(-1)^{(p+1)/2}}{2}\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \pmod{p}.$$

This relates the Zeilberger series

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6}$$

to the Ramanujan series

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi}.$$

A transformation via dual sequences

For a sequence $a_0, a_1, a_2, \ldots$ of complex numbers, define

$$a_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \quad \text{for all } n \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

and call $(a_n^*)_{n\in\mathbb{N}}$ the *dual sequence* of $(a_n)_{n\in\mathbb{N}}$. It is well known that $a_n^{**} = a_n$ for all $n \in \mathbb{N}$.

For example,

$$w_n := \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{3k}{k} \binom{2k}{k} = 3^n a_n^*$$

where

$$a_n = \begin{cases} \binom{3k}{k}\binom{2k}{k}/27^k & \text{if } n = 3k, \\ 0 & \text{if } 3 \nmid n. \end{cases}$$

On March 10, 2011, I realized that if $|m-4| > 4$ then

$$\sum_{n=0}^{\infty} (bmn+2b+(m-4)c) \frac{\binom{2n}{n} a_n^*}{(4-m)^n} = (m-4)\sqrt{\frac{m-4}{m}} \sum_{k=0}^{\infty} (bk+c) \frac{\binom{2k}{k} a_k}{m^k}.$$

Congruences for dual sequences

Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 32(2015)]: Let p be an odd prime and let m be an integer with $p \nmid m(m-4)$. Let α be a positive integer, and let $a_0, a_1, \ldots, a_{p^\alpha-1}$ be p-adic integers. Then we have the congruences

$$\sum_{k=0}^{p^\alpha-1} \frac{\binom{2k}{k}}{(4-m)^k} a_k^* \equiv \left(\frac{m(m-4)}{p^\alpha}\right) \sum_{k=0}^{p^\alpha-1} \frac{\binom{2k}{k}}{m^k} a_k \pmod{p}$$

and

$$m\sum_{k=0}^{p^\alpha-1} \frac{k\binom{2k}{k}}{(4-m)^k} a_k^* \equiv \left(\frac{m(m-4)}{p^\alpha}\right) \sum_{k=0}^{p^\alpha-1} ((m-4)k-2)\frac{\binom{2k}{k}}{m^k} a_k \pmod{p},$$

where $(\frac{\cdot}{p^\alpha})$ denotes the Jacobi symbol.

If $(-1)^k a_k = f_k := \sum_{j=0}^k \binom{k}{j}^3$ $(k = 0, 1, \ldots)$, then $a_n^* = \sum_{k=0}^n \binom{n}{k} f_k = g_n$, where $g_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$.

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases}(\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1,\end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod 4$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy\pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4}.$$

Another Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{2160^n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 105y^2 \ (x, y \in \mathbb{Z}), \\ 2x^2 - 2p \pmod{p^2} & \text{if } 2p = x^2 + 105y^2 \ (x, y \in \mathbb{Z}), \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 35y^2 \ (x, y \in \mathbb{Z}), \\ 2p - 6x^2 \pmod{p^2} & \text{if } 2p = 3x^2 + 35y^2 \ (x, y \in \mathbb{Z}), \\ 20x^2 - 2p \pmod{p^2} & \text{if } p = 5x^2 + 21y^2 \ (x, y \in \mathbb{Z}), \\ 10x^2 - 2p \pmod{p^2} & \text{if } 2p = 5x^2 + 21y^2 \ (x, y \in \mathbb{Z}), \\ 28x^2 - 2p \pmod{p^2} & \text{if } p = 7x^2 + 15y^2 \ (x, y \in \mathbb{Z}), \\ 14x^2 - 2p \pmod{p^2} & \text{if } 2p = 7x^2 + 15y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p}) = -1. \end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

One more Example Illustrating the Philosophy

Conjecture (Z.-W. Sun, Jan. 2012) (i) For any prime $p > 3$ we have

$$\sum_{n=0}^{p-1} \frac{28n+5}{576^n} \binom{2n}{n} \sum_{k=0}^{n} 5^k \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} \equiv p \left(\frac{-1}{p}\right) \left(3 + 2\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

