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ON SOME DETERMINANTS WITH LEGENDRE SYMBOL ENTRIES

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Abstract. In this paper we mainly focus on some determinants with Legendre symbol entries. For an odd prime p and an integer d, let $S(d,p)$ denote the determinant of the $(p-1)/2\times(p-1)/2$ matrix whose (i,j)-entry $(1\leqslant i,j\leqslant (p-1)/2)$ is the Legendre symbol $(\frac{i^2+dj^2}{p})$. We investigate properties of $S(d,p)$ as well as some other determinants involving Legendre symbols. In Section 3 we pose 17 open conjectures on determinants one of which states that $(\frac{-S(d,p)}{p})=1$ if $(\frac{d}{p})=1$, and $S(d,p)=0$ if $(\frac{d}{p})=-1$. This material might interest some readers and stimulate further research.

1. Introduction

For an $n\times n$ matrix $A=(a_{ij})_{1\leqslant i,j\leqslant n}$ over the field of complex numbers, we often write $\det A$ in the form $|a_{ij}|_{1\leqslant i,j\leqslant n}$. In this paper we study determinants with Legendre symbol entries.

Let p be an odd prime and let $(\frac{\cdot}{p})$ be the Legendre symbol. The circulant determinant

$$\left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=\begin{vmatrix}(\frac{0}{p}) & (\frac{1}{p}) & (\frac{2}{p}) & \ldots & (\frac{p-1}{p})\\ (\frac{p-1}{p}) & (\frac{0}{p}) & (\frac{1}{p}) & \ldots & (\frac{p-2}{p})\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (\frac{1}{p}) & (\frac{2}{p}) & (\frac{3}{p}) & \ldots & (\frac{0}{p})\end{vmatrix}$$

takes the value

$$\prod_{r=0}^{p-1}\sum_{k=0}^{p-1}\left(\frac{k}{p}\right)\left(e^{2\pi ir/p}\right)^k=0$$

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since $\sum_{k=0}^{p-1}(\frac{k}{p})=0$. (See [K99, (2.41)] for the evaluation of a general circulant determinant.) For the matrix $A=(a_{ij})_{1\leqslant i,j\leqslant p-1}$ with $a_{ij}=(\frac{i-j}{p})$, L. Carlitz [C59, Theorem 4] proved that its characteristic polynomial is

$$|xI_{p-1}-A|=\left(x^2-\left(\frac{-1}{p}\right)p\right)^{(p-3)/2}\left(x^2-\left(\frac{-1}{p}\right)\right),$$

where I_{p-1} is the $(p-1)\times(p-1)$ identity matrix. Putting $x=0$ in Carlitz's formula, we obtain that

$$(-1)^{p-1}|A|=\left(-\left(\frac{-1}{p}\right)\right)^{(p-1)/2}p^{(p-3)/2}=p^{(p-3)/2}.$$

For $m\in\mathbb{Z}$ let $\{m\}_p$ denote the least nonnegative residue of an integer m modulo p. For any integer $a\not\equiv 0 \pmod p$, $\{aj\}_p$ $(j=1,\ldots,p-1)$ is a permutation of $1,\ldots,p-1$, and its sign is the Legendre symbol $(\frac{a}{p})$ by Zolotarev's theorem (cf. [DH] and [Z]). Therefore, for any integer $d\not\equiv 0 \pmod p$ we have

$$\left|\left(\frac{i+dj}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)\left|\left(\frac{i-j}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=0 \tag{1.1}$$

and

$$\left|\left(\frac{i+dj}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)\left|\left(\frac{i-j}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)p^{(p-3)/2}. \tag{1.2}$$

Let p be an odd prime. In 2004, R. Chapman [Ch04] used quadratic Gauss sums to determine the values of

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\quad\text{and}\quad\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}.$$

Since $(p+1)/2-i+(p+1)/2-j-1\equiv-(i+j) \pmod p$, we see that

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}=\left(\frac{-1}{p}\right)\left|\left(\frac{i+j}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}$$

and

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}=\left|\left(\frac{i+j}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}.$$

Chapman [Ch12] also conjectured that

$$\left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}=\begin{cases}-r_p & \text{if } p\equiv 1 \pmod 4,\\ 1 & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{1.3}$$

where $\varepsilon_p^{(2-(\frac{2}{p}))h(p)} = r_p + s_p\sqrt{p}$ with $r_p, s_p \in \mathbb{Z}$. (Throughout this paper, ε_p and $h(p)$ stand for the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$ respectively.) As Chapman could not solve this problem for several years, he called the determinant *evil* (cf. [Ch12]). Chapman's conjecture on his "evil" determinant was recently confirmed by M. Vsemirnov [V12, V13] via matrix decomposition.

Let $p \equiv 1 \pmod 4$ be a prime. In an unpublic manuscript written in 2003 Chapman [Ch03] conjectured that

$$s_p = \left(\frac{2}{p}\right)\left|\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}. \tag{1.4}$$

Note that (1.3) and (1.4) together yield an interesting identity

$$\varepsilon_p^{(2-(\frac{2}{p}))h(p)} = \left(\frac{2}{p}\right)\left|\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\sqrt{p} - \left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}.$$

Taking the norm with respect to the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$, we are led to the identity

$$\left|\left(\frac{j-i}{p}\right)\right|^2_{0\leqslant i,j\leqslant (p-1)/2} - p\left|\left(\frac{j-i}{p}\right)\right|^2_{1\leqslant i,j\leqslant (p-1)/2} = (-1)^{h(p)}$$

since $N(\varepsilon_p) = -1$ (cf. Theorem 3 of [Co62, p. 185]). This provides an *explicit* solution to the diophantine equation $x^2 - py^2 = (-1)^{h(p)}$.

