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Combinatorial congruences via the Zeilberger algorithm

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Abstract

The well known Zeilberger algorithm is an effective tool for proving combinatorial identities. In this talk we will show how to use Zeilberger's algorithm to deduce some combinatorial congruences. In particular, we will exhibit the author's recent solution to the remaining open cases of Rodriguez-Villegas' conjectured congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\ \ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\ \ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2, where p is a prime greater than 3.

PS: For detailed materials related to this talk, the reader may download some of my preprints from my homepage `http://math.nju.edu.cn/∼zwsun`

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $(\frac{a}{p})$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

Conjectures of Rodriguez-Villegas

In 2001 Rodriguez-Villegas conjectured 22 congruences which relate truncated hypergeometric series to the number of $\mathbb{F}_p$-points of some family of Calabi-Yau manifolds. Here are some of them:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 \pmod{p^2}.$$

Progress on the above 5 congruences

All the above 5 congruences have been proved.

The last one was first established by T. Ishikawa [Nagoya Math. J. 118(1990)].

The 4 other congruences were proved by E. Mortenson [J. Number Theory 2003; Trans. Amer. Math. Soc. [355(2003)] via an advanced approach involving Gauss and Jacobi sums, p-adic Gamma functions and hypergeometric series.

Some basic facts

Let $p = 2n+1$ be an odd prime. If $n < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

An observation of van Hammer: For $k = 0, \ldots, n$ we have

$$\begin{aligned}
&\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k} \\
=&\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k} \\
=&\frac{\prod_{j=0}^{k-1}(\frac{p-1}{2}-j)(\frac{-p-1}{2}-j)}{(k!)^2} = \prod_{j=0}^{k-1}\frac{(-\frac{1}{2}-j)^2-\frac{p^2}{4}}{(k!)^2} \\
\equiv&\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}
\end{aligned}$$

An elementary proof of $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv \left(\frac{-1}{p}\right) \pmod{p^2}$

Recall the known combinatorial identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k=(-1)^n.$$

In fact,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k=\sum_{k=0}^{n}\binom{n}{n-k}\binom{-n-1}{k}$$

$$=[x^n](1+x)^n(1+x)^{-n-1}=[x^n]\frac{1}{1+x}=(-1)^n.$$

Both Zhi-Hong Sun and Roberto Tauraso [2009] observed that if $p=2n+1$ is an odd prime then

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k=(-1)^n$$

$$\Rightarrow\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^n=\left(\frac{-1}{p}\right)\pmod{p^2}.$$

F. Klein and R. Fricke (1892):

$$a(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6$$

$$= \begin{cases} 4x^2-2p & \text{if } p \equiv 1 \pmod 4 \ \&\ p = x^2+y^2 \text{ with } 2 \nmid x, \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

J. Stienstra and F. Beukers [Math. Ann. 27(1985)]:

$$c(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$

$$= \begin{cases} 4x^2-2p & \text{if } p \equiv 1,3 \pmod 8 \ \&\ p = x^2+2y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 5,7 \pmod 8; \end{cases}$$

$$b(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3$$

$$= \begin{cases} 4x^2-2p & \text{if } p \equiv 1 \pmod 3 \ \&\ p = x^2+3y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Three conjectures of Rodriguez-Villegas

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv\left(\frac{p}{3}\right)a(p)\pmod{p^2}.$$

Via an advanced approach Mortenson [2005] provided a partial solution with the following things open:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)=0\pmod{p^2}\quad\text{if } p\equiv 5\pmod{6},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2}\quad\text{if } p\equiv 3\pmod{4},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv -a(p)\pmod{p^2}\quad\text{if } p\equiv 5\pmod{6}.$$

Confirming the remaining parts

Theorem 1 (Z. W. Sun, arxiv:1012.3141). Let $p > 3$ be a prime.
(i) For each $d = 0, \ldots, (p-1)/2$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{4k}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5,7 \pmod 8,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod 4.$$

Confirming the remaining parts

(ii) When $p \equiv 3 \pmod 8$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv 4x^2 - 2p \pmod{p^2};$$

Also, when $p \equiv 5 \pmod{12}$ and $p = x^2 + y^2$ with $2 \nmid x$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \equiv 2p - 4x^2 \pmod{p^2}.$$

Find recurrence relations via Zeilberger's algorithm

A hypergeometric sum has the form $u_n = \sum_{k \geqslant 0} f(n,k)$ with both $f(n,k+1)/f(n,k)$ and $f(n+1,k)/f(n,k)$ rational functions in n and k.

