A talk given at the National Taiwan Univ. on Nov. 6, 2002.

ON THE SUM $\sum_{k\equiv r \pmod m}\binom nk$ AND RELATED RESULTS

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093
The People's Republic of China
E-mail: zwsun@nju.edu.cn
Homepage: http://pweb.nju.edu.cn/zwsun

Abstract. In the summer of 1988 Z. H. Sun and Z. W. Sun began to investigate the combinatorial sum in the title, explicit formulas for the cases $m = 8, 10, 12$ were soon obtained and some new and deep congruences followed. Partly inspired by this fruitful work, A. Granville and Z. W. Sun determined $B_{p-1}(a/m) - B_{p-1} \bmod p$ in terms of linear recurrences where a, m are positive integers with $a < m$, p is an odd prime with $p \nmid m$, and $B_n(x)$ is the Bernoulli polynomial of degree n. A connection to quadratic fields will also be introduced.

1. On the sums $\sum_{k\equiv r \pmod m}\binom nk$ and $\sum_{k\equiv r \pmod m}(-1)^{\frac{k-r}{m}}\binom nk$

In the early 1988, I read item by item the subject (11A, 11B, 11D) index of MR in the 1980's to try to find results on classical Frobenius problem which were asked by my undergraduate classmate B.-S. Liao who was a graduate student in computer science then. Suddenly, I found the title '*A note on the Fibonacci quotient $F_{p-\varepsilon}/p$*' of a paper by H. C. Williams. At that time I was familiar with Lucas' sequences which played important roles in the negative solution of Hilbert's Tenth Problem. Since I knew that $p \mid F_{p-(\frac 5p)}$ for any odd prime p where $(-)$ denotes the Jacobi-Legendre symbol, Williams' paper caught my attention. Then I copied the paper from *Canad. Math. Bull.* **25**(1982), 366–370. In the paper Williams used the 5th roots of unity to discuss the sum

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \sum_{\substack{0\leqslant k\leqslant n \\ k\equiv r \pmod m}} \binom{n}{k}$$

(where $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, $m \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ and $r \in \mathbb{Z}$) in the case $m = 5$, and then deduced the congruence

$$\frac{F_{p-(\frac{5}{p})}}{p} \equiv \frac{2}{5} \sum_{k=1}^{p-1-\lfloor p/5\rfloor} \frac{(-1)^k}{k} \pmod p.$$

($\lfloor x\rfloor$ denotes the greatest integer not exceeding x.) As I was absorbed in the study of covering systems of $\mathbb{Z}$ by residue classes and my twin brother Zhi-Hong Sun had no suitable research topic then, I gave the copy to Z. H. Sun. After Z. H. Sun had determined $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_8$, I entered this field in the summer of 1988 on his request.

The following formulas can be found in H. W. Gould's book *Combinatorial Identities* (Morgantown, W. Va., 1972).

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \frac{1}{m} \sum_{j=1}^{m} \left(2\cos\frac{j\pi}{m}\right)^n \cos\frac{(n-2r)j\pi}{m},$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_3 = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right), \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_3 = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$$

and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_4 = \frac{1}{4}\left(2^n + 2^{n/2+1}\cos\frac{n\pi}{4}\right) \quad \text{for } n > 0.$$

Thus, in the cases $m = 3, 4$, the determination of the sum $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_m$ is essentially trivial and known.

Using

$$\binom{n}{k}=\binom{n}{n-k}\quad\text{and}\quad\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1},$$

one can easily prove that

$$\begin{bmatrix}n\\ r\end{bmatrix}_m=\begin{bmatrix}n\\ n-r\end{bmatrix}_m\quad\text{and}\quad\begin{bmatrix}n+1\\ r\end{bmatrix}_m=\begin{bmatrix}n\\ r\end{bmatrix}_m+\begin{bmatrix}n\\ r-1\end{bmatrix}_m.$$

So it suffices to consider $\begin{bmatrix}n\\ r\end{bmatrix}_m$ with n odd and r even.

