# Combinatorial Number Theory in China

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**While in the past many of the basic combinatorial results were obtained mainly by ingenuity and detailed reasoning, the modern theory has grown out of this early stage, and often relies on deep, well developed tools.**

—Noga Alon (ICM, Beijing, 2002)

**Additive combinatorics is currently a highly active area of research. One remarkable feature of the field is the use of tools from many diverse fields of mathematics.**

—Terence Tao & V. H. Vu (2006)

# What is Combinatorial Number Theory ?

**Combinatorial Number Theory** studies number-theoretic problems of combinatorial favor or the combinatorics of integers.

**Topics in CNT include**: Additive combinatorics, Ramsey theory of integers, combinatorial congruences (*not* combinatorial identities), etc.

**Tools**: Elementary method, algebraic method, analytic method, and mixture of several methods.

**Two typical Problems in CNT**:

1. (Olson) *Is it true that for any* $a_1, \ldots, a_{k(n-1)+1} \in \mathbb{Z}_n^k$ *(the direct sum of* $k$ *copies of the cyclic group* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$*) we can select some of them so that their sum is zero?*

2. (Snevily's Conjecture) *Let* $G$ *be any abelian group of odd order. Then, for any subsets* $A$ *and* $B$ *of* $G$ *with cardinality* $k$ *we can write* $A = \{a_1, \ldots, a_k\}$ *and* $B = \{b_1, \ldots, b_k\}$ *with* $a_1 + b_1, \ldots, a_k + b_k$ *pairwise distinct.*

# Leading Mathematicians in CNT

Paul Erdős
Ronald Graham
Noga Alon
E. Szemeredi
W. Gowers
Terence Tao
Ben Green
I. Z. Ruzsa
M. B. Nathanson
A. Sárközy
Carl Pomerance

# Three main CNT teams in China

**Wei-Dong Gao's School**: Study zero-sum sequences for abelian groups. [I include Prof. P. Z. Yuan in Gao's school since he was led to the field of zero-sums by Prof. Gao.]

*Remark*. Gao and his students view them as combinatorists.

**Yong-Gao Chen's School**: Study sumset problems and applications of covers.

*Remark*. Chen and his students view themselves as number theorists.

**Zhi-Wei Sun's School**: Study covers of the integers or groups, restricted sumsets via Combinatorial Nullstellensatz, and various combinatorial congruences.

*Remark*. Sun and his students view themselves as combinatorists and number theorists alternatively.

# Representative work of W. D. Gao

**Gao's Theorem** [conjectured by Y. Caro]. Let $G$ be a finite additive abelian group. Let $D(G)$ (the Davenport constant for $G$) [respectively, $s(G)$] be the smallest positive integer $n$ such that for any $a_1, \ldots, a_n \in G$ there are distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$ [resp., with $k = |G|$] satisfying $a_{i_1} + \cdots + a_{i_k} = 0$.Then $s(G) = D(G) + |G| - 1$.

**References**

1. W. D. Gao, *A combinatorial problem on finite abelian groups*, J. Number Theory **58**(1996), 100–103.

2. M. Devos, *Gao's theorem for nonabelian groups*, Open Problem Garden, `http://garden.irmacs.sfu.ca/?q=op/gaos_theorem_for_nonabelian_groups`

**Conjecture**: Gao's theorem holds for any finite group $G$.

# Representative work of P. Z. Yuan

Let $G$ be a finite abelian group and $A$ a finite subset of $\mathbb{Z}$. Define $D_A(G)$ (resp., $E_A(G)$) to be the smallest positive integer $n$ such that for any $x_1, \ldots, x_n \in G$ there are $a_1, \ldots, a_k \in A$ (resp., with $k = |G|$) and distinct $j_1, \ldots, j_k \in \{1, \ldots, n\}$ with $\sum_{i=1}^{k} a_i x_{j_i} = 0$.

**Adhikari-Thangadurai Conjecture**. $E_A(G) = D_A(G) + |G| - 1$.

**A Result of P. Z. Yuan and X. Zeng (2009)**. The conjecture holds for all finite cyclic groups.

Inspired by Yuan and Zeng's preprint, D. J. Grynkiewicz et al. proved the whole conjecture.