Now we state our first theorem.

Theorem 1.1. *Let p be an odd prime. For $d \in \mathbb{Z}$ define*

$$R(d,p) := \left|\left(\frac{i+dj}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}. \tag{1.5}$$

If $p \equiv 1 \pmod 4$, then

$$R(d,p) \equiv \left(\left(\frac{d}{p}\right)d\right)^{(p-1)/4}\ \frac{p-1}{2}! \pmod p. \tag{1.6}$$

When $p \equiv 3 \pmod 4$, we have

$$R(d,p) \equiv \begin{cases} (\frac{2}{p}) \pmod p & \text{if } (\frac{d}{p}) = 1,\\ 1 \pmod p & \text{if } (\frac{d}{p}) = -1.\end{cases} \tag{1.7}$$

Also,

$$R(-d,p) \equiv \left(\frac{2}{p}\right) R(d,p) \pmod p, \tag{1.8}$$

and

$$\left|\left(\frac{i+dj+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv R(d,p)\pmod p\quad \text{for all } c\in\mathbb{Z}. \tag{1.9}$$

Remark 1.1. Let p be any odd prime. By Wilson's theorem,

$$\left(\frac{p-1}{2}!\right)^2\equiv\begin{cases}-1\pmod p & \text{if } p\equiv 1\pmod 4,\\ 1\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.10}$$

Corollary 1.1. *Let $p\equiv 1\pmod 4$ be a prime, and write $\varepsilon_p^{h(p)}=a_p+b_p\sqrt{p}$ with $a_p,b_p\in\mathbb{Q}$, where ε_p and $h(p)$ are the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$. Then we have*

$$a_p\equiv-\frac{p-1}{2}!\pmod p\quad \text{and } h(p)\equiv 1\pmod 2. \tag{1.11}$$

Proof. By (1.6) we have

$$R(1,p)\equiv\frac{p-1}{2}!\pmod p.$$

On the other hand, Chapman [Ch04, Corollary 3] proved that

$$\left|\left(\frac{i+j}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}=\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}=-\left(\frac{2}{p}\right)2^{(p-1)/2}a_p.$$

So we have the first congruence in (1.11). Taking norms (with respect to the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$) of both sides of the identity $\varepsilon_p^{h(p)}=a_p+b_p\sqrt{p}$, we obtain

$$N(\varepsilon)^{h(p)}=a_p^2-pb_p^2.$$

Since

$$a_p^2\equiv\left(\frac{p-1}{2}!\right)^2\equiv-1\pmod p,$$

we must have $N(\varepsilon)=-1$ and $2\nmid h(p)$. This proves the second congruence in (1.11). □

It is well known that for any odd prime p the $(p-1)/2$ squares

$$1^2,2^2,\ldots,\left(\frac{p-1}{2}\right)^2$$

give all the $(p-1)/2$ quadratic residues modulo p. So we think that it's natural to consider some Legendre symbol determinants involving binary quadratic forms.

Theorem 1.2. *Let p be any odd prime. For $d\in\mathbb{Z}$ define*

$$S(d,p):=\left|\left(\frac{i^2+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2} \tag{1.12}$$

and

$$T(d,p):=\left|\left(\frac{i^2+dj^2}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}. \tag{1.13}$$

(i) *For any $c\in\mathbb{Z}$ with $p\nmid c$, we have*

$$S(c^2d,p)=\left(\frac{c}{p}\right)^{(p+1)/2}S(d,p)\ \ \text{and}\ \ T(c^2d,p)=\left(\frac{c}{p}\right)^{(p+1)/2}T(d,p). \tag{1.14}$$

If $p\equiv 1\pmod 4$, then

$$S(-d,p)=\left(\frac{2}{p}\right)S(d,p)\quad\text{and}\quad T(-d,p)=\left(\frac{2}{p}\right)T(d,p). \tag{1.15}$$

When $p\equiv 3\pmod 4$, we have

$$\left(\frac{d}{p}\right)=-1\Longrightarrow S(d,p)=0. \tag{1.16}$$

(ii) *We have*

$$\left(\frac{T(d,p)}{p}\right)=\begin{cases}(\frac{2}{p}) & \text{if } (\frac{d}{p})=1,\\ 1 & \text{if } (\frac{d}{p})=-1.\end{cases} \tag{1.17}$$

Also,

$$T(-d,p)\equiv\left(\frac{2}{p}\right)T(d,p)\pmod p \tag{1.18}$$

and

$$\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv T(d,p)\pmod p\quad\text{for all } c\in\mathbb{Z}. \tag{1.19}$$

Example 1.1. Note that

$$S(1,11)=\begin{vmatrix}-1&1&-1&-1&1\\1&-1&-1&1&-1\\-1&-1&-1&1&1\\-1&1&1&-1&-1\\1&-1&1&-1&-1\end{vmatrix}=-16$$

and

$$S(2,13)=\begin{vmatrix}1&1&-1&-1&1&-1\\-1&1&1&1&-1&-1\\-1&1&1&-1&-1&1\\-1&-1&-1&1&1&1\\1&-1&1&-1&1&-1\\1&-1&-1&1&-1&1\end{vmatrix}=0.$$

Now we present our third theorem.