The Pell sequence $\{P_n\}_{n\in\mathbb{N}}$ and its companion $\{Q_n\}_{n\in\mathbb{N}}$ are defined as follows:

$$P_0=0,\ P_1=1,\ \text{and } P_{n+1}=2P_n+P_{n-1}\ \text{for } n=1,2,\cdots;$$

$$Q_0=2,\ Q_1=2,\ \text{and } Q_{n+1}=2Q_n+Q_{n-1}\ \text{for } n=1,2,\cdots.$$

By induction,

$$P_n=\frac{1}{2\sqrt{2}}\Big((1+\sqrt{2})^n-(1-\sqrt{2})^n\Big)\ \text{and } Q_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n.$$

Theorem 1.1 [Z. H. Sun, Nanjing Univ. J. Math. Biquarterly, 1993]**.** *Let $n>0$ be odd. We have*

(i) *if $n\equiv 1 \pmod 8$, then*

$$\begin{bmatrix}n\\ 2r\end{bmatrix}_8=2^{n-3}+(-1)^r2^{(n-5)/2}+(-1)^{[r/2]+(n-1)/8}2^{(n-5)/4}P_{(n+(-1)^r)/2};$$

(ii) *if $n\equiv 3 \pmod 8$, then*

$$\begin{bmatrix}n\\ 2r\end{bmatrix}_8=2^{n-3}-(-1)^r2^{(n-5)/2}+(-1)^{[r/2]+(n-3)/8}2^{(n-11)/4}Q_{(n-(-1)^r)/2};$$

(iii) *if* $n\equiv 5\pmod 8$, *then*

$$\begin{bmatrix}n\\2r\end{bmatrix}_8=2^{n-3}-(-1)^r2^{(n-5)/2}+(-1)^{[(r+1)/2]+(n+3)/8}2^{(n-5)/4}P_{(n-(-1)^r)/2};$$

(iv) *if* $n\equiv 7\pmod 8$, *then*

$$\begin{bmatrix}n\\2r\end{bmatrix}_8=2^{n-3}+(-1)^r2^{(n-5)/2}+(-1)^{[(r+1)/2]+(n+1)/8}2^{(n-11)/4}Q_{(n+(-1)^r)/2}.$$

The Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ and its companion $\{L_n\}_{n\in\mathbb{N}}$ are defined as follows:

$$F_0=0,\ F_1=1,\ \text{and } F_{n+1}=F_n+F_{n-1} \text{ for } n=1,2,\cdots;$$

$$L_0=2,\ L_1=1,\ \text{and } L_{n+1}=L_n+L_{n-1} \text{ for } n=1,2,\cdots.$$

By induction,

$$F_n=\frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right)$$

and

$$L_n=\left(\frac{1+\sqrt5}{2}\right)^n+\left(\frac{1-\sqrt5}{2}\right)^n.$$

Theorem 1.2 [Z. H. Sun & Z. W. Sun, Acta Arith. 60(1992)]**.** *Let $n>0$ be odd.*

(a) *If* $n\equiv 1\pmod 4$, *then*

$$10\begin{bmatrix}n\\(n-1)/2\end{bmatrix}_{10}=2^n+L_{n+1}+5^{(n+3)/4}F_{(n+1)/2},$$
$$10\begin{bmatrix}n\\(n+3)/2\end{bmatrix}_{10}=2^n-L_{n-1}+5^{(n+3)/4}F_{(n-1)/2},$$
$$10\begin{bmatrix}n\\(n+7)/2\end{bmatrix}_{10}=2^n-L_{n-1}-5^{(n+3)/4}F_{(n-1)/2},$$
$$10\begin{bmatrix}n\\(n+11)/2\end{bmatrix}_{10}=2^n+L_{n+1}-5^{(n+3)/4}F_{(n+1)/2}.$$

(b) *If* $n\equiv 3 \pmod 4$, *then*

$$10\begin{bmatrix} n \\ (n-1)/2\end{bmatrix}_{10} = 2^n + L_{n+1} + 5^{(n+1)/4}L_{(n+1)/2},$$
$$10\begin{bmatrix} n \\ (n+3)/2\end{bmatrix}_{10} = 2^n - L_{n-1} + 5^{(n+1)/4}L_{(n-1)/2},$$
$$10\begin{bmatrix} n \\ (n+7)/2\end{bmatrix}_{10} = 2^n - L_{n-1} - 5^{(n+1)/4}L_{(n-1)/2},$$
$$10\begin{bmatrix} n \\ (n+11)/2\end{bmatrix}_{10} = 2^n + L_{n+1} - 5^{(n+1)/4}L_{(n+1)/2}.$$

(c) *We have*

$$10\begin{bmatrix} n \\ (n-5)/2\end{bmatrix}_{10} = 2^n - 2L_n.$$

Now we define a special Lucas sequence $\{S_n\}_{n\in\mathbb{N}}$ and its companion $\{T_n\}_{n\in\mathbb{N}}$ as follows:

$$S_0=0,\ S_1=1,\ \text{and } S_{n+1}=4S_n-S_{n-1} \text{ for } n=1,2,\cdots;$$

$$T_0=2,\ T_1=4,\ \text{and } T_{n+1}=4T_n-T_{n-1} \text{ for } n=1,2,\cdots.$$

By induction $T_n=4S_n-2S_{n-1}$ and $6S_n=2T_n-T_{n-1}$, also

$$S_n=\frac{1}{2\sqrt{3}}\Big((2+\sqrt{3})^n-(2-\sqrt{3})^n\Big) \text{ and } T_n=(2+\sqrt{3})^n+(2-\sqrt{3})^n.$$

