A talk given at Florence University on June 28, 2004.

# ON VARIOUS COMBINATORIAL SUMS AND RELATED IDENTITIES

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China
*E-mail:* zwsun@nju.edu.cn
Homepage: http://pweb.nju.edu.cn/zwsun

Abstract. In this talk we give a survey of results and methods on some combinatorial sums involving binomial coefficients and related to Bernoulli and Euler polynomials. We will also talk about certain sums of minima and maxima related to Dedekind sums. Some interesting identities associated with the various sums will also be introduced.

## 1. A curious identity and the sum $\sum_{k\equiv r \pmod m}\binom{n}{k}$

In 1988 Zhi-Wei Sun proved a conjecture of his twin brother Zhi-Hong Sun, later he made a further extension by establishing the following result.

**Theorem 1.1** [Z. W. Sun, Integers 2(2002)]. *Let $m\in\mathbb{N}=\{0,1,2,\ldots\}$. Then we have the curious identity*

$$\begin{aligned}&(x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}\\&=\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i+(x-m)\binom{x}{m}.\end{aligned}\tag{1.1}$$

This identity is not obvious at all, it is somewhat strange and sophisticated.

Set

$$A_k(m,n)=\sum_{i=k}^m(-1)^i\binom{m+n+i}{m-i}\binom{2i}{k+i}\quad\text{for } k,m,n\in\mathbb{N}\text{ with } k\leqslant m$$

and

$$B(m,n)=\sum_{i=0}^m\binom{m+n+i}{m-i}(-4)^i\quad\text{for } m,n\in\mathbb{N}.$$

Z. W. Sun deduced (1.1) by showing the following result via double recursions: If $k,m,n\in\mathbb{N}$ and $k\leqslant m$, then

$$(n+2(m-k)+1)A_k(m,n)-(-1)^kB(m-k,n)=(-1)^kn\binom{m+n-k}{n}.$$

**Lemma 1.1** [Z. W. Sun, Integers 2(2002)]**.** *Let $k,m,n\in\mathbb{N}$ and $k\leqslant m$. Then we have*

$$\begin{cases}A_k(m,0)=(-1)^m\\ A_k(k,n)=(-1)^k\\ A_k(m+1,n+1)=A_k(m+1,n)+A_k(m,n+1)\end{cases}\tag{1.2}$$

*and*

$$\begin{cases}B(m,0)=(-1)^m(2m+1)\\ B(0,n)=1\\ B(m+1,n+1)=B(m+1,n)+B(m,n+1).\end{cases}\tag{1.3}$$

**Lemma 1.2** [Z. W. Sun, Integers 2(2002)]**.** *For $k,m,n\in\mathbb{N}$ with $k\leqslant m$, we have*

$$(-1)^mA_k(m,n)=\sum_{i=0}^{m-k}(-1)^i\binom{n+i-1}{i}.\tag{1.4}$$

*Equivalently, for any $l\in\mathbb{N}$ we have*

$$\sum_{j=0}^l(-1)^{l-j}\binom{x+y+j}{l-j}\binom{y+2j}{j}=\sum_{j=0}^l\binom{l-x}{j}.\tag{1.5}$$

Let us see why (1.4) and (1.5) are equivalent. Since both sides of (1.5) are polynomials in $x$ and $y$, (1.5) holds if and only if it holds for those $x = l + n$ and $y = 2k$ with $n, k \in \mathbb{N}$. Set $m = k + l$. Then

$$\begin{aligned}&\sum_{j=0}^{l}(-1)^{l-j}\binom{x+y+j}{l-j}\binom{y+2j}{j}\\ =&\sum_{i=k}^{m}(-1)^{l-(i-k)}\binom{x+2k+i-k}{l-(i-k)}\binom{2k+2(i-k)}{i-k}\\ =&(-1)^m\sum_{i=k}^{m}(-1)^i\binom{m+n+i}{m-i}\binom{2i}{k+i} = (-1)^m A_k(m,n)\end{aligned}$$

and

$$\sum_{j=0}^{m-k}(-1)^j\binom{n+j-1}{j} = \sum_{j=0}^{l}\binom{-n}{j} = \sum_{j=0}^{l}\binom{l-x}{j}.$$

So we have the equivalence of (1.4) and (1.5).

The curious identity (1.1) of Sun was reproved by A. Panholzer and H. Prodinger [Integers 2(2002)] via generating functions, they observed that

$$\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i} = [t^m]\frac{\mathcal{B}(t)^{x+m+1}}{1+4t}$$

and

$$\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i = [t^m]\frac{\mathcal{B}(t)^{x+m+1}}{(1+4t)^{3/2}},$$

where $\mathcal{B}(t) = (1+\sqrt{1+4t})/2$ and $[t^m]f(t)$ denotes the coefficient of $t^m$ in the power series of $f(t)$.

By means of Riordan arrays, D. Merlini and R. Sprugnoli [Integers 2(2002)] obtained a new proof of (1.1) by showing the following lemma which is more explicit.

