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Congruences for Franel Numbers

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Abstract

The Franel numbers $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ $(n = 0, 1, 2, \ldots)$ play important roles in both number theory and combinatorics. Surprisingly they have nice congruence properties. In this talk we introduce some fundamental congruences for Franel numbers as well as sketches of their proofs. We will also mention various open conjectures involving Franel numbers. The topic is also related to Apéry numbers and binary quadratic forms.

Part I. Introduction to Apéry numbers and Franel numbers

Apéry Numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! During his proof he used the sequence $\{B_n/A_n\}_{n=1}^{\infty}$ of rational numbers to approximate $\zeta(3)$, where

$$A_0 = 1,\ A_1 = 5,\ B_0 = 0,\ B_1 = 6,$$

and both $\{A_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$ satisfy the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}\ (n = 1, 2, \ldots).$$

In fact,

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2$$

and these numbers are called *Apéry numbers*.

Beukers' Conjecture

Dedekind eta function in the theory of modular forms:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau}$$

Note that $|q| < 1$ if $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Beukers' Conjecture (1985). For any prime $p > 3$ we have

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n.$$

An equivalent form of Beukers' conjecture

A Simple Observation. Let $p = 2n + 1$ be an odd prime. Then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$
$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}$$
$$\equiv\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

Thus Beukers' conjecture has the following equivalent form:

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^2}.$$

Ahlgren and Ono's Proof of the Beukers conjecture

Key steps in S. Ahlgren and Ken Ono's proof [2000].

(i) For an odd prime p let $N(p)$ denote the number of $\mathbb{F}_p$-points of the following Calabi-Yau threefold

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+w+\frac{1}{w}=0.$$

Then

$$a(p)=p^3-2p^2-7-N(p).$$

(ii) For any positive integer n we have

$$\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(1+2kH_{n+k}+2kH_{n-k}-4kH_k)=0,$$

where $H_k=\sum_{0<j\leqslant k}1/j$.

T. Kilbourn [Acta Arith. 123(2006)]: For any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^4}{256^k}\equiv a(p)\pmod{p^3}.$$

Some combinatorial identities

Comparing the coefficients of x^n in the expansions of

$$(x+1)^n(x+1)^n = (x+1)^{2n}$$

one finds the known identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

For $n = 1, 3, 5, \ldots$, clearly

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^3 = 0.$$

Dixon's Identity

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n}\binom{3n}{n}.$$

Franel numbers

In 1894 J. Franel introduced the Franel numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

and noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1)+2) f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

In 2008 D. Callan gave a combinatorial interpretation of the Franel numbers.

V. Strehl's Identity:

$$A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k.$$

Barrucand's Identity:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{where } g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Connection with modular forms

Don Zagier (2009) investigated what integer sequence $\{u_n\}$ satisfies $u_{-1} = 0$, $u_0 = 1$, and the Apéry-like recurrence relation

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots).$$

When $(A, B, C) = (7, 2, 8)$, u_n is just the Franel number f_n, and Zagier noted that

$$\sum_{n=0}^{\infty} f_n \left(\frac{\eta(\tau)^3 \eta(6\tau)^9}{\eta(2\tau)^3 \eta(3\tau)^9} \right)^n = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2 \eta(6\tau)^3}$$

for any complex number τ with $\mathrm{Im}(\tau) > 0$, where

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Wolstenholme-type congruences

Wolstenholme's Congruence: Let $p > 3$ be a prime. Then

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3},\ \sum_{k=1}^{p-1}\frac{1}{k} \equiv 0 \,(\mathrm{mod}\ p^2),\ \sum_{k=0}^{p-1}\frac{1}{k^2} \equiv 0 \,(\mathrm{mod}\ p).$$

Skula-Granville Congruence: For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -\left(\frac{2^{p-1}-1}{p}\right)^2 \pmod{p}.$$

A Congruence for Lucas Numbers (conjectured by R. Tauraso and proved by H. Pan and Z. W. Sun). For any prime $p > 5$, we have

$$\sum_{k=1}^{p-1}\frac{L_k}{k^2} \equiv 0 \pmod{p},$$

where the Lucas numbers $L_0, L_1, L_2, \ldots$ are given by

$$L_0 = 2,\ L_1 = 1,\ \text{and } L_{n+1} = L_n + L_{n-1}\ (n = 1, 2, 3, \ldots).$$