Theorem 1.3 [Z. W. Sun 1988; Israel J. Math. 128(2002)]**.** *Let* $n>0$ *be odd and* $r\in\mathbb{Z}$. *Then*

$$12\begin{bmatrix} n \\ r\end{bmatrix}_{12} - 2^n - 1$$
$$=\begin{cases} 3^{\frac{n+1}{2}}+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}+T_{\frac{n+1}{2}}) & \text{if } n-2r\equiv \pm1 \pmod{12},\\ -3+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}-T_{\frac{n+1}{2}}+T_{\frac{n-1}{2}}) & \text{if } n-2r\equiv \pm3 \pmod{12},\\ -3^{\frac{n+1}{2}}+(-1)^{\frac{r(n-r)}{2}}(\frac{2}{n})(2^{\frac{n+1}{2}}-T_{\frac{n-1}{2}}) & \text{if } n-2r\equiv \pm5 \pmod{12}.\end{cases}$$

Corollary 1.1. *Let n be a positive odd integer. Then*

$$\frac{(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}-1}{n}=\sum_{\substack{k=1\\2\mid k}}^{n-1}\frac{(-1)^{\frac{k}{2}}}{k}\binom{n-1}{k-1}=\sum_{\substack{k=1\\2\nmid k}}^{n-1}\frac{(-1)^{\frac{n-k}{2}}}{k}\binom{n-1}{k-1}$$

and

$$2\sum_{\substack{k=1\\4\mid k-r}}^{n-1}\frac{1}{k}\binom{n-1}{k-1}=\frac{2^{n-1}-1}{n}+(-1)^{\frac{r(n-r)}{2}}\frac{(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}-1}{n}\quad \textit{for } r\in\mathbb{Z}.$$

For an assertion A we let $[A]$ be 1 or 0 according to whether A holds or not.

Corollary 1.2. *Let n be a positive odd integer. Then*

$$\begin{bmatrix}n\\r\end{bmatrix}_6=\frac{2^{n-1}-1}{3}+\frac{[3\nmid n+r]}{2}\left((-1)^{\lfloor\frac{n-2r+1}{6}\rfloor}3^{\frac{n-1}{2}}+1\right)\quad \textit{for } r\in\mathbb{Z}.$$

Providing $n\not\equiv 3 \pmod 6$ we have

$$\frac{(\frac{3}{n})3^{\frac{n-1}{2}}-1}{n}=\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^{\lfloor\frac{k+1}{3}\rfloor}}{k}\binom{n-1}{k-1}=\frac{1}{3}\sum_{k=1}^{\lfloor n/3\rfloor}\frac{(-1)^k}{k}\binom{n-1}{3k-1}.$$

In the general case we have

Theorem 1.4 [Z. W. Sun, Israel J. Math., 128(2002)]**.** *Let $D_0(x)=2$ and*

$$D_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^{\lfloor\frac{n}{2}\rfloor-i}\quad \textit{for } n\in\mathbb{Z}^+.$$

Let $k,m\in\mathbb{Z}$ and $m>2$. Write

$$w_n(k,m)=\sum_{\substack{0<j<m/2\\(j,m)=1}}D_{|k|}\left(4\cos^2\frac{j\pi}{m}\right)\left(4\cos^2\frac{j\pi}{m}\right)^n\quad \textit{for } n\in\mathbb{Z},$$

and

$$\begin{aligned}A_m(x) = &\prod_{\substack{0<j<m/2\\(j,m)=1}}\left(x-4\cos^2\frac{j\pi}{m}\right)\\ =&x^{\varphi(m)/2}-a_1x^{\varphi(m)/2-1}-\cdots-a_{\varphi(m)/2-1}x-a_{\varphi(m)/2}.\end{aligned}$$

Then $(-1)^{s-1}a_s\in\mathbb{Z}^+$ *for* $s=1,\cdots,\varphi(m)/2$, *and*

$$w_n(k,m)=a_1w_{n-1}(k,m)+\cdots+a_{\varphi(m)/2}w_{n-\varphi(m)/2}(k,m)\quad \text{for } n\in\mathbb{Z}.$$

Whenever $n\in\mathbb{N}$ *and* $r\in\mathbb{Z}$, *we have*

$$\begin{bmatrix}n\\r\end{bmatrix}_m=\frac{2^n+(-1)^r[2\mid m\ \&\ n=0]}{m}+\frac{1}{m}\sum_{\substack{d\mid m\\d>2}}w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r,d).$$