**Lemma 1.3** [Merlini and Sprugnoli, Integers 2(2002)]**.** *Let $m\in\mathbb{N}$. Then*

$$\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}=[t^m]\frac{(1+t)^x}{1+2t} \tag{1.6}$$

*and*

$$\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=[t^m]\frac{(1+t)^x}{(1+2t)^2}. \tag{1.7}$$

In 2003 S. B. Ekhad and M. Mohammed reproved (1.1) by the Wilf-Zeilberger method.

**Lemma 1.4** [Ekhad and Mohammed, Integers 3(2003)]**.** *Let $m\in\mathbb{N}$ and*

$$T(m)=\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}.$$

*Then we have the recursion*

$$a_0(m)T(m)+a_1(m)T(m+1)+a_2(m)T(m+2)+a_3(m)T(m+3)=0,$$

*where*

$$\begin{aligned}a_0(m)=&2(x-m-1)(x-m-2),\\ a_1(m)=&(x-m-2)(x-2y-5m-11),\\ a_2(m)=&-xy-2mx+3my-5x+8y+4m^2+21m+28,\\ a_3(m)=&(m+3)(y+m+3).\end{aligned}$$

In 2003 W. Chu and L.V.D. Claudio [Integers 3(2003)] gave another proof of (1.1) by showing $T(m)=\sum_{k=0}^{m}\binom{x}{m-k}(-2)^k$ and

$$\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=\sum_{k=0}^{m}\binom{x}{m-k}(-2)^k(k+1)$$

via the so-called Jensen formula and the Chu-Vandermonde formula. However, the last two formulae are implied by (1.6) and (1.7). In fact,

$$T(m)=[t^m]\frac{(1+t)^x}{1+2t}=[t^m](1+t)^x\sum_{k=0}^{\infty}(-2t)^k=\sum_{k=0}^{m}\binom{x}{m-k}(-2)^k$$

and

$$\begin{aligned}&\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i=[t^m](1+t)^x(1+2t)^{-2}\\=&[t^m](1+t)^x\sum_{k=0}^{\infty}\binom{-2}{k}(2t)^k=\sum_{k=0}^{m}\binom{x}{m-k}(-2)^k(k+1).\end{aligned}$$

For nonnegative integer $m$, we have the following simple recursion:

$$T(m+1)+2T(m)=\binom{x}{m+1}. \tag{1.8}$$

This is because

$$\sum_{k=0}^{m+1}\binom{x}{k}(-2)^{m+1-k}=\binom{x}{m+1}-2\sum_{k=0}^{m}\binom{x}{k}(-2)^{m-k}.$$

Observe that

$$\begin{aligned}T(m)=&[t^m](1+2t)\frac{(1+2t)^x}{(1+2t)^2}=[t^m]\frac{(1+2t)^x}{(1+2t)^2}+2[t^{m-1}]\frac{(1+2t)^x}{(1+2t)^2}\\=&\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i+2\sum_{0\leqslant i<m}\binom{x+i}{m-1-i}(-4)^i.\end{aligned}$$

This plus (1.1) yields the following equivalent version of (1.1).

$$(x+m+2)T(m)=2\sum_{i=0}^{m}\binom{x+i+1}{m-i}(-4)^i+(x-m)\binom{x}{m}. \tag{1.9}$$

(Note that $\binom{x+i}{m-i}+\binom{x+i}{m-1-i}=\binom{x+i+1}{m-i}$ for $0\leqslant i<m$.)

In 2004 D. Callan [Integers 4(2004)] called (1.1) "*Sun's curious identity*" and found a very elegant combinatorial proof. As each side of (1.1) is a polynomial in $x$ of degree $m+1$ with $1/m!$ as the leading coefficient, (1.1) holds if and only if it holds for $x=0,1,\ldots,m$. Lemma 1.2 suggests that

$$(-1)^m T(m)=\sum_{i=0}^{m}\binom{m-x}{i}=2^{m-x}\quad\text{for } x=0,1,\ldots,m.$$

So Callan reduced (1.1) to the following two equalities with $0\leqslant k\leqslant m$:

$$\sum_{i=0}^{m}(-1)^i\binom{2m+y-k-i}{i}\binom{2m+y-2i}{m-i}=2^k,$$
$$\sum_{i=0}^{m}(-1)^i\binom{2m-k-i}{i}4^{m-i}=(2m-k+1)2^k.$$

Then, he supplied a nice combinatorial interpretation of these equalities. His combinatorial proof uses weight-reversing involutions on suitable configurations involving dominos and colorings, it is quite similar to the usual proof of Wilson's theorem in number theory.

Inspired by Merlini and Sprugnoli's Lemma 1.3, Z. W. Sun and his student K. J. Wu were able to extend Sun's identity (1.1) in the following way.