(ii) We have the identity

$$\sum_{n=0}^{\infty} \frac{28n+5}{576^n} \binom{2n}{n} \sum_{k=0}^{n} 5^k \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = \frac{9}{\pi}(2+\sqrt{2}).$$

Conjecture (Sun). For any prime $p > 5$, we have

$$\left(\frac{-1}{p}\right)\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{576^n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1,\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1,\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1,\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

Part III. Series for $\frac{1}{\pi}$ involving $T_n(b, c)$

Generalized central trinomial coefficients

For real numbers b and c, we define

$$T_n(b,c) := [x^n](x^2+bx+c)^n$$

(the coefficient of x^n in $(x^2+bx+c)^n$)

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k} c^k.$$

Recursion: $T_0(b,c)=1$, $T_1(b,c)=b$, and

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ (n>0),$$

where $d=b^2-4c$. It is known that if $d \neq 0$ then

$$T_n(b,c) = \sqrt{d}^n P_n\left(\frac{b}{\sqrt{d}}\right)$$

where

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the *Legendre polynomial* of degree n.

Asymptotic Behavior of $T_n(b,c)$

By the Laplace-Heine formula, for $x \notin [-1,1]$ we have

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}} \quad \text{as } n \to +\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c) \sim f_n(b,c) := \frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b,c) = (-1)^n T_n(b,c)$.

Conjecture (Sun, 2011; proved by S. Wagner): For $b, c > 0$,

$$T_n(b,c) = f_n(b,c)\left(1 + \frac{b-4\sqrt{c}}{16n\sqrt{c}} + O\left(\frac{1}{n^2}\right)\right)$$

as $n \to +\infty$. If $c>0$ and $b = 4\sqrt{c}$, then

$$\frac{T_n(b,c)}{\sqrt{c}^n} = \frac{3\times 6^n}{\sqrt{6n\pi}}\left(1 + \frac{1}{8n^2} + \frac{15}{64n^3} + \frac{21}{32n^4} + O\left(\frac{1}{n^5}\right)\right).$$

If $c<0$ and $b \in \mathbb{R}$ then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b,c)|} = \sqrt{b^2-4c}.$$

Replace $\binom{2k}{k}$ by $T_k(b,c)$

As $T_k(2,1)=\binom{2k}{k}$, in 2010 I viewed $T_k(b,c)$ as a natural extension of the central binomial coefficients. In contrast with my conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k}$ and $\sum_{k=0}^{p-1}(a+dk)\frac{\binom{2k}{k}^3}{m^k}$ modulo p^2 (with p an odd prime not dividing m), in December 2010 I formulated many conjectures with some $\binom{2k}{k}$ replaced by $T_k(b,c)$. For example, I made the following conjecture.

Conjecture (Sun, 2010-12-25). Let p be any odd prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and } p=x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv 7p\left(\frac{-1}{p}\right)\pmod{p^2}.$$

Story happened around Jan 1, 2011

$$T_n(1,16) \sim \frac{(1+2\sqrt{16})^{n+1/2}}{2\sqrt[4]{16}\sqrt{n\pi}} = \frac{9^{n+1/2}}{4\sqrt{n\pi}} = \frac{9^n}{12\sqrt{n\pi}}.$$

This is very similar to the fact that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}$. On Dec. 18, 2010, I conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2},$$

which is very similar to Ramanujan-type congruences.

Story happened around Jan 1, 2011

$$T_n(1,16) \sim \frac{(1+2\sqrt{16})^{n+1/2}}{2\sqrt[4]{16}\sqrt{n\pi}} = \frac{9^{n+1/2}}{4\sqrt{n\pi}} = \frac{9^n}{12\sqrt{n\pi}}.$$

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$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2},$$

which is very similar to Ramanujan-type congruences.