Z. W. Sun also introduced the alternating sum

$$\begin{Bmatrix}n\\r\end{Bmatrix}_m=\sum_{\substack{k=0\\k\equiv r\ (\mathrm{mod}\ m)}}^{n}(-1)^{\frac{k-r}{m}}\binom{n}{k}$$

where $m>0$, $n\geqslant 0$ and r are integers. Clearly

$$\begin{bmatrix}n\\r\end{bmatrix}_m+\begin{Bmatrix}n\\r\end{Bmatrix}_m=2\begin{bmatrix}n\\r\end{bmatrix}_{2m}\quad\text{and}\quad\begin{bmatrix}n\\r\end{bmatrix}_m-\begin{Bmatrix}n\\r\end{Bmatrix}_m=2\begin{bmatrix}n\\r+m\end{bmatrix}_{2m}.$$

Theorem 1.5 [Z. W. Sun, submitted]**.** *Let* $k,m\in\mathbb{Z}$ *and* $m>1$. *Then*

$$\sum_{i=0}^{\lfloor(m+1)/2\rfloor}(-1)^ic_m(i)\begin{bmatrix}n-2i\\k-i\end{bmatrix}_m=2(-1)^k[m=n\equiv 0\ (\mathrm{mod}\ 2)]$$

for each integer $n\geqslant 2\lfloor(m+1)/2\rfloor$, *and*

$$\sum_{i=0}^{\lfloor m/2\rfloor}(-1)^id_m(i)\begin{Bmatrix}n-2i\\k-i\end{Bmatrix}_m=(-1)^k[m-1=n\equiv 0\ (\mathrm{mod}\ 2)]$$

for any integer $n\geqslant 2\lfloor m/2\rfloor$, *where*

$$c_m(i)=\frac{m^2+m-2i}{(m-i)(m+1-i)}\binom{m+1-i}{i}\in\mathbb{Z}$$

and

$$d_m(i)=\frac{m}{m-i}\binom{m-i}{i}\in\mathbb{Z}.$$

Corollary 1.3 (Fleck). *Let p be an odd prime. Then* $p^{\lfloor \frac{n-1}{p-1} \rfloor} \mid \left\{ {n \atop r} \right\}_p$.

Proof. Since $p \mid d_p(i)$ for $i = 1, 2, \cdots, (p-1)/2$, by induction Fleck's result follows from Theorem 1.5.

2. Applications in Number Theory

Lemma 2.1 [Z. W. Sun, Proc. Amer. Math. Soc. 123(1995)]. *Let $n > 0$ be odd. Then*

$$\begin{aligned}&2^{n-1} + n2^{n-2} \sum_{k=1}^{(n-1)/2} \frac{1}{k2^k}\binom{n-1}{2k-1} = 2^{(n-1)/2} P_n \\ =&\Big(\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4k}(-1)^k\Big)^2 + \Big(\sum_{k=1}^{\lfloor (n+1)/4 \rfloor} \binom{n}{4k-2}(-1)^k\Big)^2.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}2^{(n+3)/2} P_n =& \sqrt{2}^n \left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) \\ =&(2+\sqrt{2})^n + (2-\sqrt{2})^n = 2\sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} 2^{n-k}\sqrt{2}^k \\ =&2\sum_{k=0}^{(n-1)/2}\binom{n}{2k}2^{n-2k}2^k = 2^{n+1} + n2^n \sum_{k=1}^{(n-1)/2} \frac{1}{k2^k}\binom{n-1}{2k-1}.\end{aligned}$$

So the first equality holds.

Now we give a simple proof of the second equality due to Z. Shan and Edward T. H. Wang [Proc. Amer. Math. Soc. 127(1999)]. Let $\varepsilon = e^{2\pi i/8} = e^{\pi i/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = (1+i)/\sqrt{2}$. Clearly $\varepsilon^2 = e^{\pi i/2} = i$