**Theorem 1.2** [Sun and Wu, arXiv:math.CO/0401057]**.** *For any* $m=0,1,2,\ldots$ *we have*

$$\begin{aligned}&(x+(m+1)z)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\=&z\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l}{m-n}(1+z)^{n+l}(1-z)^{n-l}\\&+(x-m)\binom{x}{m};\end{aligned}\tag{1.10}$$

*equivalently,*

$$\begin{aligned}&(x+(m+1)z+1)\sum_{n=0}^{m}(-1)^n\binom{x+y+nz}{m-n}\binom{y+n(z+1)}{n}\\=&(z+1)\sum_{0\leqslant l\leqslant n\leqslant m}(-1)^n\binom{n}{l}\binom{x+l+1}{m-n}(1+z)^{n+l}(1-z)^{n-l}\\&+(x-m)\binom{x}{m}.\end{aligned}\tag{1.11}$$

Clearly (1.10) in the case $z=1$ gives Sun's identity (1.1), and (1.11) in the case $z=1$ yields the equivalent form (1.9) of (1.1).

For $m\in\mathbb{Z}^+=\{1,2,3,\dots\}$, $n\in\mathbb{N}$ and $r\in\mathbb{Z}$ we set

$$\begin{bmatrix}n\\r\end{bmatrix}_m=\sum_{\substack{k=0\\k\equiv r\ (\mathrm{mod}\ m)}}^{n}\binom{n}{k}\quad\text{and}\quad\begin{Bmatrix}n\\r\end{Bmatrix}_m=\sum_{\substack{k=0\\k\equiv r\ (\mathrm{mod}\ m)}}^{n}(-1)^{\frac{k-r}{m}}\binom{n}{k}.\tag{1.12}$$

As $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ for any $k\in\mathbb{Z}^+$, in 1988 Z. W. Sun observed the following useful recursions:

$$\begin{bmatrix}n+1\\r\end{bmatrix}_m=\begin{bmatrix}n\\r\end{bmatrix}_m+\begin{bmatrix}n\\r-1\end{bmatrix}_m\quad\text{and}\quad\begin{Bmatrix}n+1\\r\end{Bmatrix}_m=\begin{Bmatrix}n\\r\end{Bmatrix}_m+\begin{Bmatrix}n\\r-1\end{Bmatrix}_m.\tag{1.13}$$

The study of the sum $\left[{n\atop r}\right]_m$ dates back to 1876 when C. Hermite showed that if $n$ is odd and $p$ is an odd prime then $\left[{n\atop 0}\right]_{p-1}\equiv 1\pmod p$. The sum $\left\{{n\atop r}\right\}_m$ was introduced by Z. W. Sun in 1988 when he studied $\left[{n\atop r}\right]_{12}$. In the modern investigations made by the speaker and his twin brother Z. H. Sun (cf., e.g., [Sun and Sun, Acta Arith. 60(1992)] and [Z. W. Sun, Israel J. Math. 128(2002)]), the sum $\left[{n\atop r}\right]_m$ was expressed in terms of linear recurrences and then applied to produce congruences for primes.

Now we apply Sun's identity (1.1) or the key auxiliary identity (1.5) to study the combinatorial sum $\left[{n \atop r}\right]_m$.

**Theorem 1.3** [Z. W. Sun, arXiv:math.NT/0404385]**.** *Let $m$ be a positive integer. Then, for any integers $k$ and $n \geqslant 2\lfloor (m-1)/2 \rfloor$, we have*

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_m = 2^{n-m+1} + \delta_{m-2,\,n} \frac{(-1)^k}{2}, \quad (1.14)$$

*where $\delta_{l,n}$ is the Kronecker symbol.*

Using (1.13) and induction on $n$, it suffices to show (1.14) in the case $n = 2h$ and $0 \leqslant k < m$ where $h = \lfloor (m-1)/2 \rfloor$. For any $i \in \mathbb{N}$ with $i \leqslant h$, we have $k - i + m > n - 2i$ since $n - m < 0 \leqslant k + i$, thus

$$\begin{bmatrix} n-2i \\ k-i \end{bmatrix}_m = \begin{cases} \binom{n-2i}{k-i} & \text{if } i \leqslant k, \\ 0 & \text{if } i > k. \end{cases}$$

Let $x = m - 1 - n + k$, $y = n - 2k$, and $\Sigma$ denote the left hand side of (1.14). Then

$$\begin{aligned} \Sigma &= \sum_{i=0}^{k} (-1)^i \binom{m-1-i}{i} \binom{n-2i}{k-i} \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{x+y+j}{k-j} \binom{y+2j}{j} \\ &= \sum_{j=0}^{k} \binom{k-x}{j} = \sum_{j=0}^{k} \binom{n-(m-1)}{j} \end{aligned}$$

with the help of Lemma 1.2. If $m$ is odd, then $n = m - 1$ and hence $\Sigma = \sum_{j=0}^{k} \binom{0}{j} = 1 = 2^{n-m+1}$. If $m$ is even, then $n = m - 2$ and

$$\Sigma = \sum_{j=0}^{k} \binom{-1}{j} = \sum_{j=0}^{k} (-1)^j = \frac{1 + (-1)^k}{2} = 2^{n-m+1} + \frac{(-1)^k}{2}.$$

So we do have $\Sigma = 2^{n-m+1} + \delta_{m-2,\,n}(-1)^k/2$ as required.