Theorem 1.3. (i) *For any odd prime p, we have*

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv\begin{cases}(\frac{2}{p})\pmod p & \text{if } p\equiv 1\pmod 4,\\ ((p-1)/2)!\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{1.20}$$

(ii) *Let $p\equiv 3\pmod 4$ be a prime. Then*

$$\left|\frac{1}{i^2+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv\left(\frac{2}{p}\right)\pmod p.\tag{1.21}$$

We are going to prove Theorems 1.1-1.3 in the next section, and pose over ten new conjectures on determinants in Section 3.

2. Proof of Theorems 1.1-1.3

Lemma 2.1 ([K05, Lemma 9])**.** *Let $P(z)=\sum_{k=0}^{n-1}a_kz^k$ be a polynomial with complex number coefficients. Then we have*

$$|P(x_i+y_j)|_{1\leqslant i,j\leqslant n}=a_{n-1}^n\prod_{k=0}^{n-1}\binom{n-1}{k}\times\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_j-y_i).\tag{2.1}$$

Proof of Theorem 1.1. Set $n=(p-1)/2$. For any $c\in\mathbb{Z}$, we have

$$\left|\left(\frac{i+dj+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv|(i+dj+c)^n|_{0\leqslant i,j\leqslant n}\pmod p.$$

In light of Lemma 2.2,

$$\begin{aligned}&|(i+dj+c)^n|_{0\leqslant i,j\leqslant n}=|(i+dj+c-d-1)^n|_{1\leqslant i,j\leqslant n+1}\\ =&\prod_{k=0}^{n}\binom{n}{k}\times\prod_{1\leqslant i<j\leqslant n+1}(i-j)(dj+c-d-1-(di+c-d-1))\\ =&\frac{(n!)^{n+1}}{\prod_{k=0}^n k!(n-k)!}(-d)^{n(n+1)/2}\prod_{1\leqslant i<j\leqslant n+1}(j-i)^2=(-d)^{n(n+1)/2}(n!)^{n+1}.\end{aligned}$$

Therefore (1.9) holds, and also

$$R(d,p)\equiv(-d)^{(p^2-1)/8}\left(\frac{p-1}{2}!\right)^{(p+1)/2}\pmod p.\tag{2.2}$$

In the case $p \equiv 1 \pmod 4$, from (2.2) we obtain

$$\begin{aligned} R(d,p) \equiv& (-d)^{(p-1)/4\ (p+1)/2}\ \frac{p-1}{2}!\left(\frac{p-1}{2}!\right)^{2(p-1)/4} \\ \equiv& \left(d^{(p+1)/2}\right)^{(p-1)/4}\ \frac{p-1}{2}! \equiv \left(\left(\frac{d}{p}\right)d\right)^{(p-1)/4}\ \frac{p-1}{2}! \pmod p. \end{aligned}$$

In the case $p \equiv 3 \pmod 4$, (2.2) yields

$$R(d,p) \equiv (-d)^{(p-1)/2\times(p+1)/4}\left(\frac{p-1}{2}!\right)^{2(p+1)/4} \equiv \left(\frac{-d}{p}\right)^{(p+1)/4} \pmod p$$

and hence (1.7) follows.

Now it remains to show (1.8). If $p \equiv 1 \pmod 4$, then by (1.6)

$$R(-d,p) \equiv \left(\left(\frac{-d}{p}\right)(-d)\right)^{(p-1)/4}\ \frac{p-1}{2}! \equiv \left(\frac{2}{p}\right)R(d,p) \pmod p.$$

If $p \equiv 3 \pmod 4$, then $(\frac{-d}{p}) = -(\frac{d}{p})$ and hence we get (1.8) from (1.7).

The proof of Theorem 1.1 is now complete. □

Lemma 2.2. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\left(\frac{((p-1)/2)!}{p}\right) = \left(\frac{2}{p}\right). \tag{2.3}$$

Proof. Since

$$(-4)^{(p-1)/4} = (-1)^{(p-1)/4}2^{(p-1)/2} = \left(\frac{2}{p}\right)2^{(p-1)/2} \equiv 1 \pmod p,$$

for some $x \in \mathbb{Z}$ we have

$$x^4 \equiv -4 \equiv 4\left(\frac{p-1}{2}!\right)^2 \pmod p, \quad \text{i.e., } x^2 \equiv \pm 2\times\frac{p-1}{2}! \pmod p.$$

Therefore (2.3) holds. □

Proof of Theorem 1.2(i). Let $c \in \mathbb{Z}$ with $p \nmid c$. For each $j = 1,\ldots,(p-1)/2$ let $\sigma_c(j)$ be the unique $r \in \{1,\ldots,(p-1)/2\}$ such that $cj \equiv r$ or $-r \pmod p$. By a result of H. Pan [P06], the sign of the permutation σ_c equals $(\frac{c}{p})^{(p+1)/2}$. Thus

$$S(c^2d,p) = \left|\left(\frac{i^2+d\sigma_c(j)^2}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2} = \left(\frac{c}{p}\right)^{(p+1)/2}S(d,p).$$

Similarly the second equality in (1.14) also holds.