**References**


1. P. Yuan and X. Zeng, *Davenport constant with weights*, preprint, 2009.

2. D. J. Grynkiewicz, L. E. Marchan and O. Ordaz, *A weighted generalization of two theorems of Gao*, preprint, `arXiv:0903.2810`.

# Representative work of Y. G. Chen

For $n = 1, 2, 3, \ldots$ define

$$g(n) = \max\{|S| : \ S \subseteq \mathbb{Z}^+ \text{ and } [x, y] \leqslant n \text{ for all } x, y \in S\}.$$

In 1951 P. Erdős asked for values of $g(n)$.

In 1998 Y. G. Chen successfully provided an asymptotic formula.

**Chen's Theorem**. $g(n) = \sqrt{\frac{9}{8}n} + o(\sqrt{n})$.

Let $A$ be an infinite set of positive integers. In 1962 P. Erdős conjectured that if $A(n) = |\{a \in A : \ a \leqslant n\}| > c\sqrt{n}$ (where $c$ is a positive constant), then $\{\sum_{a \in S} a : \ S \subseteq A\}$ contains an infinite arithmetic progression.

**Y. G. Chen gave the first proof of this conjecture.**

**References**


1. Y. G. Chen, *Sequences with bounded l.c.m. of each pair of terms*, Acta Arith. **84**(1998), 71–95.

2. Y. G. Chen, *On subset sums of a fixed set*, Acta Arith. **106**(2003), 207–211.

# Representative work of Z. W. Sun

A main contribution of Sun to CNT is the discovery of connections between zero-sum problems for abelian groups and covers of the integers by residue classes.

**Theorem [Z. W. Sun, Israel J. Math. 2009]**. Let

$$A = \{a_1(\mathrm{mod}\ n_1), \ldots, a_k(\mathrm{mod}\ n_k)\}$$

be a finite collection of residue classes, and let $G$ be any abelian $p$-group. If $A$ covers each integer either (exactly) $2|G|-1$ or $2|G|$ times, then for any $c_1, \ldots, c_k \in G$ there exists an $I \subseteq \{1, \ldots, k\}$ such that $\sum_{i\in I} 1/n_i = |G|$ and $\sum_{i\in I} c_i = 0$.

*Remark*. It is interesting to view $1/n_i$ as a weight of $i$. In the case $n_1 = \cdots = n_k = 1$, the result of Sun yields the classical Erdős-Ginzburg-Ziv theorem. By a recent paper of H. Pan and L. L. Zhao [Adv. in Appl. Math. 2009], for each $m = 2, 3, \ldots$ there are infinitely many exact $m$-covers which cannot split into two covers of $\mathbb{Z}$.

# Representative work of Z. W. Sun (continued)

In the direction of Snevily's conjecture, Sun established a 3-dimensional analogue of Snevily's conjecture.

**Theorem [Z. W. Sun, Math. Res. Lett. 2008]** Let $G$ be any additive abelian group with cyclic torsion subgroup, and let $A$, $B$ and $C$ be finite subsets of $G$ with cardinality $n > 0$. Then there is a numbering $\{a_i\}_{i=1}^n$ of the elements of $A$, a numbering $\{b_i\}_{i=1}^n$ of the elements of $B$ and a numbering $\{c_i\}_{i=1}^n$ of the elements of $C$, such that all the sums $a_i + b_i + c_i$ $(1 \leqslant i \leqslant n)$ are (pairwise) distinct. Consequently, each subcube of the Latin cube formed by the Cayley addition table of $\mathbb{Z}/N\mathbb{Z}$ contains a Latin transversal.

**Remark.** We don't require that $|G|$ is odd. The theorem does not hold for general abelian groups. It even fails for the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

# Representative work of Z. W. Sun (continued)

In the direction of covers of groups by cosets, Sun made an important progress on the Herzog-Schönheim conjecture.

