A talk given at the National Center for Theoretical Sciences (Hsinchu, Taiwan; August 4, 2010)

Arithmetic Properties of Combinatorial Quantities

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August 4, 2010

Abstract

In combinatorics there are many combinatorial quantities arising from enumeration problems, e.g., Catalan numbers, central trinomial coefficients, Motzkin numbers, central Delannoy numbers, Bell numbers and harmonic numbers. In this talk we introduce various conjectures and results on $\sum_{k=0}^{p-1} a_k/m^k$ modulo powers of p, where a_k is the kth certain combinatorial quantity, p is a prime and m is an integer not dividing by p.

Congruences for p-integers

Let p be a prime. A rational number is called a *p-integer* (or a rational p-adic integer) if it can be written in the form a/b with $a, b \in \mathbb{Z}$ and $(b, p) = 1$. All p-integers form a ring R_p which is a subring of the ring $\mathbb{Z}_p$ of all p-adic integers. For a p-integer a/b, an integer c and a nonnegative integer n if $a/b = c + p^n q$ for some $q \in R_p$ (equivalently, $a \equiv bc \pmod{p^n}$), then we write

$$\frac{a}{b} \equiv c \pmod{p^n}.$$

An Example for Congruences involving p-Integers:

$$1 + \frac{1}{2} \equiv 1 - 4 = -3 \pmod{3^2}.$$

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $(\frac{a}{p})$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

Classical congruences for central binomial coefficients

A central binomial coefficient has the form

$$\binom{2k}{k} \ (k = 0, 1, 2, \ldots).$$

Wolstenholme's Congruence. For any prime $p > 3$ we have

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Remark. In 1900 Glaiser proved that for any prime $p > 3$ we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4},$$

where B_n denotes the nth Bernoulli number.

Classical congruences for central binomial coefficients

Morley's Congruence. For any prime $p > 3$ we have

$$\binom{p-1}{(p-1)/2} \equiv \left(\frac{-1}{p}\right) 4^{p-1} \pmod{p^3}.$$

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod{p}.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

A theorem of Stienstra and Beukers

J. Stienstra and F. Beukers [Math. Ann. 27(1985)]:

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6$$
$$=\begin{cases}4x^2-2p & \text{if } p\equiv 1 \pmod 4 \ \&\ p=x^2+y^2 \text{ with } 2\nmid x \ \&\ 2\mid y,\\ 0 & \text{if } p\equiv 3 \pmod 4;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{n})^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$
$$=\begin{cases}4x^2-2p & \text{if } p\equiv 1,3 \pmod 8 \ \&\ p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 5,7 \pmod 8;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3$$
$$=\begin{cases}4x^2-2p & \text{if } p\equiv 1 \pmod 3 \ \&\ p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Catalan numbers

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

Recursion.

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n = 0, 1, 2, \ldots).$$

Generating Function.

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

Combinatorial Interpretations. The Catalan numbers arise in many enumeration problems. For example, C_n is the number of binary parenthesizations of a string of $n+1$ letters, and it is also the number of ways to triangulate a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

Recent results on $\sum_{k=0}^{p-1}\binom{2k}{k}$ and $\sum_{k=0}^{p-1}C_k \bmod p^2$

Let p be a prime and let $a\in\mathbb{Z}^+=\{1,2,3,\ldots\}$.
H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\quad\text{for } p>3.$$

Sun & R. Tauraso [Int. JNT, Adv. in Appl. Math. 2010].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p^a-1}C_k\equiv\frac{3(\frac{p^a}{3})-1}{2}\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3.$$

Determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$

Let p be an odd prime. If $p/2 < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \pmod{p},$$

where m is an integer with $p \nmid m$.