Now we handle the case $p\equiv 1 \pmod 4$. As $((p-1)/2)!^2\equiv -1 \pmod p$, by applying (1.14) with $c=((p-1)/2)!$ and using (2.3) we immediately get (1.15).

Below we assume that $p\equiv 3 \pmod 4$. As the transpose of $S(-1,p)$ equals $(\frac{-1}{p})^{(p-1)/2}S(-1,p)=-S(-1,p)$, we have $S(-1,p)=0$. If $(\frac{d}{p})=-1$, then $d\equiv -c^2 \pmod p$ for some integer $c\not\equiv 0 \pmod p$, and hence

$$S(d,p)=S(-c^2,p)=\left(\frac{c}{p}\right)^{(p+1)/2}S(-1,p)=0.$$

This proves (1.16).

So far we have proved the first part of Theorem 1.2. $\square$

Proof of Theorem 1.2(ii). Set $n=(p-1)/2$. For any $c\in\mathbb{Z}$, we have

$$\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv |(i^2+dj^2+c)^n|_{0\leqslant i,j\leqslant n} \pmod p.$$

In light of Lemma 2.2,

$$\begin{aligned}
&|(i^2+dj^2+c)^n|_{0\leqslant i,j\leqslant n}=|((i-1)^2+d(j-1)^2+c)^n|_{1\leqslant i,j\leqslant n+1}\\
=&\prod_{k=0}^n\binom{n}{k}\times\prod_{1\leqslant i<j\leqslant n+1}((i-1)^2-(j-1)^2)(d(j-1)^2+c-d(i-1)^2-c)\\
=&\frac{(n!)^{n+1}}{\prod_{k=0}^n k!(n-k)!}(-d)^{n(n+1)/2}\prod_{0\leqslant i<j\leqslant n}(j-i)^2(j+i)^2\\
=&(-d)^{n(n+1)/2}(n!)^{n+1}\prod_{0\leqslant i<j\leqslant n}(i+j)^2.
\end{aligned}$$

Therefore (1.19) holds, and also

$$\begin{aligned}
T(d,p)\equiv&(-d)^{(p^2-1)/8}\left(\frac{p-1}{2}!\right)^{(p+1)/2}\prod_{0\leqslant i<j\leqslant (p-1)/2}(i+j)^2\\
\equiv&R(d,p)\prod_{0\leqslant i<j\leqslant (p-1)/2}(i+j)^2 \pmod p
\end{aligned}$$

with the help of (2.2). Combining this with (1.8) we obtain (1.18). Note that

$$\left(\frac{T(d,p)}{p}\right)=\left(\frac{R(d,p)}{p}\right).$$

If $(\frac{d}{p})=1$, then by Theorem 1.1 we have

$$R(d,p)\equiv\begin{cases}d^{(p-1)/4}((p-1)/2)! \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (\frac{2}{p}) \pmod p & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

and hence $(\frac{R(d,p)}{p}) = (\frac{2}{p})$ with the help if Lemma 2.2. In the case $(\frac{d}{p}) = -1$, by Theorem 1.1 we have

$$R(d,p) \equiv \begin{cases} (-d)^{(p-1)/4}((p-1)/2)! \pmod p & \text{if } p \equiv 1 \pmod 4, \\ 1 \pmod p & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and hence $(\frac{R(d,p)}{p}) = 1$ with the help if Lemma 2.2. Therefore (1.17) also holds. We are done. □

Lemma 2.3. *Let p be any odd prime. For any $d \in \mathbb{Z}$ with $(\frac{d}{p}) = -1$, we have the new congruence*

$$\prod_{x=1}^{(p-1)/2} (x^2 - d) \equiv (-1)^{(p+1)/2}\, 2 \pmod p. \tag{2.4}$$

Proof. For any integer a, it is well known that

$$\begin{aligned} a^{(p-1)/2} \equiv 1 \pmod p \iff & \left(\frac{a}{p}\right) = 1 \\ \iff & a \equiv x^2 \pmod p \quad \text{for some } x = 1, \ldots, \frac{p-1}{2}. \end{aligned}$$

Therefore

$$\prod_{x=1}^{(p-1)/2} (z - x^2) \equiv z^{(p-1)/2} - 1 \pmod p$$

and hence

$$\prod_{x=1}^{(p-1)/2} (y + d - x^2) \equiv (y + d)^{(p-1)/2} - 1 \pmod p. \tag{2.5}$$

Comparing the constant terms of both sides of the congruence (2.5), we obtain

$$\prod_{x=1}^{(p-1)/2} (d - x^2) \equiv d^{(p-1)/2} - 1 \equiv -2 \pmod p$$

and hence (2.4 follows. □

Remark 2.1. Under the condition of Lemma 2.3, we could also prove the following congruences

$$\sum_{x=1}^{(p-1)/2} \frac{1}{x^2 - d} \equiv \frac{1}{4d} \pmod p \quad \text{and} \quad \sum_{x=1}^{(p-1)/2} \frac{1}{(x^2 - d)^2} \equiv -\frac{5}{16d^2} \pmod p \tag{2.6}$$

by comparing coefficients of y and y^2 in the congruence (2.5).