Supercongruences involving Franel numbers

Theorem (Z. W. Sun [Adv. Appl. Math., 2013]). Let $p > 3$ be a prime. For any p-adic integer r we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{k+r}{k}f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{k+r}{k}^2 \pmod{p^2}.$$

In particular,

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\ \sum_{k=0}^{p-1}(-1)^k kf_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k \equiv 0 \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k \equiv 0 \pmod{p}.$$

The polynomials $f_n(x)=\sum_{k=0}^n\binom nk^2\binom{2k}{n}x^k$

Motivated by Strehl's identity $\sum_{k=0}^n\binom nk^2\binom{2k}{n}=f_n$, we define the polynomials

$$f_n(x):=\sum_{k=0}^n\binom nk^2\binom{2k}{n}x^k=\sum_{k=0}^n\binom nk\binom{k}{n-k}\binom{2k}{k}x^k\ (n=0,1,\ldots).$$

Theorem (Sun [Adv. Appl. Math. 51(2013)]) Let p be an odd prime and let r be any p-adic integer. Then

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k\binom{k+r}{k}^2\pmod{p^2}.$$

To prove this we need an auxiliary identity

$$\sum_{l=k}^{2k}(-1)^l\binom lk\binom{k}{l-k}\binom{x+l}{l}=\binom{x+k}{x}^2.$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

$$\sum_{l=1}^{p-1}(-1)^l\frac{f_l(x)}{l} = \sum_{l=1}^{p-1}\frac{(-1)^l}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k$$
$$= \sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}x^k\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}.$$

If $1 \leqslant k \leqslant (p-1)/2$, then

$$\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} = \sum_{l=k}^{2k}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}$$
$$= \sum_{j=0}^{k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}$$
$$= (-1)^k\sum_{j=0}^{k}\binom{-k}{j}\binom{k}{k-j} = 0$$

by the Chu-Vandermonde identity.

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

If $(p+1)/2 \leqslant k \leqslant p-1$, then

$$\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} = \sum_{j=0}^{p-1-k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}$$
$$=(-1)^k\sum_{j=0}^{p-1-k}\binom{-k}{j}\binom{k}{k-j}.$$

Applying Andersen's identity

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{m-k} = \frac{m-n}{m}\binom{x-1}{n}\binom{-x}{m-n} \quad (0 \leqslant n \leqslant m),$$

we obtain

$$\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} = (-1)^{k-1}\binom{k}{p-k} \equiv \frac{1}{2}\binom{2(p-k)}{p-k} \pmod{p}$$

Note that $\binom{2k}{k} \equiv 0 \pmod{p}$ for $k = (p+1)/2, \ldots, p-1$.

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k \equiv 0 \pmod{p^2}$

By the above,

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}f_l(x) \equiv \sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{k}x^k\frac{\binom{2(p-k)}{p-k}}{2} \equiv p\sum_{k=(p+1)/2}^{p-1}\frac{x^k}{k^2} \pmod{p^2}.$$

To prove $\sum_{l=1}^{p-1}(-1)^l f_l/l \equiv 0 \pmod{p^2}$, it suffices to note that

$$2\sum_{k=(p+1)/2}^{p-1}\frac{1}{k^2} \equiv \sum_{k=(p+1)/2}^{p-1}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right) = \sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 0 \pmod{p}.$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k^2 \equiv 0 \pmod{p}$

Now we show

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)} \equiv 0 \pmod{p}, \text{ where } f_k^{(r)} := \sum_{j=0}^{k}\binom{k}{j}^r.$$

Clearly,

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}} = \sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^{kr}}{k^{r-1}} + \frac{(-1)^{(p-k)r}}{(p-k)^{r-1}}\right) \equiv 0 \pmod{p}.$$

Thus

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} &\equiv \sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{l=k}^{p-1}(-1)^{lr}\binom{l-1}{k-1}^{r-1}\binom{l}{k} \\ &= \sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{j=0}^{p-1-k}(-1)^{(k+j)r}\binom{k+j-1}{j}^{r-1}\binom{k+j}{j} \\ &= \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-1-k}\binom{-k}{j}^{r-1}\binom{-k-1}{j} \pmod{p}.\end{aligned}$$

Proof of $\sum_{k=1}^{p-1}(-1)^k f_k/k^2 \equiv 0 \pmod{p}$

By the above,

$$\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-k-1}\binom{p-k}{j}^{r-1}\binom{p-k-1}{j} \pmod{p}.$$