Sun [Sci. China Math. 2010]: Let p be an odd prime and let $a, m \in \mathbb{Z}$ with $a > 0$ and $p \nmid m$. Then

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p^a}\right) + \left(\frac{m^2-4m}{p^{a-1}}\right) u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $(-)$ is the Jacobi symbol and $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ \ u_1 = 1, \ \ \text{and } u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

On $\sum_{k=0}^{p-1}\binom{3k}{k}/m^k \bmod p$

In 2009 Sun determined $\sum_{k=0}^{p-1}\binom{3k}{k}/m^k \bmod p$ where $p>3$ is a prime and m is an integer not divisible by p.

Some particular congruences (Sun, 2009):

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{8^k}\equiv\frac{3}{4}\left(\left(\frac{p}{5}\right)-1\right)\pmod p,$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{7^k}\equiv\begin{cases}-2\pmod p & \text{if } p\equiv\pm2\pmod 7,\\ 1\pmod p & \text{otherwise.}\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k}}{5^k}\equiv\begin{cases}1\pmod p & \text{if } p\equiv1\pmod 5\ \&\ p\neq11,\\ -1/11\pmod p & \text{if } p\equiv2,3\pmod 5,\\ -9/11\pmod p & \text{if } p\equiv4\pmod 5.\end{cases}$$

If $p\equiv1\pmod 3$ then

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{6^k}\equiv2^{(p-1)/3}\pmod p.$$

On $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k$ modulo p^2

A Conjecture of Rodriguez-Villegas proved by E. Mortenson. If p is an odd prime, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}\pmod{p^2}.$$

Remark. (a) By Stirling's formula,

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n\to+\infty.$$

It follows that

$$\binom{2k}{k}^2\sim\frac{16^k}{k\pi}.$$

(b) Mortenson'proof involves Gauss and Jacobi sums and the p-adic Gamma function. In fact, now there are elementary proofs.

Euler numbers and some congruences mod p^3

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}\ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453].

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) \frac{p^2}{4} E_{p-3} \pmod{p^3}.$$

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Some congruences related to Euler numbers

Theorem (Sun, 2010). For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}.$$

Remark. Note that

$$\lim_{k \to +\infty} \frac{k \binom{2k}{k}^2}{16^k} = \frac{1}{\pi} \ \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} = \frac{\pi^2}{18}.$$

Some auxiliary results needed for the proof

A Lemma (Sun, 2010). (i) If $p = 2n+1$ is an odd prime, then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1-\frac{p}{4}(H_{n+k}-H_{n-k})\right)\equiv\frac{\binom{2k}{k}^2}{16^k}\pmod{p^4}.$$

(ii) We have

$$(-1)^n\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^k(H_{n+k}-H_{n-k})=\frac{3}{2}\sum_{k=1}^n\frac{\binom{2k}{k}}{k}.$$

Some auxiliary identities:

$$\sum_{k=1}^n\frac{\binom{2k}{k}}{k}=\frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^n\frac{1}{k^2\binom{n}{k}^2}\ \text{(Staver)},$$

$$\sum_{k=1}^n\frac{(-1)^k}{k^2\binom{n}{k}\binom{n+k}{k}}=(-1)^{n-1}\left(3\sum_{k=1}^n\frac{1}{k^2\binom{2k}{k}}+2\sum_{k=1}^n\frac{(-1)^k}{k^2}\right)\ \text{(Apéry)}$$

$$\sum_{k=1}^n\frac{1}{k^2\binom{n+k}{k}}=3\sum_{k=1}^n\frac{1}{k^2\binom{2k}{k}}-\sum_{k=1}^n\frac{1}{k^2}\ \ \text{(Sun)}.$$

A conjecture involving Euler numbers

Conjecture (Sun, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{(2k-1)\binom{2k}{k}} \equiv E_{p-3} + (-1)^{(p-1)/2} - 1 \pmod{p}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{16^k}{k(2k-1)\binom{2k}{k}} \equiv \frac{8}{3} E_{p-3} \pmod{p}.$$