Proof of Theorem 1.3. (i) Set $n=(p-1)/2$. Clearly

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv |(i+j)^{n-1}|_{1\leqslant i,j\leqslant n}\pmod p.$$

By Lemma 2.1,

$$\begin{aligned}|(i+j)^{n-1}|_{1\leqslant i,j\leqslant n}=&\prod_{k=0}^{n-1}\binom{n-1}{k}\times\prod_{1\leqslant i<j\leqslant n}(i-j)(j-i)\\ =&\frac{(n-1)!^n}{\prod_{k=0}^{n-1}k!(n-1-k)!}(-1)^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}(j-i)^2\\ =&(-1)^{n(n-1)/2}(n-1)!^n\\ =&(-1)^{(p-1)(p-3)/8}\left(\frac{2}{p-1}\right)^{(p-1)/2}\left(\frac{p-1}{2}!\right)^{(p-1)/2}.\end{aligned}$$

Therefore

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv\left(\frac{p-1}{2}!\right)^{(p-1)/2}\pmod p. \tag{2.7}$$

In the case $p\equiv 1\pmod 4$, this yields

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv(-1)^{(p-1)/4}=\left(\frac{2}{p}\right)\pmod p.$$

If $p\equiv 3\pmod 4$, then by (2.7) we have

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2}\equiv\frac{p-1}{2}!\left(\frac{p-1}{2}!\right)^{2(p-3)/4}\equiv\frac{p-1}{2}!\pmod p.$$

So (1.20) always holds.

(ii) It is known (cf. [K05, (5.5)]) that

$$\left|\frac{1}{x_i+y_j}\right|_{1\leqslant i,j\leqslant n}=\frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{i=1}^n\prod_{j=1}^n(x_i+y_j)}.$$

Taking $n=(p-1)/2$ and $x_i=y_i=i^2$ for $i=1,\ldots,n$, we get

$$\left|\frac{1}{i^2+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2}=\frac{\prod_{1\leqslant i<j\leqslant (p-1)/2}(j^2-i^2)^2}{\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2)}. \tag{2.8}$$

Observe that

$$\begin{aligned}\prod_{1\leqslant i<j\leqslant (p-1)/2}(j^2-i^2)^2 &= \prod_{j=1}^{(p-1)/2}((j-1)!(j+1)\cdots(2j-1))^2\\ &= \prod_{j=1}^{(p-1)/2}\frac{(2j-1)!^2}{j^2} = \frac{\prod_{j=1}^{(p-1)/2}(2j-1)!(p-2j)!}{((p-1)/2)!^2}\\ &= \frac{\binom{p-1}{(p-1)/2}}{(p-1)!}\prod_{j=1}^{(p-1)/2}\frac{(p-1)!}{\binom{p-1}{2j-1}}\\ &\equiv \frac{(-1)^{(p-1)/2}}{-1}\prod_{j=1}^{(p-1)/2}\frac{-1}{(-1)^{2j-1}} = 1 \pmod p\end{aligned}$$

with the help of Wilson's theorem. Also,

$$\begin{aligned}\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2) &= \prod_{i=1}^{(p-1)/2}\left(\prod_{j=1}^{(p-1)/2} i^2\left(1+\frac{j^2}{i^2}\right)\right)\\ &\equiv \prod_{i=1}^{(p-1)/2}\left(i^{p-1}\prod_{x=1}^{(p-1)/2}(1+x^2)\right) \pmod p.\end{aligned}$$

As -1 is a quadratic non-residue modulo p, applying (2.4) with $d=-1$ we get

$$\prod_{x=1}^{(p-1)/2}(x^2+1) \equiv (-1)^{(p+1)/2}2 = 2 \pmod p.$$

Therefore

$$\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2) \equiv \prod_{i=1}^{(p-1)/2} 2 = 2^{(p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod p.$$

So the desired congruence (1.21) follows from (2.8). We are done. □

3. Some open conjectures on determinants

Wilson's theorem implies that

$$\frac{p-1}{2}! \equiv \pm 1 \pmod p \quad \text{for any prime } p\equiv 3 \pmod 4.$$

Conjecture 3.1 (2013-08-05)**.** *Let p be an odd prime. Then we have*

$$\left|\left(\frac{i^2-((p-1)/2)!j}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}=0 \iff p\equiv 3 \pmod 4. \tag{3.1}$$

Remark 3.1. See [Su13, A226163] for the sequence

$$\left|\left(\frac{i^2-((p_n-1)/2)!j}{p_n}\right)\right|_{1\leqslant i,j\leqslant (p_n-1)/2}\quad (n=2,3,\ldots),$$

where p_n denotes the nth prime. In 1961 L. J. Mordell [M61] proved that for any prime $p>3$ with $p\equiv 3\pmod 4$ we have

$$\frac{p-1}{2}!\equiv(-1)^{(h(-p)+1)/2}\pmod p, \tag{3.2}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Conjecture 3.2 (2013-07-18)**.** *Let p be an odd prime. For $d\in\mathbb{Z}$ let $S(d,p)$ be given by* (1.12)*. Then*

$$\left(\frac{-S(d,p)}{p}\right)=1\quad \textit{if}\ \left(\frac{d}{p}\right)=1, \tag{3.3}$$

and

$$S(d,p)=0\quad \textit{if}\ \left(\frac{d}{p}\right)=-1. \tag{3.4}$$

Remark 3.2. See [Su13, A227609] for the sequence $S(1,p_n)$ $(n=2,3,\ldots)$. Let p be any odd prime and let $d\in\mathbb{Z}$ with $p\nmid d$. The sum of entries in each row or column of the determinant $S(d,p)$ actually equals $-(1+(\frac{d}{p}))/2$. Indeed, for any $i_0,j_0=1,\ldots,(p-1)/2$ we have

$$\sum_{j=1}^{(p-1)/2}\left(\frac{i_0^2+dj^2}{p}\right)=\sum_{i=1}^{(p-1)/2}\left(\frac{i+dj_0^2}{p}\right)=\begin{cases}0 & \text{if } (\frac{d}{p})=-1,\\ -1 & \text{if } (\frac{d}{p})=1.\end{cases}$$