**Corollary 1.1.** *Let $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ set*

$$u_n = \begin{bmatrix} n \\ \lfloor (k+n)/2 \rfloor \end{bmatrix}_m \text{ and } v_n = mu_n - 2^n - \delta_{n,0}\delta_{(-1)^m,1}(-1)^{\lfloor k/2 \rfloor}. \quad (1.15)$$

*Then we have*

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} u_{n-2i} = 2^{n-m+1} - \delta_{m-2,\,n}\frac{(-1)^{\lfloor (k+m)/2 \rfloor}}{2} \quad (1.16)$$

*for every integer $n \geqslant 2\lfloor (m-1)/2 \rfloor$. Also, $\{v_n\}_{n\in\mathbb{N}}$ is a linear recurrent sequence with the following recursion:*

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} v_{n-2i} = 0 \quad \text{for all } n \geqslant 2\left\lfloor \frac{m-1}{2} \right\rfloor. \quad (1.17)$$

Using Theorem 1.3 and the basic idea in its proof, Z. W. Sun also proved the following result.

**Theorem 1.4** [Z. W. Sun, arXiv:math.NT/0404385]**.** *Let $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. Then*

$$\sum_{i=0}^{\lfloor (m+1)/2 \rfloor} (-1)^i c_m(i) \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_m = 2(-1)^k \delta_{m,\,n} \quad (1.18)$$

*for each integer $n \geqslant 2\lfloor (m+1)/2 \rfloor$, and*

$$\sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i d_m(i) \begin{Bmatrix} n-2i \\ k-i \end{Bmatrix}_m = (-1)^k \delta_{m-1,\,n} \quad (1.19)$$

*for any integer $n \geqslant 2\lfloor m/2 \rfloor$, where $c_1(1) = 4$, and*

$$c_m(i) = \frac{m^2+m-2i}{(m-i)(m+1-i)}\binom{m+1-i}{i} \in \mathbb{Z}$$

*and*

$$d_m(i) = \frac{m}{m-i}\binom{m-i}{i} \in \mathbb{Z}$$

*for every $i = 0, 1, \ldots, m-1$.*

Another way to obtain Theorem 1.4 is to use the following lemmas.

**Lemma 1.5** [Z. W. Sun, Israel J. Math. 128(2002)]. *Let $k \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. Then*

$$\frac{1}{m}\sum_{\gamma^m=1}\gamma^k(2+\gamma+\gamma^{-1})^n=\begin{bmatrix}2n\\k+n\end{bmatrix}_m \tag{1.20}$$

*and*

$$\frac{1}{m}\sum_{\gamma^m=-1}\gamma^k(2+\gamma+\gamma^{-1})^n=\begin{Bmatrix}2n\\k+n\end{Bmatrix}_m. \tag{1.21}$$

*Also, for $\varepsilon \in \{1,-1\}$ we have*

$$\sum_{\gamma^m=\varepsilon}\gamma^k(2-\gamma-\gamma^{-1})^n=(-1)^k m\times\begin{cases}\left[{2n\atop k+n}\right]_m & \text{if } \varepsilon=(-1)^m,\\ \left\{{2n\atop k+n}\right\}_m & \text{otherwise.}\end{cases} \tag{1.22}$$

**Lemma 1.6.** *Let $m$ be a positive integer. Then*

$$\prod_{\substack{0<j<m\\2\mid j-\delta}}\left(x-4\cos^2\frac{j\pi}{2m}\right)=\begin{cases}C_m(x) & \text{if } \delta=0,\\ D_m(x) & \text{if } \delta=1,\end{cases} \tag{1.23}$$

*where*

$$C_m(x)=\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^i\binom{m-1-i}{i}x^{\lfloor\frac{m-1}{2}\rfloor-i} \tag{1.24}$$

*and*

$$D_m(x)=\sum_{i=0}^{\lfloor m/2\rfloor}(-1)^i\frac{m}{m-i}\binom{m-i}{i}x^{\lfloor\frac{m}{2}\rfloor-i}. \tag{1.25}$$

**Theorem 1.5** [Z. W. Sun, Israel J. Math. 128(2002)]. *Let $k, m \in \mathbb{Z}$ and $m > 2$. Write*

$$w_n(k,m)=\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}\left(4\cos^2\frac{j\pi}{m}\right)\left(4\cos^2\frac{j\pi}{m}\right)^n \quad \text{for } n\in\mathbb{Z}^+, \tag{1.26}$$

*where* $D_0(x)$ *is regarded as* 2. *Then* $\{w_n(k,m)\}_{n=1}^{\infty}$ *is a linear recurrent sequence of integers with order* $\varphi(m)/2$. *Whenever* $n \in \mathbb{Z}^+$ *and* $r \in \mathbb{Z}$, *we have*

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \frac{2^n}{m} + \frac{1}{m} \sum_{\substack{d \mid m \\ d>2}} w_{\lfloor \frac{n+1}{2} \rfloor}(n-2r, d). \tag{1.27}$$