Remark. Let $p > 3$ be a prime. The speaker has shown that

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p},$$

$$\sum_{k=2}^{(p-1)/2} \frac{4^k}{(k-1)^2\binom{2k}{k}} \equiv 8E_{p-3} - 4 - 12\left(\frac{-1}{p}\right) \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{4^k}{(k+1)\binom{2k}{k}} \equiv \left(\frac{-1}{p}\right)(4 - 2E_{p-3}) - 2 \pmod{p}.$$

On $\sum_{k=0}^{p-1}\binom{2k}{k}^3 \bmod p^2$

Conjecture (Z. W. Sun, 2009): Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \& \ p=x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1. \end{cases}$$

Moreover,

$$\sum_{k=0}^{(p-1)/2}(21k+8)\binom{2k}{k}^3 \equiv 8p+\left(\frac{-1}{p}\right)32p^3E_{p-3} \pmod{p^4}.$$

Remark. In number theory it is known that if p is a prime with $(\frac{p}{7})=1$ (i.e., $p\equiv 1,2,4 \pmod 7$) then there are unique positive integers x and y such that $p=x^2+7y^2$.

The speaker's conjecture involving $x^2 + 11y^2$

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases}$$

Furthermore,

$$\sum_{k=0}^{p-1} (11k + 3) \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv 3p + \frac{7}{2} p^4 B_{p-3} \pmod{p^5},$$

$$p \sum_{k=1}^{(p-1)/2} \frac{(11k - 3) 64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \equiv 32 \frac{2^{p-1} - 1}{p} - \frac{64}{3} p^2 B_{p-3} \pmod{p^3}.$$

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11}) = 1$ we can write $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$.

Conjecture involving $x^2 + 163y^2$

Conjecture (Sun, 2010). Let $p > 5$ be a prime with $p \neq 23, 29$.

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-640320)^{3k}}$$

$$\equiv \begin{cases} (\frac{-10005}{p})(x^2 - 2p) \pmod{p^2} & \text{if } (\frac{p}{163}) = 1 \ \&\ 4p = x^2 + 163y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{163}) = -1. \end{cases}$$

Remark. It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For each of the 9 values of d we have corresponding conjectures similar to the above one.

Conjecture for $\mathbb{Q}(\sqrt{-d})$ with class number two

Let $d > 0$ be a squarefree integer. It is known that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if d is among

$5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$

Except for $d = 35, 91, 115, 187, 235, 403, 427$ we have found explicit conjectures involving $x^2 + dy^2$.

Conjecture for $\mathbb{Q}(\sqrt{-15})$ (Sun). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 5x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{15k+4}{(-27)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4 \left(\frac{p^a}{3}\right) \pmod{p^2}.$$

Six conjectured series for π^2 and other constants

Conjecture (Z. W. Sun, 2010): We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

A conjecture motivated by some series for $\zeta(3)$ and $\zeta(4)$

Conjecture (Sun, 2010). Let $p > 7$ be a prime and let $H_{p-1} = \sum_{k=1}^{p-1} 1/k \equiv -p^2 B_{p-3}/3 \pmod{p^3}$. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} H_{p-1} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - \frac{H_{p-1}}{p^3} \equiv -\frac{7}{45} p B_{p-5} \pmod{p^2}.$$

Also,

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -2B_{p-3} \pmod{p}.$$

Motivation.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

Harmonic numbers

Stirling numbers of the first kind:

$$s(n,k) = |\{\sigma \in S_n :\ \sigma \text{ has exactly } k \text{ cycles}\}|$$

Harmonic numbers:

$$H_n = \sum_{0<k\leqslant n} \frac{1}{k} = \frac{s(n+1,2)}{n!} \quad (n = 0,1,2,\ldots).$$

It is known that

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \quad \text{(S. W. Coffman, 1987)}$$

and

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{360}\pi^4 \quad \text{(D. Borwein and J. M. Borwein, 1995).}$$

Congruences on harmonic numbers

Theorem (Z. W. Sun, 2009): Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p},$$

$$\sum_{k=1}^{p-1} k^2 H_k^2 \equiv -\frac{4}{9} \pmod{p},$$

$$\sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

$$\sum_{k=1}^{p-1} H_k^2 \equiv 2p - 2 \pmod{p^2}.$$

When $p > 5$ we have

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

Congruences on harmonic numbers of even order

Conjecture (Z. W. Sun, 2009) Let m be any positive even integer. If p is a prime with $p-1\nmid 3m$, then

$$\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m}\equiv 0 \pmod p,$$

where $H_{k,m}=\sum_{0<j\leqslant k}1/j^m$.