Conjecture (Z. W. Sun, Jan. 2, 2011). We have

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 T_k(1,16) = \frac{24}{\pi},$$

```
T[n_]:=If[n>0,Coefficient[(x^2+x+16)^n,x^n],1]
S[n_]:=Sum[(30k+7)Binomial[2k,k]^2*T[k]/(-256)^k,{k,0,n}]
Print[N[S[200]Pi,20]]
```

Output: 24.000000000000000000

New series for $1/\pi$ involving $T_k(b,c)$

For $b, c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a, b, c, d, m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

`Type I.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

`Type II.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

`Type III.` $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

`Type IV.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

`Type V.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

New series for $1/\pi$ involving $T_k(b,c)$

For $b, c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a, b, c, d, m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

Type I. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In October 2011, I found 10 conjectural series for $1/\pi$ of two new types:

Type VI. $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

Type VII. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, W. Zudilin.

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

My unsolved conjectural series of type VII

Conjecture (Sun, 2011). (i) For any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n\binom{2n-1}{n-1}}\sum_{k=0}^{n-1}(2800512k+435257)434^{2(n-1-k)}\binom{2k}{k}T_k(73,576)^2$$

is an odd integer, and

$$n\binom{2n-1}{n-1} \,\Big|\, \sum_{k=0}^{n-1}(24k+5)28^{2(n-1-k)}\binom{2k}{k}T_k(4,9)^2.$$

(ii) We have

$$\sum_{k=0}^{p-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2=\frac{10406669}{2\sqrt{6}\,\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}).$$

Conjecture (Sun). (i) If $p>3$ is a prime with $p\neq 7,11,17,31$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(73,576)^2}{434^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1,\ p=x^2+102y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{p}{17})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=2x^2+51y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{17})=-1,\ p=3x^2+34y^2,\\ 24x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{17})=-1,\ p=6x^2+17y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-102}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) For any odd prime $p\neq 7,31$, we have

$$\sum_{k=0}^{p-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2$$
$$\equiv p\left(466752\left(\frac{-6}{p}\right)-31495\right)\pmod{p^2}.$$

Conjecture (Sun). (i) For any prime $p > 7$, we have

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{T_k(4,9)^2}{28^{2k}}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1,\ p=x^2+30y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{5})=-1,\ p=3x^2+10y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{5})=-1,\ p=2x^2+15y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=5x^2+6y^2,\\ p\delta_{p,7}\pmod{p^2} & \text{if } (\frac{-30}{p})=-1.\end{cases}$$

where x and y are integers.

(ii) For any odd prime $p \neq 7$, we have

$$\sum_{k=0}^{p-1}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2\equiv p\left(\frac{-6}{p}\right)\left(4+\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

Duality Principle

Duality Principle (conjectured by Z.-W. Sun in 2011; appeared in arXiv:1911.05456). Let $(a_k)_{k\geqslant 0}$ be an integer sequence such that

$$a_k \equiv \left(\frac{d}{p}\right) D^k a_{p-1-k} \pmod{p}$$

for any prime $p \nmid 6dD$ and $k \in \{0, \ldots, p-1\}$, where d and D are fixed nonzero integers. If m is a nonzero integer such that

$$\sum_{k=0}^{\infty} \frac{bk+c}{m^k} a_k = \frac{\lambda_1\sqrt{d_1} + \lambda_2\sqrt{d_2} + \lambda_3\sqrt{d_3}}{\pi}$$

for some $b, d_1, d_2, d_3 \in \mathbb{Z}^+$, $c \in \mathbb{Z}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}$, then m divides D, and

$$\sum_{k=0}^{p-1} \frac{a_k}{m^k} \equiv \left(\frac{d}{p}\right) \sum_{k=0}^{p-1} \frac{a_k}{(D/m)^k} \pmod{p^2}$$

for any prime $p > 3$ with $p \nmid dD$.

Remark. For any $b, c \in \mathbb{Z}$ and odd prime $p \nmid b^2 - 4c$, we have $T_k(b,c) \equiv (\frac{b^2-4c}{p})(b^2-4c)^k T_{p-1-k}(b,c) \pmod{p}$ for $0 \leqslant k < p$.