and $\varepsilon\bar{\varepsilon}=1$. Thus,

$$\begin{aligned}\frac{(1+\varepsilon)^n+(1-\varepsilon)^n}{2} &= \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}\varepsilon^k\\ &= \sum_{0\le 4k\le n}\binom{n}{4k}\varepsilon^{4k}+\sum_{0\le 4k-2\le n}\binom{n}{4k-2}\varepsilon^{4k-2}\\ &= \sum_{k=0}^{\lfloor (n-1)/4\rfloor}\binom{n}{4k}(-1)^k - i\sum_{k=1}^{\lfloor (n+1)/4\rfloor}\binom{n}{4k-2}(-1)^k.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Bigg(\sum_{k=0}^{\lfloor (n-1)/4\rfloor}\binom{n}{4k}(-1)^k\Bigg)^2+\Bigg(\sum_{k=1}^{\lfloor (n+1)/4\rfloor}\binom{n}{4k-2}(-1)^k\Bigg)^2\\ =&\frac{(1+\varepsilon)^n+(1-\varepsilon)^n}{2}\cdot\frac{(1+\bar{\varepsilon})^n+(1-\bar{\varepsilon})^n}{2}\\ =&\frac{(2+\varepsilon+\bar{\varepsilon})^n+(2-\varepsilon-\bar{\varepsilon})^n}{4}=\frac{(2+\sqrt{2})^n+(2-\sqrt{2})^n}{4}=2^{(n-1)/2}P_n.\end{aligned}$$

This ends the proof. □

The following congruence was first conjectured by Z. H. Sun in 1988, but he could only show it in the case $p\equiv 1 \pmod 8$.

Theorem 2.1 [Z. W. Sun, Proc. Amer. Math. Soc. 123(1995)]**.** *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k2^k}\equiv\sum_{k=1}^{\lfloor \frac{3}{4}p\rfloor}\frac{(-1)^{k-1}}{k} \pmod p. \tag{2.1}$$

Proof. For $k=1,2,\cdots,p-1$ clearly

$$\binom{p}{k}=\frac{p}{k}\binom{p-1}{k-1}\equiv p\frac{(-1)^{k-1}}{k} \pmod{p^2}.$$

Applying Lemma 2.1 we find that

$$\begin{aligned}&2^{p-1}+\frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{(-1)^{2k-1}}{k2^k}\\ \equiv&2^{(p-1)/2}P_p\equiv 1+2\sum_{k=1}^{\lfloor p/4\rfloor}\binom{p}{4k}(-1)^k\\ \equiv&1+\frac{p}{2}\sum_{k=1}^{\lfloor p/4\rfloor}\frac{(-1)^{k-1}}{k}\ (\mathrm{mod}\ p^2).\end{aligned}$$

Thus

$$\frac{2^p-2}{p}-\sum_{k=1}^{(p-1)/2}\frac{1}{k2^k}\equiv\frac{2}{p}\left(2^{(p-1)/2}P_p-1\right)\equiv\sum_{k=1}^{\lfloor p/4\rfloor}\frac{(-1)^{k-1}}{k}\pmod p.$$

Observe that

$$\frac{2^p-2}{p}=\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}\equiv\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\ (\mathrm{mod}\ p)$$

(due to G. Eisentein) and that

$$\sum_{k=1}^{\lfloor p/4\rfloor}\frac{(-1)^{k-1}}{k}\equiv\sum_{k=1}^{\lfloor p/4\rfloor}\frac{(-1)^{p-k-1}}{p-k}\equiv\sum_{3p/4<k<p}\frac{(-1)^{k-1}}{k}\ (\mathrm{mod}\ p).$$

So the desired congruence follows. $\square$

Remark. Let p be an odd prime. By the proof of Theorem 2.1, $2(2^{\frac{p-1}{2}}P_p-1)/p\equiv\sum_{0<k<p/4}(-1)^{k-1}/k\ (\mathrm{mod}\ p)$. As

$$\left(\frac{2}{p}\right)Q_{p-(\frac{2}{p})}=4\left(\frac{2}{p}\right)P_p-Q_p\equiv 4-(1+\sqrt{2}+(1-\sqrt{2}))^p\equiv 2\ (\mathrm{mod}\ p),$$

$Q^2_{p-(\frac{2}{p})}-4=8P^2_{p-(\frac{2}{p})}\equiv 0\ (\mathrm{mod}\ p^2)$ and hence $(\frac{2}{p})P_{p-(\frac{2}{p})}=P_p-\frac{1}{2}Q_{p-(\frac{2}{p})}\equiv P_p-(\frac{2}{p})\ (\mathrm{mod}\ p^2)$. Thus

$$\begin{aligned}\frac{P_{p-(\frac{2}{p})}}{p}&\equiv\frac{(\frac{2}{p})P_p-1}{p}\equiv\sum_{0<k<\frac{p}{4}}\frac{(-1)^{k-1}}{2k}-\frac{2^{p-1}-1}{2p}\\ &\equiv\frac{1}{2}\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{(-1)^k}{k}\ (\mathrm{mod}\ p).\end{aligned}\tag{2.2}$$

Theorem 2.2 [Z. W. Sun, 1988; Israel J. Math. 128(2002)]**.** *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{3^k}{k}\equiv\sum_{k=1}^{\lfloor p/6\rfloor}\frac{(-1)^k}{k}=-\sum_{p/12<k<p/6}\frac{1}{k}\pmod p. \tag{2.3}$$