Remark. Sun proved the conjecture in the case $2p/3<m<p$ via Bernoulli numbers. Later his former student Li-Lu Zhao (Hong Kong Univ.) fully confirmed the conjecture.

Beukers' conjecture on Apéry numbers

In his proof of the irrationality of $\zeta(3)$, Apéry introduced

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Dedekind eta function in the theory of modular forms:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau}$$

Note that $|q| < 1$ if $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$.

Beukers' Conjecture (1985). For any prime $p > 3$ we have

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n.$$

Beukers' conjecture on Apéry numbers

A Simple Observation. Let $p = 2n+1$ be an odd prime. Then

$$\begin{aligned}&\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}\\ =&\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k} \equiv \binom{-1/2}{k}^2\\ =&\left(\binom{2k}{k}/(-4)^k\right)^2 = \binom{2k}{k}^2/16^k \pmod{p^2}.\end{aligned}$$

Thus Beukers' conjecture has the following equivalent form:

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^2}.$$

Ahlgren and Ono's Proof of the Beukers conjecture

S. Ahlgren and Ken Ono [J. Reine Angew. Math. 518(2000)]: The Beukers conjecture is true!

Outline of their proof. First show that $a(p)$ can be expressed as a special value of the Gauss hypergeometric function ${}_4F_3(\lambda)$ defined in terms of Jacobi sums. Then rewrite Jacobi sums in terms of Gauss' sums and apply the Gross-Koblitz formula to express Gauss sums in terms of the p-adic Gamma function $\Gamma_p(x)$. Finally use combinatorial properties of $\Gamma_p(x)$ and some sophisticated combinatorial identities involving harmonic numbers $H_n = \sum_{0<k\leqslant n} 1/k$.

Two key points in Ahlgren and Ono's Proof

Two key points in S. Ahlgren and Ken Ono's proof [2000].

(i) For an odd prime p let $N(p)$ denote the number of $\mathbb{F}_p$-points of the following Calabi-Yau threefold

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+w+\frac{1}{w}=0.$$

Then

$$a(p)=p^3-2p^2-7-N(p).$$

(ii) For any positive integer n we have

$$\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(1+2kH_{n+k}+2kH_{n-k}-4kH_k)=0,$$

where $H_k=\sum_{0<j\leqslant k}1/j$.

T. Kilbourn [Acta Arith. 123(2006)]: For any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^4}{256^k}\equiv a(p)\pmod{p^3}.$$

Congruences on Apéry numbers

Recall that Apéry numbers are those integers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Theorem (Sun, 2010). For any $n \in \mathbb{Z}^+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k} \in \mathbb{Z}.$$

If $p > 3$ is a prime, then $\sum_{k=0}^{p-1}(2k+1)A_k \equiv p \pmod{p^4}$.

Conjecture (Sun, 2010). For any positive integer n we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \in \mathbb{Z}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3}.$$

Congruences on Apéry numbers

Conjecture (Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} A_k$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod 8 \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod 8; \end{cases}$$

and

$$\sum_{k=0}^{p-1} (-1)^k A_k$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

On central Delannoy numbers

$$D_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}.$$

In combinatorics, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Theorem (Sun, 2010). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}.$$

When $p > 3$ we also have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k D_k \equiv p \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(2k+1)D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

On central Delannoy numbers

Conjecture (Sun, 2010). (i) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)D_k^2 \equiv p^2 - 4p^3 q_p(2) - 2p^4 q_p(2)^2 \pmod{p^5}.$$

(ii) Let p be any odd prime. Then

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2} \equiv 2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p} \text{ and } \sum_{k=0}^{p-1}D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Remark. I can show that $n \mid \sum_{k=0}^{n-1}(2k+1)(-1)^k D_k^2$ for $n \in \mathbb{Z}^+$.