To see this we note that

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}\left(\frac{i_0^2+dj^2}{p}\right)&\equiv\sum_{j=1}^{(p-1)/2}(i_0^2+dj^2)^{(p-1)/2}\\ &\equiv\frac{p-1}{2}i_0^{p-1}+\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}i_0^{p-1-2k}\frac{d^k}{2}\sum_{j=1}^{p-1}j^{2k}\\ &\equiv-\frac12+\frac12\left(\frac{d}{p}\right)(p-1)\equiv-\frac{1+(\frac{d}{p})}{2}\pmod p.\end{aligned}$$

The following conjecture can be viewed as a supplement to Conjecture 3.2.

Conjecture 3.3 (2013-08-07). *Let p be an odd prime, and let $c,d\in\mathbb{Z}$ with $p\nmid cd$. Define*

$$S_c(d,p)=\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}. \tag{3.5}$$

Then

$$\left(\frac{S_c(d,p)}{p}\right)=\begin{cases}1 & \text{if } (\frac{c}{p})=1\ \&\ (\frac{d}{p})=-1,\\ (\frac{-1}{p}) & \text{if } (\frac{c}{p})=(\frac{d}{p})=-1,\\ (\frac{-2}{p}) & \text{if } (\frac{-c}{p})=(\frac{d}{p})=1,\\ (\frac{-6}{p}) & \text{if } (\frac{-c}{p})=-1\ \&\ (\frac{d}{p})=1.\end{cases} \tag{3.6}$$

Remark 3.3. See [Su13, A228005] for the sequence $S_1(1,p_n)$ $(n=2,3,\ldots)$. Let p be an odd prime and let $b,c,d\in\mathbb{Z}$ with $p\nmid bcd$. It is easy to see that

$$\begin{aligned}\left|\left(\frac{i^2+dj^2+b^2c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}&=\left|\left(\frac{(bi)^2+d(bj)^2+b^2c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}\\&=\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}.\end{aligned}$$

Conjecture 3.4 (2013-08-12). *Let p be an odd prime. For $c,d\in\mathbb{Z}$ define*

$$(c,d)_p:=\left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}. \tag{3.7}$$

(i) *If d is nonzero, then there are infinitely many odd primes q with $(c,d)_q=0$. Also,*

$$\left(\frac{d}{p}\right)=-1\implies (c,d)_p=0. \tag{3.8}$$

When $(c,d)_p$ is nonzero, its p-adic valuation (i.e., p-adic order) must be even.

(ii) *We have*

$$\begin{aligned}(6,1)_p&=0\quad \text{if } p\equiv 3 \pmod 4,\\ (3,2)_p=(4,2)_p&=0\quad \text{if } p\equiv 7 \pmod 8,\\ (3,3)_p&=0\quad \text{if } p\equiv 11 \pmod{12},\\ (10,9)_p&=0\quad \text{if } p\equiv 5 \pmod{12}.\end{aligned} \tag{3.9}$$

Remark 3.4. See [Su13, A225611] for the sequence $(6,1)_{p_n}$ $(n=2,3,\ldots)$. It is easy to see that $(-c,d)_p=(\frac{-1}{p})(c,d)_p$ for any odd prime p and integers c and d.

Conjecture 3.5 (2013-08-12). *Let p be an odd prime. For $c,d\in\mathbb{Z}$ define*

$$[c,d]_p:=\left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}. \tag{3.10}$$

(i) *If d is nonzero, then there are infinitely many odd primes q with $[c,d]_q=0$. When $[c,d]_p$ is nonzero, its p-adic valuation (i.e., p-adic order) must be even.*

(ii) *If $p\nmid d$ and $(c,d)_p\neq 0$, then*

$$\frac{[c,d]_p}{(c,d)_p}=\begin{cases}(p-1)/2 & \text{if } p\nmid c^2-4d,\\ (1-p)/(p-2) & \text{if } p\mid c^2-4d.\end{cases} \tag{3.11}$$

(iii) *We have*

$$\begin{aligned}[6,1]_p=[3,2]_p&=0 \quad \text{if } p\equiv 3 \pmod 4,\\ [3,3]_p&=0 \quad \text{if } p\equiv 5 \pmod 6,\\ [4,2]_p&=0 \quad \text{if } p\equiv 5,7 \pmod 8,\\ [5,5]_p&=0 \quad \text{if } p\equiv 13,17 \pmod{20}.\end{aligned} \tag{3.12}$$

Remark 3.5. See [Su13, A228095] for the sequence $[3,3]_{p_n}$ $(n=2,3,\ldots)$. It is easy to see that $[-c,d]_p=(\frac{-1}{p})[c,d]_p$ for any odd prime p and integers c and d.