Equivalently,

$$\frac{S_{(p-(\frac{3}{p}))/2}}{p}\equiv\frac{1}{6}\left(\frac{2}{p}\right)\sum_{p/6<k<p/2}\frac{(-1)^k}{k}\pmod p. \tag{2.4}$$

Theorem 2.3 [Z. H. Sun & Z. W. Sun, Acta Arith. 60(1992)]**.** *Let $p\neq 2,5$ be a prime.*

(i) *We have*

$$\frac{F_{p-(\frac{5}{p})}}{p}\equiv-2\sum_{\substack{0<k<p\\ 5\mid k-2p}}\frac{1}{k}\equiv 2\sum_{\substack{0<k<p\\ 5\mid k+p}}\frac{1}{k}\ (\mathrm{mod}\ p). \tag{2.5}$$

If $p\equiv 1\ (\mathrm{mod}\ 4)$, then

$$L_{p-(\frac{5}{p}))/2}\equiv(-1)^{\lfloor (p-5)/10\rfloor}\left(\frac{5}{p}\right)5^{(p-1)/4}\left(\frac{5^{p-1}-1}{2}-2\right)\ (\mathrm{mod}\ p^2). \tag{2.6}$$

If $p\equiv 3\ (\mathrm{mod}\ 4)$, then

$$F_{p-(\frac{5}{p}))/2}\equiv(-1)^{\lfloor (p-5)/10\rfloor}\left(\frac{5}{p}\right)5^{(p-3)/4}\left(\frac{5^{p-1}-1}{2}-2\right)\ (\mathrm{mod}\ p^2). \tag{2.7}$$

(ii) *If $p\equiv 1,9\ (\mathrm{mod}\ 20)$ and hence $p=x^2+5y^2$ for some $x,y\in\mathbb{Z}$, then $p\mid F_{(p-1)/4}$ if and only if $4\mid xy$.*

(iii) *If $p^2\nmid F_{p-(\frac{5}{p})}$ (this is equivalent to an open conjecture of D. D. Wall), then there are no $x,y,z\in\mathbb{Z}$ such that $x^p+y^p=z^p$ and $p\nmid xyz$.*

Remark. Odd primes p with $p^2 \mid F_{p-(\frac{5}{p})}$ are named as Wall-Sun-Sun primes by R. Crandall, K. Dilcher and C. Pomerance [Math. Comp., 66(1997)]. See also C. Caldwell, *The Prime Glossary: Wall-Sun-Sun prime*, `http://primes.utm.edu/glossary/page.php/WallSunSunPrime.html`, and R. Crandall and C. Pomerance, *Prime Numbers: A Computational Perspective*, Springer-Verlag, New York, 2001.

3. On Bernoulli and Euler polynomials

For $n = 0, 1, 2, \cdots$ the nth Bernoulli polynomial $B_n(x)$ is given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

where the Bernoulli numbers $B_0, B_1, B_2, \cdots$ is defined by the power series

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \quad (0 < |z| < 2\pi).$$

Let $n \in \mathbb{N}$. It is known that

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n} - 1\right) B_n.$$

Also, if $2 \mid n$ then

$$\begin{aligned} B_n\left(\frac{1}{3}\right) = B_n\left(\frac{2}{3}\right) =& (3^{1-n} - 1)\frac{B_n}{2}, \\ B_n\left(\frac{1}{4}\right) = B_n\left(\frac{3}{4}\right) =& 2^{-n}(2^{1-n} - 1)B_n, \\ B_n\left(\frac{1}{6}\right) = B_n\left(\frac{5}{6}\right) =& (2^{1-n} - 1)(3^{1-n} - 1)\frac{B_n}{2}. \end{aligned}$$

Observe that $\varphi(1) = \varphi(2) = 1$ and $\varphi(3) = \varphi(4) = \varphi(6) = 2$.

Inspired by the work on the sum $\left[{n \atop r}\right]_m$, in 1996 A. Granville and Z. W. Sun proved the following surprising result for Bernoulli polynomials.