Congruences involving Schröder numbers

$$D_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}.$$

The nth Schröder number is given by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}$$

which is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ that never rise above the line $y=x$.

Conjecture (Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} D_k S_k \equiv 1 + 4pq_p(2) - 2p^2 q_p(2)^2 \pmod{p^3},$$

and

$$\sum_{k=1}^{(p-1)/2} D_k S_k \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \ \&\ p = x^2+y^2 \ (2 \nmid x), \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

On central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}\binom{2k}{k}.$$

Theorem (H. Q. Cao and Sun, 2010). For any prime $p > 3$ we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

Conjecture (Sun, 2010) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (8k+5) T_k^2 \equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1} (8k+5) T_k^2 \equiv 3p \left(\frac{p}{3}\right) \pmod{p^2}.$$

Mod p^2 congruences on Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ in an $n \times n$ grid using only steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Sun, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1 - 9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and we introduce the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Continued

$$D_n = T_n(3,2), \quad \text{but } M_n(3,2) \text{ is different from } S_n.$$

H. S. Wilf observed that

$$\sum_{n=0}^{\infty} T_n(b,c)x^n = \frac{1}{\sqrt{1-2bx+(b^2-4c)x^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c)+(4c-b^2)nT_{n-1}(b,c) \ \ (n \in \mathbb{Z}^+).$$

Theorem (Sun, 2010). Let p be an odd prime and let $b, c, m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{p}$. Then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod{p}$$

and

$$2c\sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} \equiv (m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right) \pmod{p}.$$

Continued

Theorem (Sun, 2010). Let $b, c \in \mathbb{Z}$.

(i) For any $n \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k} \equiv 0 \pmod{n},$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c).$$

(ii) Suppose that $b^2-4c=1$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c) = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k-1}{k-1}\left(\frac{b-1}{2}\right)^{k-1} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}^+$. If p is a prime not dividing c, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c) \equiv p \pmod{p^2}.$$

Conjecture on generalized central trinomial coefficients

Conjecture (Sun, 2010). Let $b, c \in \mathbb{Z}$.

(i) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(b^2-4c)^{n-1-k} \equiv 0 \pmod{n^2}.$$

(ii) Suppose that $b^2-4c=1$. Then

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^m \equiv 0 \pmod{n}$$

for all $m, n \in \mathbb{Z}^+$. If p is a prime not dividing c, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)^3 \equiv p\left(\frac{-2b-1}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)^4 \equiv p \pmod{p^2}.$$

A new kind of numbers

For $b, c \in \mathbb{Z}$ we introduce a new kind of numbers:

$$W_n(b,c) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n-k}{k}^2 b^{n-2k} c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{2k}{k}^2 b^{n-2k} c^k.$$

Note that $W_n(-b,c) = (-1)^n W_n(b,c)$.

Conjecture (Sun, 2010). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} W_k(1,1)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,3 \pmod 8 \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

If $p \equiv 1,3 \pmod 8$, then

$$\sum_{k=0}^{p-1} (16k+3) W_k(1,1) \equiv 8p \pmod{p^2}.$$

If $p \equiv 5,7 \pmod 8$ and $p \neq 7$,then $\sum_{k=0}^{p-1} \frac{W_k(1,1)}{(-7)^k} \equiv 0 \pmod{p^2}$.