For $m \in \{5, 8, 10, 12\}$ the sum $\left[{n \atop r}\right]_m$ can be expressed in terms of linear recurrences of orders not exceeding $\varphi(m)/2 = 2$. For example, Z. H. Sun and Z. W. Sun [Acta Arith. 60(1992)] expressed the sum $\left[{n \atop r}\right]_{10}$ explicitly in terms of Fibonacci numbers and Lucas numbers. Z. W. Sun [Israel J. Math. 128(2002)] obtained an explicit closed formula for the sum $\left[{n \atop r}\right]_{12}$ in terms of the second recurrence $\{T_n\}_{n=0}^{\infty}$ where

$$T_0 = 2,\ T_1 = 4,\ \text{and } T_{n+1} = 4T_n - T_{n-1} \text{ for } n = 1, 2, 3, \ldots .$$

As a consequence he discovered and proved the following congruence in 1988:

$$\sum_{0<k<p/2} \frac{3^k}{k} \equiv \sum_{0<k<p/6} \frac{(-1)^k}{k} \pmod p \quad \text{for any prime } p. \tag{1.28}$$

This was first announced in [Z. W. Sun, Proc. Amer. Math. Soc. 123(1995)] and the original proof (given by Z. W. Sun in 1988) appeared in [Z. W. Sun, Israel J. Math. 128(2002)].

## 2. On Bernoulli and Euler polynomials

For $n = 0, 1, 2, \cdots$ the $n$th Bernoulli polynomial $B_n(x)$ is given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

where the Bernoulli numbers $B_0, B_1, B_2, \cdots$ is defined by the power series

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \quad (0 < |z| < 2\pi).$$

Let $n \in \mathbb{N}$. It is known that

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n} - 1\right) B_n.$$

Also, if $2 \mid n$ then

$$\begin{aligned}
B_n\left(\frac{1}{3}\right) = B_n\left(\frac{2}{3}\right) =&(3^{1-n} - 1)\frac{B_n}{2}, \\
B_n\left(\frac{1}{4}\right) = B_n\left(\frac{3}{4}\right) =&2^{-n}(2^{1-n} - 1)B_n, \\
B_n\left(\frac{1}{6}\right) = B_n\left(\frac{5}{6}\right) =&(2^{1-n} - 1)(3^{1-n} - 1)\frac{B_n}{2}.
\end{aligned}$$

Observe that $\varphi(1) = \varphi(2) = 1$ and $\varphi(3) = \varphi(4) = \varphi(6) = 2$.

**Lemma 2.1.** *Let $p$ be an odd prime not dividing $m \in \mathbb{Z}^+$, and let $n \in \{0, \ldots, m-1\}$. Let $\{\alpha\}$ denote the fractional part of a real number $\alpha$.*

(i) [A. Granville & Z. W. Sun, Pacific J. Math. 1996]

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1} \equiv -\sum_{k=1}^{\lfloor pn/m \rfloor} \frac{1}{k} \pmod{p}. \tag{2.1}$$

(ii) [Z. W. Sun, arXiv:math.NT/0404385]

$$\frac{(-1)^{\lfloor pn/m \rfloor}}{2} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) + \frac{2^{p-1} - 1}{p} \equiv \sum_{k=1}^{\lfloor pn/m \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}. \tag{2.2}$$

The following results are inspired by Lemma 2.1 and the work on the sum $\left[{n \atop r}\right]_m$.

**Theorem 2.1.** *Let $p$ be an odd prime and $m,n$ be positive integers with $(m,pn)=1$. For $a\in\mathbb{Z}$ with $p\nmid a$, let $q_p(a)$ stand for the Fermat quotient $(a^{p-1}-1)/p$.*

(i) [A. Granville & Z. W. Sun, Pacific J. Math. 1996]

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right)-B_{p-1}\equiv\begin{cases}(\frac{n}{5})\frac{5}{4p}F_{p-(\frac{5}{p})}+\frac{5}{4}q_p(5)\pmod p & \text{if } m=5,\\ (\frac{2}{n})\frac{2}{p}P_{p-(\frac{2}{p})}+4q_p(2)\pmod p & \text{if } m=8,\\ (\frac{n}{5})\frac{15}{4p}F_{p-(\frac{5}{p})}+\frac{5}{4}q_p(5)+2q_p(2)\pmod p & \text{if } m=10,\\ (\frac{3}{pn})\frac{3}{p}S_{p-(\frac{3}{p})}+3q_p(2)+\frac{3}{2}q_p(3)\pmod p & \text{if } m=12,\end{cases}\tag{2.3}$$

*where $(-)$ denotes the Jacobi symbol, and the sequences $\{F_k\}_{k\in\mathbb{N}}$, $\{P_k\}_{k\in\mathbb{N}}$ and $\{S_k\}_{k\in\mathbb{N}}$ are defined as follows:*

$$F_0=0,\ F_1=1,\ \text{and } F_{k+2}=F_{k+1}+F_k \text{ for } k\in\mathbb{N};$$

$$P_0=0,\ P_1=1,\ \text{and } P_{k+2}=2P_{k+1}+P_k \text{ for } k\in\mathbb{N};$$

$$S_0=0,\ S_1=1,\ \text{and } S_{k+2}=4S_{k+1}-S_k \text{ for } k\in\mathbb{N}.$$