Theorem 3.1 [A. Granville & Z.W. Sun, Pacific J. Math. 172(1996)]**.** *Let p be an odd prime, and a and m be integers with $1 \leqslant a < m$ and $p \nmid m$. Then*

$$B_{p-1}\left(\frac{a}{m}\right) - B_{p-1} \equiv \frac{m}{2p}(U_p - 1) \pmod p, \tag{3.1}$$

where $\{U_n\}_{n=0}^{+\infty}$ is a certain linearly recurrent sequence of order $\lfloor m/2 \rfloor$ which depends only on a, m and p mod m, namely

$$U_n = \frac{1}{2m}\sum_{\substack{\gamma^m=1\\ \gamma\neq 1}} \frac{2-\gamma^a-\gamma^{-a}}{2-\gamma^p-\gamma^{-p}}(2-\gamma-\gamma^{-1})^n \quad \textit{for } n = 0, 1, 2, \cdots. \tag{3.2}$$

Moreover we can express the sequence $\{U_n\}_{n=0}^{+\infty}$ in terms of linear recurrences with orders in $\{1\} \cup \{\varphi(d)/2 \colon d \mid m \ \& \ d > 2\}$, namely

$$mU_n = [2 \nmid a(m-1)]2^{2n-1} + \sum_{\substack{d\mid m\\ d>2}} u_n(d; a, p) \tag{3.3}$$

where

$$\begin{aligned} u_n(d; a, p) &= \sum_{\substack{0<c<d/2\\ (c,d)=1}} \frac{2 - e^{2\pi i \frac{c}{d} a} - e^{-2\pi i \frac{c}{d} a}}{2 - e^{2\pi i \frac{c}{d} p} - e^{-2\pi i \frac{c}{d} p}} \left(2 - e^{2\pi i \frac{c}{d}} - e^{-2\pi i \frac{c}{d}}\right)^n \\ &= \sum_{\substack{0<c<d/2\\ (c,d)=1}} \left(\frac{\sin(\pi ac/d)}{\sin(\pi pc/d)}\right)^2 \left(4\sin^2\frac{\pi c}{d}\right)^n. \end{aligned} \tag{3.4}$$

(*Note that $u_n(m; a, p) = mU_n$ if m is an odd prime.*)

A basic idea in the proof is that if m is a positive integer not divisible by p then

$$-\sum_{0<k\leqslant \lfloor pr/m \rfloor} \frac{1}{k} \equiv B_{p-1}\left(\left\{\frac{pr}{m}\right\}\right) - B_{p-1} \pmod p \text{ for } r = 0, 1, \ldots, m-1 \tag{3.5}$$

where $\{x\}$ stands for $x-\lfloor x\rfloor$.

Here is a remarkable consequence of Theorem 3.1.

Theorem 3.2 [A. Granville & Z.W. Sun, Pacific J. Math. 172(1996)]. *Let p be an odd prime relatively prime to a fixed $m\in\{5,8,10,12\}$. Then we can determine $B_{p-1}(a/m)-B_{p-1}$ mod p (with $1\le a\le m$ and $(a,m)=1$) as follows:*

$$\begin{aligned} B_{p-1}\left(\frac{a}{5}\right)-B_{p-1}&\equiv\left(\frac{ap}{5}\right)\frac{5}{4p}F_{p-(\frac{5}{p})}+\frac{5^p-5}{4p}\pmod p;\\ B_{p-1}\left(\frac{a}{8}\right)-B_{p-1}&\equiv\left(\frac{2}{ap}\right)\frac{2}{p}P_{p-(\frac{2}{p})}+4\cdot\frac{2^{p-1}-1}{p}\pmod p;\\ B_{p-1}\left(\frac{a}{10}\right)-B_{p-1}&\equiv\left(\frac{ap}{5}\right)\frac{15}{4p}F_{p-(\frac{5}{p})}+\frac{5^p-5}{4p}+\frac{2^p-2}{p}\pmod p;\\ B_{p-1}\left(\frac{a}{12}\right)-B_{p-1}&\equiv\left(\frac{3}{a}\right)\frac{3}{p}S_{p-(\frac{3}{p})}+3\cdot\frac{2^{p-1}-1}{p}+\frac{3^p-3}{2p}\pmod p. \end{aligned}$$

The Euler polynomials $E_0(x),E_1(x),\cdots$ are defined by means of

$$\frac{2e^{xz}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}.$$

From Theorem 3.2 we can deduce

Theorem 3.3 [Z. W. Sun, submitted]. *Let p be an odd prime, and let m,n be positive integers with $(m,pn)=1$. Then*

$$(-1)^{\lfloor pn/m\rfloor}E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right)\equiv\begin{cases}(\frac{2}{n})\frac{4}{p}P_{p-(\frac{2}{p})}\pmod p & \text{if } m=4,\\ (\frac{n}{5})\frac{5}{p}F_{p-(\frac{5}{p})}+\frac{2^p-2}{p}\pmod p & \text{if } m=5,\\ (\frac{3}{pn})\frac{6}{p}S_{p-(\frac{3}{p})}\pmod p & \text{if } m=6.\end{cases}\tag{3.6}$$