**Theorem [Z. W. Sun, J. Algebra 2004]** Let $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group $G$ with all the $G_i$ *subnormal* in $G$ but not all equal to $G$. Suppose that $\mathcal{A}$ covers all the elements of $G$ the same number of times, and that among the indices

$$n_1 = [G : G_1] \leqslant \ldots \leqslant n_k = [G : G_k].$$

each occurs at most $M \in \mathbb{Z}^+$ times. Then $M > 1$, i.e, $[G : G_i] = [G : G_j]$ for some $1 \leqslant i < j \leqslant k$. Furthermore,

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log\log M),$$

where $\gamma$ is Euler's constant and the $O$-constant is absolute.

**Remark.** A subgroup $H$ of $G$ is called subnormal if there is a chain of subgroups $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ such that $H_i$ is normal in $H_{i+1}$ for all $i = 0, \ldots, n-1$.

# Representative work of Z. W. Sun (continued)

**References**

1. Z. W. Sun, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announ. Amer. Math. Soc. **9**(2003), 51–60.

2. Z. W. Sun, *Zero-sum problems for abelian p-groups and covers of the integers by residue classes*, Israel. J. Math. **170**(2009), 235–252.

3. Z. W. Sun, *An additive theorem and restricted sumsets*, Math. Res. Lett. **15**(2008), 1263–1276.

4. Z. W. Sun, *On the Herzog-Schönheim conjecture for uniform covers of groups*, J. Algebra **273**(2004), 153–175.

# Some conjectures of Z. W. Sun

**Conjecture 1** (Sun, 2003) If $A = \{a_i (\mathrm{mod}\ n_i)\}_{i=1}^k$ covers every integer at least $m$ times then $\sum_{i \in I} 1/n_i \in m\mathbb{Z}$ for some $\emptyset \neq I \subseteq \{1, \ldots, k\}$.

**Conjecture 2** (Sun, 2004) If $a_1 G_1, \ldots, a_k G_k$ $(k > 1)$ are disjoint left cosets in a group $G$ with $[G : G_i] < \infty$ for all $i = 1, \ldots, k$, then $\gcd([G : G_i], [G : G_j]) \geqslant k$ for some $1 \leqslant i < j \leqslant k$.

**Conjecture 3** (Sun, 2008). Let $a_1, \ldots, a_n$ be nonzero elements of a field $F$. Then, for any finite $A \subseteq F$ we have

$$\begin{aligned}&|\{a_1 x_1 + \cdots + a_n x_n : \ x_1, \ldots, x_n \text{ are distinct elements of } A\}|\\ &\quad \geqslant \min\{p(F) - \delta,\ n(|A| - n) + 1\},\end{aligned}$$

where $\delta = 1$ if $n = 2$ and $a_1 + a_2 = 0$, and $\delta = 0$ otherwise.

**Conjecture 4** (Sun, 2008). Any $n \times n \times n$ Latin cube contains a Latin transversal.

**Conjecture 5** (Sun, 2009) Let $r(n)$ be the number of ways to write $n$ in the form $x^2 + (y^2 - 3y)/2 + (2z^2 - z)$ with $x, y, z \in \mathbb{N}$. Then $\{r(n) : n \in \mathbb{Z}^+\} = \mathbb{Z}^+$.

# The second generation

W. D. Gao, Y. G. Chen and Z. W. Sun can be viewed as the representatives of the first generation of combinatorial number theorists in China. They taught CNT by themselves. Gao is good at the tool of group-rings, Chen is good at analytic method, and Sun is familiar with algebraic methods and combinatorics. Now they are all over forties.

The second generation of Chinese combinatorial number theorists consists some young professors (or associate prof.) with ages between 30-37. Below is a list of main representatives of the second generation.

**Gao's School**: Ju-Juan Zhuang (Dalian Martime Univ.), et al.

**Chen's School**: Li-Xia Dai (Nanjing Normal Univ.), Min Tang (Anhui Normal Univ.), Shu-Guang Guo (Yancheng Normal College), Xue-Gong Sun (Huaihai Institute of Technology), et al.

**Sun's School**: Hao Pan (Nanjing Univ.), Hui-Qin Cao (Nanjing Auditing College), Song Guo (Huaiyin Normal College), et al.

# The third generation

The third generation is among the graduate students participating this conference.

Thank you!