More conjectures

Conjecture (Sun, 2010). (i) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}(-1)^k W_k(1,-1)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and } p = x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(6k+5)(-1)^k W_k(1,-1) \equiv 0 \pmod n.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(6k+5)(-1)^k W_k(1,-1) \equiv p\left(2+3\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{W_k(2,-1)}{(-2)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

If $p\equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1}(4k+3)\frac{W_k(2,-1)}{(-2)^k}\equiv 0 \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010) (i) Let p be an odd prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{W_k(2,1)}{(-2)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and } p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

(ii) For any $n\in\mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(4k+3)W_k(2,-1)(-2)^{n-1-k}\equiv 0 \pmod n.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(4k+3)\frac{W_k(2,-1)}{(-2)^k}\equiv p\left(2\left(\frac{2}{p}\right)+\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010) (i) Let $p \neq 2, 5$ be a prime. Then we have

$$\sum_{k=0}^{p-1} W_k(1,-4)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \&\ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (20k+17) W_k(1,-4) \equiv 0 \pmod{n}.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1} (20k+17) W_k(1,-4) \equiv p\left(10\left(\frac{-1}{p}\right) + 7\right) \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010) (i) For any prime $p > 5$, we have

$$\sum_{k=0}^{p-1} W_k(1,81)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,9,11,19 \pmod{40} \text{ \& } p = x^2 + 10y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } p \equiv 7,13,23,37 \pmod{40} \text{ \& } 2p = x^2 + 10y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-10}{p}) = -1. \end{cases}$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(10k+9)W_k(1,81) \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(10k+9)W_k(1,81) \equiv p\left(4\left(\frac{-2}{p}\right)+5\right) \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010) (i) For any prime $p \neq 7$, we have

$$\sum_{k=0}^{p-1}(-1)^k W_k(1,-16)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p \equiv 1,2,4 \pmod 7 \ \&\ p = x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3,5,6 \pmod 7. \end{cases}$$

(ii) For all $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(42k+37)(-1)^k W_k(1,-16) \equiv 0 \pmod n.$$

If p is a prime, then

$$\sum_{k=0}^{p-1}(42k+37)(-1)^k W_k(1,-16) \equiv p\left(21\left(\frac{p}{7}\right)+16\right) \pmod{p^2}.$$

More conjectures

Conjecture (Sun, 2010) (i) For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} W_k(1,-324)$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{13}{p}) = (\frac{-1}{p}) = 1 \ \&\ p = x^2+13y^2, \\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{13}{p}) = (\frac{-1}{p}) = -1 \ \&\ 2p = x^2+13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-13}{p}) = -1. \end{cases}$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(260k+237)W_k(1,-324) \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(260k+237)W_k(1,-324) \equiv p\left(130\left(\frac{-1}{p}\right)+107\right) \pmod{p^2}.$$

Bell numbers

For $n = 1, 2, 3, \ldots$, the nth Bell number b_n denotes the number of partitions of a set of cardinality n. In addition, $b_0 := 1$. Here are values of $b_1, \ldots, b_{10}$:

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975.$$

Recursion:

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k \quad (n = 0, 1, 2, \ldots).$$

Exponential Generating Function:

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Touchard's Congruence: For any prime p and $m, n = 0, 1, 2, \ldots$ we have

$$b_{p^m+n} \equiv m b_n + b_{n+1} \pmod{p}.$$

A conjecture on Bell numbers

Conjecture (Sun, July 17, 2010). For any positive integer n there is a unique integer $s(n)$ such that

$$\sum_{k=0}^{p-1}\frac{b_k}{(-n)^k}\equiv s(n) \pmod p \ \text{ for any prime } p\nmid n.$$

In particular,

$$s(2)=1,\ s(3)=2,\ s(4)=-1,\ s(5)=10,\ s(6)=-43,$$
$$s(7)=266,\ s(8)=-1853,\ s(9)=14834,\ s(10)=-133495.$$

Remark. It is easy to see that $s(1)=2$. In fact, if p is a prime then

$$\sum_{k=0}^{p-1}(-1)^k b_k \equiv \sum_{k=0}^{p-1}\binom{p-1}{k} b_k = b_p$$
$$\equiv b_0+b_1=2 \pmod p \text{ (by Touchard's congruence).}$$

More Conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*, `arXiv:0911.5665`

which contains **100 unsolved conjectures** raised by me.

You are welcome to solve my conjectures!

Thank you!