(ii) [Z. W. Sun, arXiv:math.NT/0404385]

$$(-1)^{\lfloor pn/m\rfloor}E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right)\equiv\begin{cases}(\frac{2}{n})\frac{4}{p}P_{p-(\frac{2}{p})}\pmod p & \text{if } m=4,\\ (\frac{n}{5})\frac{5}{p}F_{p-(\frac{5}{p})}+2q_p(2)\pmod p & \text{if } m=5,\\ (\frac{3}{pn})\frac{6}{p}S_{p-(\frac{3}{p})}\pmod p & \text{if } m=6.\end{cases}\tag{2.4}$$

Let $m,n\in\mathbb{Z}^+$, $q\in\mathbb{Z}$ and $(q,m)=1$. We define a linear recurrence $\{U_l^{(q)}(m,n)\}_{l\in\mathbb{N}}$ of order $\lfloor m/2\rfloor$ by

$$U_l^{(q)}(m,n)=\frac{1}{2m}\sum_{\substack{\gamma^m=1\\ \gamma\neq 1}}\frac{2-\gamma^{qn}-\gamma^{-qn}}{2-\gamma^q-\gamma^{-q}}(2-\gamma-\gamma^{-1})^l.\tag{2.5}$$

Note that

$$mU_l^{(q)}(m,n) = (1-(-1)^{(m-1)n})2^{2l-2} + \sum_{\substack{d\mid m\\ d>2}} u_l^{(q)}(d,n),$$

where $\{u_l^{(q)}(d,n)\}_{l\in\mathbb{N}}$ is a linear recurrence of order $\varphi(d)/2$ given by

$$\begin{aligned} u_l^{(q)}(d,n) &= \sum_{\substack{0<c<d/2\\ (c,d)=1}} \frac{2-e^{2\pi i\frac{c}{d}qn}-e^{-2\pi i\frac{c}{d}qn}}{2-e^{2\pi i\frac{c}{d}q}-e^{-2\pi i\frac{c}{d}q}}\left(2-e^{2\pi i\frac{c}{d}}-e^{-2\pi i\frac{c}{d}}\right)^l \\ &= \sum_{\substack{0<c<d/2\\ (c,d)=1}} \left(\frac{\sin(\pi nqc/d)}{\sin(\pi qc/d)}\right)^2\left(4\sin^2\frac{\pi c}{d}\right)^l. \end{aligned}$$

When $(q,2m)=1$, for $l\in\mathbb{N}$ we also define

$$V_l^{(q)}(m,n) = \frac{1}{2m}\sum_{\gamma^m=-1}\frac{2-\gamma^{qn}-\gamma^{-qn}}{2-\gamma^q-\gamma^{-q}}(2-\gamma-\gamma^{-1})^l. \tag{2.6}$$

**Theorem 2.2.** *Let $p$ be an odd prime, and let $m,n>0$ be integers with $p\nmid m$ and $m\nmid n$.*

(i) [A. Granville & Z.W. Sun, Pacific J. Math. 172(1996)] *If $p\equiv \pm q \pmod m$ where $q\in\mathbb{Z}$, then*

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1} \equiv \frac{m}{2p}\left(U_p^{(q)}(m,n)-1\right) \pmod p \tag{2.7}$$

(ii) [Z. W. Sun, arXiv:math.NT/0404385] *If $m$ does not divide $n$, and $p$ is an odd prime with $p\equiv\pm q \pmod{2m}$, then*

$$(-1)^{\lfloor pn/m\rfloor}E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right)+2q_p(2) \equiv \frac{m}{p}\left(V_p^{(q)}(m,n)-1\right) \pmod p. \tag{2.8}$$

(iii) [Z. W. Sun, arXiv:math.NT/0404385] *For any $q\in\mathbb{Z}$ with $(q,m)=1$, we have the recursion*

$$U_l^{(q)}(m,n) = \sum_{0<i\leqslant\lfloor m/2\rfloor}(-1)^{i-1}a_m(i)U_{l-i}^{(q)}(m,n) \quad \text{for } l\geqslant\left\lfloor\frac{m}{2}\right\rfloor, \tag{2.9}$$

*and*

$$V_l^{(q)}(m,n)=\sum_{j=1}^{\lfloor (m+1)/2\rfloor}(-1)^{j-1}b_m(j)V_{l-j}^{(q)}(m,n)\quad \text{for } l\geqslant\left\lfloor\frac{m+1}{2}\right\rfloor \tag{2.10}$$

*providing* $(q,2m)=1$, *where integers* $a_m(i)$ *and* $b_m(j)$ *are given by*

$$a_m(i)=\begin{cases}c_m(i) & \text{if } 2\mid m,\\ d_m(i) & \text{if } 2\nmid m,\end{cases}\quad \text{and}\quad b_m(j)=\begin{cases}d_m(j) & \text{if } 2\mid m,\\ c_m(j) & \text{if } 2\nmid m.\end{cases}$$