For any positive integer n, we have

$$f_n^{(r)} = \sum_{k=0}^{n}\left(\frac{k}{n}+\frac{n-k}{n}\right)\binom{n}{k}^r = 2\sum_{k=0}^{n}\frac{n-k}{n}\binom{n}{k}^r$$
$$=2\sum_{k=0}^{n-1}\binom{n}{k}^{r-1}\binom{n-1}{k}.$$

Therefore

$$\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\cdot\frac{f_{p-k}^{(r)}}{2} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{(p-k)r}f_k^{(r)}}{(p-k)^{r-1}}$$
$$\equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)} \pmod{p}.$$

A further conjecture

Conjecture (Z. W. Sun). For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) + \frac{2}{3}p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{f_k}{8^k} \equiv \sum_{n=0}^{p-1}(-1)^n F_n \equiv \left(\frac{p}{3}\right) - \frac{p^2}{12}B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

where $F_n := \sum_{k=0}^{n}\binom{n}{k}^3(-8)^k$, and $B_{p-2}(x)$ denotes the Bernoulli polynomial of degree $p-2$. Also,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)(-1)^k F_k$$

is an odd integer for any positive integer n.

Part II. Connections to Binary Quadratic Forms

Gauss' congruence

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Determining x mod p^2 with $p = x^2 + y^2$ and $4 \mid x - 1$

Z. W. Sun [Acta Arith. 156(2012)]: Let $p \equiv 1 \pmod 4$ be a prime. Write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$(-1)^{(p-1)/4} x \equiv \sum_{k=0}^{p-1} \frac{k+1}{8^k} \binom{2k}{k}^2 \equiv \sum_{k=0}^{p-1} \frac{2k+1}{(-16)^k} \binom{2k}{k}^2 \pmod{p^2}.$$

Z. W. Sun [Finite Fields Appl. 22(2013)]: For any prime $p \equiv 3 \pmod 4$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \equiv -\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \frac{(-1)^{(p+1)/4} 2p}{\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}.$$

Yeung's result on $\binom{(p-1)/2}{(p-1)/3}$ mod p^2

For a prime p and an integer $a \not\equiv 0 \pmod p$, the *Fermat quotient*

$$q_p(a) := \frac{a^{p-1} - 1}{p} \in \mathbb{Z}.$$

K. M. Yeung [J. Number Theory 33(1989)]: Let $p \equiv 1 \pmod 3$ be a prime and write $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$. Then we have

$$\binom{(p-1)/2}{(p-1)/3} \equiv \left(2x - \frac{p}{2x}\right)\left(1 - \frac{2}{3}p\,q_p(2) + \frac{3}{4}p\,q_p(3)\right) \pmod{p^2}.$$

Remark. Yeung's result is an analogue of Gauss' congruence but it is less elegant since the right-hand side contains the unpleasant expression $1 - \frac{2}{3}p\,q_p(2) + \frac{3}{4}p\,q_p(3)$.

A conjecture on Apéry numbers

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2, \\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8; \end{cases}$$

also,

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3. \end{cases}$$

Remark. In 2011 I proved the mod p version of both congruences and that

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv 0 \pmod{p^2} \ \text{ for any prime } p\equiv 2 \pmod 3.$$

Apéry polynomials

Define Apéry polynomials by

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \ (n = 0, 1, 2, \ldots).$$

Z. W. Sun [J. Number Theory 132(2012)]. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}.$$

Also, for any p-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}.$$

An auxiliary theorem

Theorem (Z. W. Sun [J. N. Number Theory 132(2012)]) Let p be an odd prime and let x be any p-adic integer.
(i) If $x \equiv 2k \pmod{p}$ with $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.$$

(ii) If $x \equiv k \pmod{p}$ with $k \in \{0, \ldots, p-1\}$, then

$$\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}.$$

It is interesting to compare parts (i)-(ii) with the known identities

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

A general conjecture on supercongruences

Conjecture (Z. W. Sun). Let p be an odd prime and let n be a positive integer. Suppose that x is a p-adic integer with $x \equiv -2k \pmod{p}$ for some $k \in \{1, \ldots, \lfloor (p+1)/(2n+1) \rfloor\}$. Then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}.$$

We proved this for $n = 1$ via the Zeilberger algorithm (see Z. W. Sun [J. Number Theory 132(2012)]).