Comments from Shaun Cooper

In 2017, Prof. Shaun Cooper published the following book:
`S. Cooper, Ramanujan Theta Functions, Springer, Cham, 2017.`

In his Notes for Chapter 14 (Ramanujan's series for $1/\pi$), he wrote the following comments:

"The theory of Ramanujan's series for $1/\pi$ was extended significantly by the announcement of a large number of conjectures by Z.-W. Sun that are summarized in [279]. Sun's conjectures have stimulated and inspired works by W. Zudilin and coauthors, including work with H.H. Chan and J. Wan [93], the paper [125], works with J. Guillera [172], with J. Wan [294] and the paper [308]. See also the work of M. Rogers and A. Straub [258] and the works of J. Wan [292], [293]."

My 2019 conjectural series of type VIII

In November 2019, I introduced a new type series for $1/\pi$.

`Type VIII.` $\sum_{k=0}^{\infty}(a+dk)T_k(b_1,c_1)T_k(b_2,c_2)^2/m^k = C/\pi$.

Conjecture (Sun, Nov. 2019). We have

$$\sum_{k=0}^{\infty}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2=\frac{55\sqrt{15}}{9\pi}, \qquad \text{(VIII1)}$$

$$\sum_{k=0}^{\infty}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2=\frac{1452\sqrt{5}}{\pi}, \qquad \text{(VIII2)}$$

$$\sum_{k=0}^{\infty}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2=\frac{189\sqrt{15}}{2\pi}, \qquad \text{(VIII3)}$$

$$\sum_{k=0}^{\infty}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2=\frac{6795\sqrt{5}}{\pi}. \qquad \text{(VIII4)}$$

A congruence related to the identity (VIII4)

Conjecture (Sun, 2019). Let $p>5$ be a prime with $p\neq 11$. Then

$$\sum_{k=0}^{p-1}\frac{T_k(14,1)T_k(11,-11)^2}{(-29700)^k}$$

$$\equiv\begin{cases}4x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=1,\ p=x^2+165y^2\\ 2x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=x^2+165y^2,\\ 12x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{11})=1,\ p=3x^2+55y^2,\\ 6x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1,\ 2p=3x^2+55y^2,\\ 2p-20x^2 & \text{if } (\frac{-1}{p})=(\frac{p}{11})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=5x^2+33y^2,\\ 2p-10x^2 & \text{if } (\frac{-1}{p})=(\frac{p}{11})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=5x^2+33y^2,\\ 44x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{11})=1,\ p=11x^2+15y^2,\\ 22x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=11x^2+15y^2,\\ 0 & \text{if } (\frac{-165}{p})=-1,\end{cases}$$

modulo p^2, where x and y are integers.

Remark. The quadratic field $\mathbb{Q}(\sqrt{-165})$ has class number 8.

Congruences related to the identity (VIII4)

Conjecture (Sun, 2019). Let $p > 5$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{39480k+7321}{(-29700)^k} T_k(14,1) T_k(11,-11)^2 \equiv p\left(5738\left(\frac{-5}{p}\right)+70\left(\frac{3}{p}\right)+1513\right) \pmod{p^2}.$$

If $(\frac{3}{p}) = (\frac{-5}{p}) = 1$, then

$$\sum_{k=0}^{pn-1} \frac{39480k+7321}{(-29700)^k} T_k(14,1) T_k(11,-11)^2 - p\sum_{k=0}^{n-1} \frac{39480k+7321}{(-29700)^k} T_k(14,1) T_k(11,-11)^2$$

divided by $(pn)^2$ is a p-adic integer for each $n \in \mathbb{Z}^+$.

Main References:

1. Z.-W. Sun, *List of conjectural series for powers of π and other constants*, preprint, `arXiv:1102.5649`, 2011-2014.

2. Z.-W. Sun, *Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$*, in: Number Theory and Related Area (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Adv. Lect. Math. 27, Higher Education Press and Internat. Press, Beijing-Boston, 2013, pp. 149–197.

3. Z.-W. Sun, *New series for powers of π and related congruences*, accepted by Electron. Res. Arch. (See also arXiv:1911.05456.)

Thank you!