In 1995 M. Kaneko [Proc. Japan Acad. Ser. A. Math. Sci. 71(1995)] found the following new recursion formula for Bernoulli numbers:

$$\sum_{j=0}^{k}\binom{k+1}{j}(k+j+1)B_{k+j}=0\ \text{ for } k=1,2,\ldots.$$

By means of the Volkenborn integral, H. Momiyama [Fibonacci Quart. 39(2001)] got the following symmetric extension: If $k,l\in\mathbb{N}$ and $k+l>0$, then

$$(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)B_{l+j}+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}(k+j+1)B_{k+j}=0.$$

Wu, Sun and Pan [Fibonacci Quart. 42(2004)] extended this to Bernoulli polynomials. Motivated by the above work, the speaker was able to get the following result.

**Theorem 2.3** [Z. W. Sun, European J. Combin. 24(2003)]**.** *Let* $k$ *and* $l$ *be nonnegative integers, and* $f(x)=(-x)^{k+l+1}/((k+l+1)\binom{k+l}{k})$.

(i) *If* $x+y+z=0$ *then*

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{y^{l+j+1}}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{z^{k+j+1}}{k+j+1}=f(x).$$

*In particular,*

$$\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{2^j(k+j+1)}=\frac{2^k}{(2k+1)\binom{2k}{k}}.$$

(ii) *Suppose that $x+y+z=1$. Then*

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{B_{l+j+1}(y)}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{B_{k+j+1}(z)}{k+j+1}=f(x). \tag{2.11}$$

*Also,*

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}B_{l+j}(y)=(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}B_{k+j}(z) \tag{2.12}$$

*and*

$$\begin{aligned}&(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}x^{k-j+1}(l+j+1)B_{l+j}(y)\\&+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}x^{l-j+1}(k+j+1)B_{k+j}(z)\\=&(-1)^k(k+l+2)\left(B_{k+l+1}(x+y)-B_{k+l+1}(y)\right).\end{aligned} \tag{2.13}$$

(iii) *The second part remains valid if we replace all the Bernoulli polynomials in* (2.11) – (2.13) *by corresponding Euler polynomials.*

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. We call the sequence $\{a_n^*\}_{n\in\mathbb{N}}$ given by $a_n^*=\sum_{i=0}^{n}\binom{n}{i}(-1)^i a_i$ the *dual sequence* of $\{a_n\}_{n\in\mathbb{N}}$. It is well known that $a_n^{**}=a_n$ for all $n\in\mathbb{N}$. It is easy to see that the sequence $\{(-1)^nB_n\}_{n\in\mathbb{N}}$ is self-dual as observed by Z. H. Sun. Like the definition of Bernoulli polynomials, we introduce

$$A_n(x)=\sum_{i=0}^{n}\binom{n}{i}(-1)^i a_i x^{n-i}\quad\text{and}\quad A_n^*(x)=\sum_{i=0}^{n}\binom{n}{i}(-1)^i a_i^* x^{n-i}.$$

The following result is more general than Theorem 2.3.

**Theorem 2.4** [Z. W. Sun, European J. Combin. 24(2003)]**.** *Let* $k, l \in \mathbb{N}$ *and* $x + y + z = 1$*. Then*

$$\begin{aligned}(-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \frac{A_{l+j+1}(y)}{l+j+1} + (-1)^l \sum_{j=0}^{l} \binom{l}{j} x^{l-j} \frac{A^*_{k+j+1}(z)}{k+j+1} \\ = \frac{a_0(-x)^{k+l+1}}{(k+l+1)\binom{k+l}{k}}.\end{aligned} \tag{2.14}$$

*Also,*

$$(-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} A_{l+j}(y) = (-1)^l \sum_{j=0}^{l} \binom{l}{j} x^{l-j} A^*_{k+j}(z) \tag{2.15}$$

*and*

$$\begin{aligned}&(-1)^k \sum_{j=0}^{k} \binom{k+1}{j} x^{k-j+1} (l+j+1) A_{l+j}(y) \\ &+ (-1)^l \sum_{j=0}^{l} \binom{l+1}{j} x^{l-j+1} (k+j+1) A^*_{k+j}(z) \\ =&(k+l+2) \left((-1)^{k+1} A_{k+l+1}(y) + (-1)^{l+1} A^*_{k+l+1}(z)\right).\end{aligned} \tag{2.16}$$

In the proof of Theorem 2.4 Zhi-Wei Sun used the following lemma which is (2.15) in the special case $a_j = (-1)^i \delta_{ij}$ $(j \in \mathbb{N})$, $x = 1$, $y = t$ and $z = -t$.