Zeilberger's algorithm of telescoping provides a linear recurrence relation for u_n with the recurrent coefficients polynomials in n.

An Example. The Apéry numbers are given by

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n = 0, 1, 2, \ldots).$$

By the Zeilberger algorithm via `Mathematica`, we find the recurrence relation

$$(n+1)^3 A_n - (2n+3)(17n^2 + 51n + 39)A_{n+1} + (n+2)^3 A_{n+2} = 0.$$

Detailed proof of the first congruence in Theorem 1

For $d = 0, 1, 2, \ldots$, we define

$$f(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k}.$$

By the Zeilberger algorithm, we find the recursive relation:

$$\begin{aligned}&(3d+1)(6d+1)f(d) - (3d+2)(6d+5)f(d+1)\\ =&\frac{(3p-1)(3p-2)(2d+1)}{2^{2p-1}27^{p-1}p}\binom{2p}{p+2d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1}.\end{aligned}$$

Note that

$$\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod p$$

and

$$(3p-2)\binom{3p-3}{p-1} = p\binom{3p-2}{p} \equiv 0 \pmod p.$$

Detailed proof of the first congruence in Theorem 1

If $d < (p-1)/2$, then

$$\binom{2p}{p+2d+1} = \frac{2p}{p-1-2d}\binom{2p-1}{p-2-2d} \equiv 0 \pmod{p}.$$

Thus, for each $d = 0, \ldots, (p-1)/2$ we have

$$(3d+1)(6d+1)f(d) \equiv (3d+2)(6d+5)f(d+1) \pmod{p^2}.$$

Suppose that $p \equiv 5 \pmod 6$ and $0 \leqslant d < (p-1)/2$. Then $3d+1, 6d+1 \neq p, 2p$ and hence $3d+1, 6d+1 \not\equiv 0 \pmod p$. So

$$f(d+1) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.$$

Therefore

$$f\left(\frac{p-1}{2}\right) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}$$

$$\implies f(0) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

Detailed proof of the first congruence in Theorem 1

Note that

$$
\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{2k}{k+p-1}\binom{2k}{k}\binom{3k}{k}}{108^k} \\
=&\frac{\binom{2p-2}{p-1}\binom{3p-3}{p-1}}{108^{p-1}} \\
=&108^{1-p}\frac{p}{2p-1}\binom{2p-1}{p}\frac{p}{3p-2}\binom{3p-2}{p} \\
\equiv&0 \pmod{p^2}.
\end{aligned}
$$

This concludes the proof of

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2}.$$

Gauss' congruence and its refinement

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

An elementary proof of a known result

It is known that for any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

Zhi-Hong Sun found an elementary proof of this result. Write $p=2n+1$ and recall Bell's identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}2^{2n-2k}=\begin{cases}\binom{n}{n/2}^2 & \text{if } 2\mid n,\\ 0 & \text{if } 2\nmid n.\end{cases}$$

Thus

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{16^k}\binom{2k}{k}\frac{2^{p-1}}{4^k}\equiv\begin{cases}\binom{(p-1)/2}{(p-1)/4}^2 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

And the desired result follows.

A new result via the Zeilberger algorithm

Theorem 2 (Z. W. Sun, `arxiv:1012.3141`) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0 \pmod{p^2}$$

for all $d\in\{0,\ldots,p-1\}$ with $d\equiv (p+1)/2 \pmod 2$.

We prove the theorem via the Zeilberger algorithm.