A mod p^3 conjecture

Conjecture (Z. W. Sun [J. Number Theory 132(2012)]). Let $p>3$ be a prime. If $p\equiv 1 \pmod 3$, then

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}.$$

If $p\equiv 1,3 \pmod 8$, then

$$\sum_{k=0}^{p-1} A_k \equiv \sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}.$$

Remark. It was conjectured by Rodriguez-Villegas and proved by E. Mortenson and Z. W. Sun that for any odd prime p we have

$$\frac{\binom{4k}{k,k,k,k}}{256^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \ \&\ p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Arithmetic means involving Apéry numbers

Theorem. Let n be a positive integer.

(i) (Z. W. Sun [J. Number Theory 132(2012)]) We have

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}.$$

For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.
(ii) (Conjectured by Z. W. Sun and proved by V.J.W. Guo and J. Zeng)

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \equiv 0 \pmod{n}.$$

Connection between $p = x^2 + 3y^2$ and Franel numbers

Z.W. Sun [J. Number Theory 133(2013)]: Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Conjecture (Z. W. Sun): For any prime $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$, we have

$$x \equiv \frac{1}{4}\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k} \equiv \frac{1}{2}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} \pmod{p^2}.$$

Two auxiliary identities

MacMahon's Identity:

$$\sum_{k=0}^{n} \binom{n}{k}^3 z^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{3k} \binom{2k}{k} \binom{3k}{k} z^k (1+z)^{n-2k}.$$

In particular,

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{3k} \binom{2k}{k} \binom{3k}{k} 2^{n-2k}.$$

A New Identity:

$$f_n = \sum_{k=0}^{n} \binom{n+2k}{3k} \binom{2k}{k} \binom{3k}{k} (-4)^{n-k}.$$

This can be proved by obtaining the recurrence relation for the right-hand side via the Zeilberger algorithm.

Three more identities needed

Shi-Chieh Chu's Identity:

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}.$$

A Simple Identity:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{3k+1} = \prod_{k=1}^{n} \frac{3k}{3k+1}.$$

An Identity for Combinatorial Sums: If $p > 3$ is odd, then

$$\sum_{k \equiv r \pmod 6} \binom{p}{k} = \frac{2^{p-1}-1}{3} + \frac{\delta_r}{2}\left((-1)^{\lfloor (p+1-2r)/6 \rfloor} 3^{(p-1)/2} + 1\right),$$

where δ_r takes 1 or 0 according as $3 \nmid p + r$ or not.

Some congruences needed

Lemma 1. Let $p>3$ be a prime and let $\varepsilon=(\frac{p}{3})$. Then

$$\sum_{k=1}^{(p-\varepsilon)/3}\frac{1}{2k-1}\equiv-\frac{3}{4}q_p(3)\pmod{p}$$

and

$$\binom{2(p-\varepsilon)/3}{(p-\varepsilon)/3}2^{-2(p-\varepsilon)/3}$$
$$\equiv\frac{1}{2-\varepsilon}\binom{(p-\varepsilon)/2}{(p-\varepsilon)/3}\left(1-\frac{3}{4}p\,q_p(3)\right)\pmod{p^2}.$$

Lemma 2. Let $p\equiv 1\pmod{3}$ be a prime. Then

$$\binom{p+2(p-1)/3}{(p-1)/3}\equiv\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{1}{3k-1}\equiv-\frac{2}{3}q_p(2)\pmod{p}.$$

Conjecture involving $g_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$

Recall that $\sum_{k=0}^n \binom{n}{k} f_k = g_n$ where $g_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$.

Conjecture (Z. W. Sun): Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$x \equiv \sum_{k=0}^{p-1} (k+1) \frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} (k+1) \frac{g_k}{(-3)^k} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$2 \sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv -\sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

On $\sum_{k=0}^{p-1} g_k/(\pm 3)^k$ modulo p

Let m be 3 or -3. Then $m-1 \in \{2,-4\}$. Observe that

$$\sum_{n=0}^{p-1}\frac{g_n}{m^n} = \sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{n=k}^{p-1}\binom{n}{k}\frac{1}{m^{n-k}}$$

$$= \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{k+j}{j}\frac{1}{m^j} = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{-k-1}{j}\frac{1}{(-m)^j}$$

$$\equiv \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{p-1-k}{j}\left(-\frac{1}{m}\right)^j = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(1-\frac{1}{m}\right)^{p-1-k}$$

$$\equiv \sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(\frac{m}{m-1}\right)^k = \sum_{k=0}^{p-1}\frac{f_k}{(m-1)^k} \pmod p.$$

So we obtain $\sum_{k=0}^{p-1} g_k/(\pm 3)^k$ modulo p since

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \equiv \begin{cases} 2x \pmod p & \text{if } p = x^2+3y^2\ (3 \mid x-1),\\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3).\end{cases}$$

Conjectures involving $\sum_{k=0}^n \binom{n}{k}^4 x^k$

In 2011 the author introduced the polynomials $S_n(x) = \sum_{k=0}^n \binom{n}{k}^4 x^k$ $(n = 0, 1, 2, \ldots)$ and posed 13 related conjectures. Here is one of them.