For integers $q,m,n\in\mathbb{Z}$ with $m>1$ and $n>0$, we define recurrent sequences $\{U_l^{(q)}(m,n)\}_{l=0}^{+\infty}$ and $\{V_l^{(q)}(m,n)\}_{l=0}^{+\infty}$ as follows:

$$U_l^{(q)}(m,n)=\sum_{r=0}^{n-1}\frac{n-r}{1+[r=0]}\left(\binom{2l}{\{l+qr\}_m}(-1)^{\{l+qr\}_m-l}-\frac{[l=0]}{m}\right) \tag{3.7}$$

for $l=0,1,\cdots,\lfloor m/2\rfloor-1$ (where $\{a\}_m$ denotes the least nonnegative residue of a modulo m), and

$$V_l^{(q)}(m,n)=\sum_{r=0}^{n-1}\frac{n-r}{1+[r=0]}\binom{2l}{\{l+qr\}_m}(-1)^{qr+(m-1)\lfloor\frac{l+qr}{m}\rfloor} \tag{3.8}$$

for $l=0,1,\cdots,\lfloor(m-1)/2\rfloor$; and the recursions are

$$U_l^{(q)}(m,n)=\sum_{0<i\leqslant\lfloor m/2\rfloor}(-1)^{i-1}a_m(i)U_{l-i}^{(q)}(m,n)\quad\text{for } l\geqslant\left\lfloor\frac{m}{2}\right\rfloor \tag{3.9}$$

and

$$V_l^{(q)}(m,n)=\sum_{0<j\leqslant\lfloor(m+1)/2\rfloor}(-1)^{j-1}b_m(j)V_{l-j}^{(q)}(m,n)\quad\text{for } l\geqslant\left\lfloor\frac{m+1}{2}\right\rfloor \tag{3.10}$$

where integers $a_m(i)$ and $b_m(j)$ are given by

$$a_m(i)=\begin{cases}c_m(i) & \text{if } 2\mid m,\\ d_m(i) & \text{if } 2\nmid m,\end{cases}\quad\text{and}\quad b_m(j)=\begin{cases}d_m(j) & \text{if } 2\mid m,\\ c_m(j) & \text{if } 2\nmid m,\end{cases} \tag{3.11}$$

where $c_m(i)$ and $d_m(j)$ are as in Theorem 1.5.

(3.1) can be made more explicitly, with help of Theorem 1.5 we have

Theorem 3.4 [Z. W. Sun, submitted]**.** *Let p be an odd prime, and q,m,n be positive integers with $m>1$ and $p\nmid m$. If $p\equiv\pm q \pmod m$, then*

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right)-B_{p-1}\equiv\frac{m}{2p}\left(U_p^{(q)}(m,n)-[m\nmid n]\right)\pmod p. \tag{3.12}$$

If $p\equiv \pm q \pmod{2m}$, *then*

$$\begin{aligned}&(-1)^{\lfloor pn/m\rfloor}E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right)+\frac{2^p-2}{p}\\ \equiv&\frac{m}{p}\left(V_p^{(q)}(m,n)-[m\nmid n]-2[2m\mid m+n]\right) \pmod{p}.\end{aligned}\tag{3.13}$$

Here is another related result obtained recently.

Theorem 3.5 [Z. W. Sun, `arXiv:math.NT/0401228`]. *Let E be a quadratic field with discriminant $d=2^{\alpha}p_1\cdots p_r$ where $\alpha\in\{0,2,3\}$ and $p_1,\cdots,p_r$ are distinct odd primes. Let $\varepsilon=(a+b\sqrt{d})/2$ be the fundamental unit of the field E where $a,b\in\mathbb{Z}$, and $N(\varepsilon)$ be the norm $(a^2-b^2d)/4$ of ε. Let h be the class number of the field E, and p be an odd prime not dividing d. Let $(\frac{d}{p})$ be the Kronecker symbol, and set $u_0=0$, $u_1=1$ and $u_{n+1}=au_n-N(\varepsilon)u_{n-1}$ for $n=1,2,\cdots$. Then, for $\rho=\pm1$ we have*

$$\begin{aligned}\prod_{\substack{0<c<d\\ (\frac{d}{c})=\rho}}\binom{p-1}{\lfloor pc/d\rfloor}\equiv&1+\frac{\varphi(d)}{2}\left((\alpha+[\alpha>0])(2^{p-1}-1)+\sum_{0<i\leqslant r}\frac{p_i^p-p_i}{p_i-1}\right)\\ &+\frac{\rho}{2}\left(\frac{d}{p}\right)^{[N(\varepsilon)=1]}u_{p-(\frac{d}{p})}bdh \pmod{p^2}.\end{aligned}$$