For $d=0,1,2,\ldots$ set

$$u_d=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}=\sum_{d\leqslant k<p}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2u_d-(2d+3)^2u_{d+2}=\frac{(2p-1)^2(d+1)}{64^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1}=pC_{p-1}\equiv 0 \pmod p,\ \ p\Big|(d+1)\binom{2p}{p+d+1}\ (d<p).$$

A new result via the Zeilberger algorithm

So, if $0 \leqslant d < p-2$ then

$$(2d+1)^2 u_d \equiv (2d+3)^2 u_{d+2} \pmod{p^2}.$$

For $d \in \{0, \ldots, p-3\}$ with $d \equiv (p+1)/2 \pmod 2$, clearly $p \neq 2d+1 < 2p$ and hence

$$u_{d+2} \equiv 0 \pmod{p^2} \implies u_d \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$ then $p-1 \equiv (p+1)/2 \pmod 2$;
if $p \equiv 1 \pmod 4$ then $p-2 \equiv (p+1)/2 \pmod 2$.
Thus, if $d \in \{p-1, p-2\}$ and $d \equiv (p+1)/2 \pmod 2$, then $d \geqslant (p+1)/2$ and hence $u_d \equiv 0 \pmod{p^2}$.

It follows that $u_d \equiv 0 \pmod{p^2}$ for all $d \in \{0, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$.

One more theorem via the Zeilberger algorithm

Theorem 3 (Z. W. Sun, `arXiv:1101.1946`). Let p be an odd prime and let x be any p-adic integer.

(i) If $x \equiv 2k \pmod p$ with $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^2 \equiv (-1)^k\binom{x}{k} \pmod{p^2}.$$

(ii) If $x \equiv k \pmod p$ with $k \in \{0, \ldots, p-1\}$, then

$$\sum_{r=0}^{p-1}\binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}.$$

(iii) If $p > 3$ and $x \equiv -2k \pmod p$ for some $k \in \{1, \ldots, \lfloor p/3 \rfloor\}$. Then we have

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^3 \equiv 0 \pmod{p^2}.$$

A corollary

Corollary (Z. W. Sun, 2011) Let $a_n := \sum_{k=0}^{n}\binom{n}{k}^2 C_k$ for $n = 0, 1, 2, \ldots$. Then, for any odd prime p we have

$$a_1 + \cdots + a_{p-1} \equiv 0 \pmod{p^2}.$$

Proof of the Corollary. Observe that

$$\sum_{n=0}^{p-1} a_n = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1}\binom{n}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1-k}\binom{k+j}{k}^2.$$

If $0 \leqslant k \leqslant p-1$ and $p-k \leqslant j \leqslant p-1$, then

$$\binom{k+j}{k} = \frac{(k+j)!}{k!j!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1}\binom{k+j}{k}^2 = \sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\binom{x_k}{j}^2,$$

where $x_k = -k-1 \equiv p-1-k \pmod{p}$.

Proof of the Corollary

Applying Theorem 3(ii) we get

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \binom{2x_k}{p-1-k} = \sum_{k=0}^{p-1} (-1)^k \binom{p+k}{2k+1} C_k = 1 \pmod{p^2}.$$

since we can easily prove the identity

$$\sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k = 1$$

by induction on n.

Remark. We find no prime $p \leqslant 5,000$ with $\sum_{k=1}^{p-1} a_k \equiv 0 \pmod{p^3}$ and no composite number $n \leqslant 70,000$ satisfying $\sum_{k=1}^{n-1} a_k \equiv 0 \pmod{n^2}$. We conjecture that there is no composite number n such that

$$a_1 + \cdots + a_{n-1} \equiv 0 \pmod{n^2}.$$

Some conjectures of mine

I have hundreds of conjectures (cf. my preprints posted to arXiv).

Conjecture 1 (Z. W. Sun, 2011) Let p be an odd prime and let $n \geqslant 2$ be an integer. If x is a p-adic integer with $x \equiv -2k \pmod{p}$ for some $k \in \{1, \ldots, \lfloor (p+1)/(2n+1) \rfloor\}$, then

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}.$$

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$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}.$$

Conjecture 2 (Z. W. Sun, Nov. 13, 2009). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \&\ p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

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Conjecture 3 (Z. W. Sun, Jan. 2, 2011). We have

$$\sum_{k=0}^{\infty} \frac{30k+7}{(-256)^k} \binom{2k}{k}^2 a_k = \frac{24}{\pi},$$

where a_k is the coefficient of x^k in $(x^2 + x + 16)^k$.

Thank you!