Conjecture (Z. W. Sun) For any prime $p > 2$ we have

$$\sum_{n=0}^{p-1} S_n(12)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \ \&\ p = x^2 + y^2\ (3 \nmid x), \\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \ \&\ p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\sum_{k=0}^{p-1}(4k+3)S_k(12) \equiv p\left(1 + 2\left(\frac{3}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)S_k(12) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjectures involving $\sum_{k=0}^{n}\binom{n}{k}^4x^k$

Here is another conjecture.

Conjecture (Z. W. Sun). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(-20)$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+y^2\ (5\nmid x),\\ 4xy\pmod{p^2} & \text{if } p\equiv 13,17\,(\mathrm{mod} 20),\ p=x^2+y^2\ (5\mid x-y)\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod{4}.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(6k+5)S_k(-20)\equiv p\left(\frac{-1}{p}\right)\left(2+3\left(\frac{-5}{p}\right)\right)\pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)S_k(-20)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

Part III. More conjectures on Apéry numbers and Franel numbers

A conjecture on f_n and g_n

Conjecture [Z. W. Sun, JNT 133(2013)]. For each $n = 1, 2, 3, \ldots$,

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k \in \mathbb{Z} \quad \text{and} \quad \frac{1}{n^2}\sum_{k=0}^{n-1}(4k+1)g_k 9^{n-1-k} \in \mathbb{Z}.$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k \equiv 2p^2(2^p-1)^2 \pmod{p^5},$$

$$\sum_{k=0}^{p-1}(4k+1)\frac{g_k}{9^k} \equiv \frac{p^2}{2}\left(3-\left(\frac{p}{3}\right)\right) - p^2(3^p-3) \pmod{p^4}.$$

Remark. The part for Franel numbers has been confirmed by V. J. W. Guo.

More conjectures for f_n and g_n

Conjecture (Z. W. Sun) (i) For any integer $n > 1$, we have

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{(n-1)n^2},$$

$$\sum_{k=0}^{n-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv 0 \pmod{4(n-1)n^3},$$

$$\sum_{k=0}^{n-1}(12k^3+34k^2+30k+9)g_k \equiv 0 \pmod{3n^3}.$$

(ii) For each odd prime p we have

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1)-16p^3q_p(2) \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv -4p^3 \pmod{p^4}.$$

More conjectures for Apéry numbers

Conjecture (Z. W. Sun) (i) For any positive integer n, we have

$$\sum_{k=0}^{n-1}(6k^3+9k^2+5k+1)(-1)^kA_k \equiv 0 \pmod{n^3},$$

$$\sum_{k=0}^{n-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k \equiv 0 \pmod{n^4}.$$

(ii) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}(6k^3+9k^2+5k+1)A_k \equiv p^3+2p^4H_{p-1}-\frac{2}{5}p^8B_{p-5} \pmod{p^9},$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. If $p>5$, then

$$\sum_{k=0}^{p-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k$$

$$\equiv -2p^4+3p^5+(6p-8)p^5H_{p-1}-\frac{12}{5}p^9B_{p-5} \pmod{p^{10}}.$$

A conjecture involving 3-adic valuations

For a rational number a/b, its 3-adic valuation (or 3-adic order) $\nu_3(a/b)$ is defined as $\nu_3(a) - \nu_3(b)$, where $\nu_3(m) := \sup\{n \in \mathbb{N} : \ p^n \mid m\}$ for any nonzero integer m.

Conjecture (Z. W. Sun) Let n be any positive integer. Then

$$\nu_3\left(\sum_{k=0}^{n-1}(-1)^k f_k\right) \geqslant 2\nu_3(n), \quad \nu_3\left(\sum_{k=0}^{n-1}(-1)^k kf_k\right) \geqslant 2\nu_3(n),$$

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k\right) = 3\nu_3(n) \leqslant \nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right).$$

If n is a positive multiple of 3, then

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right) = 3\nu_3(n) + 2.$$

Thank you!