**Lemma 2.2** [Z. W. Sun, European J. Combin. 24(2003)]**.** *Let* $i, k, l \in \mathbb{N}$*. Then*

$$\sum_{j=0}^{l} \binom{l}{j} \binom{k+j}{i} (-1)^{l-j} (1+t)^{k+j-i} = \sum_{j=0}^{k} \binom{k}{j} \binom{l+j}{i} t^{l+j-i}. \tag{2.17}$$

It should be mentioned that an identity obtained by S. Simons [Math. Gaz. 85(2001)] and reproved by R. Chapman [Math. Gaz. 87(2003)] and

H. Prodinger [Math. Gaz. 88(2004)] is just our (2.17) in the special case $i = k = l$. In 2004 R. Chapman showed the speaker his new proofs of Theorem 2.4 and Lemma 2.2 via exponential generating functions.

## 3. Sums of minima and maxima related to Dedekind sums

Let $h$ and $k$ be positive integers. In 1892 R. Dedekind introduced the classical Dedekind sum

$$s(h,k) = \sum_{r=0}^{k-1} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{3.1}$$

where $((x)) = \{x\} - 1/2$ if $x \notin \mathbb{Z}$, and $((x)) = 0$ otherwise. When $h$ and $k$ are coprime, Dedekind was able to express $s(h,k) + s(k,h)$ as an explicit rational function in $h$ and $k$; the result is now known as the reciprocity law for Dedekind sums.

Let $h_1, \cdots, h_n \in \mathbb{Z}^+$. Zhi-Wei Sun [Discrete Math. 257(2002)] introduced the sums

$$m(h_1,\cdots,h_n) = m(\{h_i\}_{i=1}^n) = \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \min\left\{\frac{r_1}{h_1},\cdots,\frac{r_n}{h_n}\right\} \tag{3.2}$$

and

$$M(h_1,\cdots,h_n) = M(\{h_i\}_{i=1}^n) = \sum_{r_1=0}^{h_1-1} \cdots \sum_{r_n=0}^{h_n-1} \max\left\{\frac{r_1}{h_1},\cdots,\frac{r_n}{h_n}\right\}, \tag{3.3}$$

the first of which times $h_1 \cdots h_n$ is the number of lattice points in a pyramid of dimension $n+1$. The sums $m(h_1,h_2)$ and $M(h_1,h_2)$ are closely connected with the reciprocity law for Dedekind sums (see, Section 3 of Z. W. Sun [Discrete Math. 257(2002)]).

**Theorem 3.1** [Z. W. Sun, Discrete Math. 257(2002)]**.** *Let* $h_1, \ldots, h_n$ *be positive integers. Then*

$$\frac{m(h_1,\cdots,h_n)}{(h_1-1)\cdots(h_n-1)} = 1 + \sum_{\emptyset\neq I\subseteq\{1,\cdots,n\}} (-1)^{|I|} \frac{m(\{h_i\}_{i\in I})}{\prod_{i\in I}(h_i-1)} \tag{3.4}$$

*if* $h_1,\cdots,h_n > 1$. *Also,*

$$\frac{M(h_1,\cdots,h_n) - h_1\cdots h_n + 1}{(h_1+1)\cdots(h_n+1)} = \sum_{\emptyset\neq I\subseteq\{1,\cdots,n\}} (-1)^{|I|} \frac{M(\{h_i\}_{i\in I})}{\prod_{i\in I}(h_i+1)}. \tag{3.5}$$

For a finite set $S$ and functions $f$ and $g$ from the power set of $S$ to an additive abelian group, it is well known that

$$f(J) = \sum_{I\subseteq J}(-1)^{|I|}g(I) \text{ for } J\subseteq S \iff g(J) = \sum_{I\subseteq J}(-1)^{|I|}f(I) \text{ for } J\subseteq S.$$

Thus, (3.4) is of particular interest.

**Theorem 3.2** [Z. W. Sun, Discrete Math. 257(2002)]**.** *Let* $h, k, l$ *be positive integers. Then*

$$\begin{aligned} m(h,k,l) =& \frac{hkl}{4} - \frac{hk+hl+kl}{6} + \frac{h+k+l-1}{8} \\ &+ \frac{h+k-2hk}{24l} + \frac{h+l-2hl}{24k} + \frac{k+l-2kl}{24h} \\ &+ \frac{(h-1)(k,l)^2}{24kl} + \frac{(k-1)(h,l)^2}{24hl} + \frac{(l-1)(h,k)^2}{24hk} \end{aligned} \tag{3.6}$$

*and*

$$\begin{aligned} M(h,k,l) =& \frac{3}{4}hkl - \frac{hk+hl+kl}{6} - \frac{h+k+l+1}{8} \\ &+ \frac{h+k+2hk}{24l} + \frac{h+l+2hl}{24k} + \frac{k+l+2kl}{24h} \\ &- \frac{(h+1)(k,l)^2}{24kl} - \frac{(k+1)(h,l)^2}{24hl} - \frac{(l+1)(h,k)^2}{24hk}. \end{aligned} \tag{3.7}$$

Let $h_1, h_2, h_3, h_4 \in \mathbb{Z}^+$. In the same paper Z. W. Sun also determined $m(h_1,h_2,h_3,h_4) + M(h_1,h_2,h_3,h_4)$ explicitly; but the determination of $m(h_1,h_2,h_3,h_4)$ and $M(h_1,h_2,h_3,h_4)$ remains open.