Let p be any odd prime. For $a,b,c\in\mathbb{Z}$ with $p\nmid a$, it is known (cf. [BEW]) that

$$\sum_{x=0}^{p-1}\left(\frac{ax^2+bx+c}{p}\right)=\begin{cases}-(\frac{a}{p}) & \text{if } p\nmid b^2-4ac,\\ (p-1)(\frac{a}{p}) & \text{if } p\mid b^2-4ac.\end{cases}$$

Thus, for any $c,d\in\mathbb{Z}$ we can easily calculate the sum of all entries in a row or a column of $(c,d)_p$ or $[c,d]_p$.

Conjecture 3.6 (2013-08-11). *Let $p>5$ be a prime with $p\equiv 1 \pmod 4$. Define*

$$D_p^+:=\left|(i+j)\left(\frac{i+j}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2} \text{ and } D_p^-:=\left|(j-i)\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}. \tag{3.13}$$

Then

$$\left(\frac{D_p^+}{p}\right)=\left(\frac{D_p^-}{p}\right)=1. \tag{3.14}$$

Remark 3.6. It is known that a skew-symmetric $2n\times 2n$ determinant with integer entries is always a square (cf. [St90] and [K99]).

Conjecture 3.7 (2013-08-20). *For any prime $p > 3$, we have*

$$\left|(i^2+j^2)\left(\frac{i^2+j^2}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.15}$$

Remark 3.7. This conjecture is somewhat curious.

Conjecture 3.8 (2013-08-12). *Let $p \equiv 5 \pmod 6$ be a prime. Then*

$$\mathrm{ord}_p \left|\frac{1}{i^2-ij+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2} = \frac{p+1}{6}, \tag{3.16}$$

where $\mathrm{ord}_p\, x$ denotes the p-adic order of a rational number x. Also, we have

$$\left|\frac{1}{i^2-ij+j^2}\right|_{1\leqslant i,j\leqslant p-1} \equiv 2x^2 \pmod p \tag{3.17}$$

for some $x \in \{1, \ldots, (p-1)/2\}$.

Remark 3.8. Compare this conjecture with Theorem 1.3(ii).

The $(n+1)\times(n+1)$ Hankel determinant associated with a sequence $a_0, a_1, \ldots$ of numbers is defined by $|a_{i+j}|_{0\leqslant i,j\leqslant n}$. The evaluation of this determinant is known for some particular sequences including Catalan numbers and Bell numbers (cf. [K99]).

Conjecture 3.9 (2013-08-17). (i) *For any positive integers m and n, we have*

$$(-1)^n \left|H_{i+j}^{(m)}\right|_{0\leqslant i,j\leqslant n} > 0, \tag{3.18}$$

where $H_k^{(m)}$ denotes the m-th order harmonic number $\sum_{0<j\leqslant k} 1/j^m$.

(ii) *For any prime $p \equiv 1 \pmod 4$ and $m = 2, 4, 6, \ldots$ we have*

$$\left|H_{i+j}^{(m)}\right|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.19}$$

Remark 3.9. For any odd prime p and $i, j \in \{0, \ldots, (p-1)/2\}$, the number $H_{i+j}^{(m)}$ is an p-adic integer for each positive integer m.

Conjecture 3.10. (i) (2013-08-14) *For Franel numbers $f_n := \sum_{k=0}^{n} \binom{n}{k}^3$ ($n = 0, 1, \ldots$), the number $6^{-n}|f_{i+j}|_{0\leqslant i,j\leqslant n}$ is always a positive odd integer. In general, for any integer $r > 1$ and the r-th order Franel numbers $f_n^{(r)} := \sum_{k=0}^{n} \binom{n}{k}^r$ ($n = 0, 1, \ldots$), the number $2^{-n}|f_{i+j}^{(r)}|_{0\leqslant i,j\leqslant n}$ is always a positive odd integer.*

(ii) (2013-08-20) *For any prime $p \equiv 1 \pmod 4$ with $p \not\equiv 1 \pmod{24}$, we have*

$$|f_{i+j}|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.20}$$

Remark 3.10. See [Su13, A225776] for the sequence $|f_{i+j}|_{0\leqslant i,j\leqslant n}$ ($n = 0, 1, 2, \ldots$).

Conjecture 3.11. (i) (2013-08-14) *For two kinds of Apéry numbers*

$$b_n:=\sum_{k=0}^n\binom nk^2\binom{n+k}{k}\ \text{and}\ A_n:=\sum_{k=0}^n\binom nk^2\binom{n+k}{k}^2\quad(n=0,1,2,\ldots),$$

both

$$\frac{|b_{i+j}|_{0\leqslant i,j\leqslant n}}{10^n}\quad\text{and}\quad\frac{|A_{i+j}|_{0\leqslant i,j\leqslant n}}{24^n}$$

are always positive integers.

(ii) (2013-08-20) *For any prime p with $2\nmid\lfloor p/10\rfloor$ and $p\not\equiv 31,39\pmod{40}$, we have*

$$|b_{i+j}|_{0\leqslant i,j\leqslant (p-1)/2}\equiv 0\pmod p.\tag{3.21}$$

Remark 3.11. See [Su13, A228143] for the sequence $|A_{i+j}|_{0\leqslant i,j\leqslant n}$ $(n=0,1,2,\ldots)$.

Conjecture 3.12 (2013-08-20)**.** *For $n=0,1,2,\ldots$ define*

$$c_n:=\sum_{k=0}^n(-1)^k\binom nk^4\quad\text{and}\quad d_n:=\sum_{k=0}^n(-1)^k\binom nk^2\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

Then, for any odd prime p we have

$$|c_{i+j}|_{0\leqslant i,j\leqslant p-1}\equiv\left(\frac{-1}{p}\right)\pmod p\quad\text{and}\quad|d_{i+j}|_{0\leqslant i,j\leqslant p-1}\equiv 1\pmod p.\tag{3.22}$$

Remark 3.12. See [Su13, A228304] for the sequence c_n $(n=0,1,2,\ldots)$.

Conjecture 3.13. (i) (2013-08-14) *For Catalan-Larcombe-French numbers*

$$P_n:=\sum_{k=0}^n\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom nk}=2^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}^24^{n-2k}\ (n=0,1,\ldots),$$

the number $2^{-n(n+3)}|P_{i+j}|_{0\leqslant i,j\leqslant n}$ is always a positive odd integer.

(ii) (2013-08-20) *For any odd prime p, we have the supercongruence*

$$|P_{i+j}|_{0\leqslant i,j\leqslant p-1}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}.\tag{3.23}$$

Remark 3.13. See [Sl, A053175] for some basic properties of Catalan-Larcombe-French numbers.

Conjecture 3.14. (i) (2013-08-14) *For Domb numbers*

$$D_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \ (n = 0, 1, \ldots),$$

the number $12^{-n}|D_{i+j}|_{0\leqslant i,j\leqslant n}$ *is always a positive odd integer.*

(ii) (2013-08-20) *For any prime* p, *we have*

$$|D_{i+j}|_{0\leqslant i,j\leqslant p-1} \equiv \begin{cases} (\frac{-1}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{3.24}$$

Remark 3.14. See [Sl, A002895] for some basic properties of Domb numbers, and [Su13, A228289] for the sequence $|D_{i+j}|_{0\leqslant i,j\leqslant p_n-1}$ $(n = 1, 2, 3, \ldots)$. It is known that any prime $p \equiv 1 \pmod 3$ can be written uniquely in the form $x^2 + 3y^2$ with x and y positive integers.

Conjecture 3.15 (2013-08-22)**.** *For* $n = 0, 1, 2, \ldots$ *let*

$$s_n := \sum_{k=0}^{n} \binom{n}{k}^2 C_k \quad \text{and} \quad S(n) = |s_{i+j}|_{0\leqslant i,j\leqslant n},$$

where C_k *denotes the Catalan number* $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$.

(i) $S(n)$ *is always positive and odd, and not congruent to* 7 *modulo* 8.

(ii) *Let* p *be an odd prime. If* $p \equiv 1 \pmod 3$ *and* $p = x^2 + 3y^2$ *with* $x, y \in \mathbb{Z}$ *and* $x \equiv 1 \pmod 3$, *then*

$$S(p-1) \equiv \left(\frac{-1}{p}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{3.25}$$

If $p \equiv 2 \pmod 3$, *then*

$$S(p-1) \equiv -\left(\frac{-1}{p}\right) \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}. \tag{3.26}$$

Remark 3.15. See [Sl, A086618] for the sequence s_n $(n = 0, 1, 2, \ldots)$, and [Su13, A228456] for the sequence $S(n)$ $(n = 0, 1, 2, \ldots)$.

Conjecture 3.16 (2013-08-21)**.** *For* $n = 0, 1, 2, \ldots$ *let*

$$w_n := \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{2k}{k} \binom{3k}{k} \ \text{and} \ W(n) = |w_{i+j}|_{0\leqslant i,j\leqslant n}.$$

Then

$$n \equiv 1 \pmod 3 \implies W(n) = 0. \tag{3.27}$$

When $n \equiv 0, 2 \pmod 3$, *the number* $(-1)^{\lfloor (n+1)/3 \rfloor} W(n)/6^n$ *is always a positive odd integer.*

Remark 3.16. See [Sl, A006077] for the sequence w_n $(n = 0, 1, 2, \ldots)$.

Conjecture 3.17 (2013-08-15). *For any positive integer n, we have*

$$\left|B_{i+j}^2\right|_{0\leqslant i,j\leqslant n}<0 \quad and \quad \left|E_{i+j}^2\right|_{0\leqslant i,j\leqslant n}>0, \tag{3.28}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers and $E_0, E_1, E_2, \ldots$ are Euler numbers.

Remark 3.17. We have many similar conjectures with Bernoulli or Euler numbers replaced by some other classical numbers.

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References

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[C59] L. Carlitz, *Some cyclotomic matrices*, Acta Arith. **5** (1959), 293–308.

[Ch03] R. Chapman, *Determinants of Legendre matrices*, preprint August 5, 2003.

[Ch04] R. Chapman, *Determinants of Legendre symbol matrices*, Acta Arith. **115** (2004), 231–244.

[Ch12] R. Chapman, *My evil determinant problem*, preprint, Dec. 12, 2012, available from `http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/evildet.pdf`.

[Co62] H. Cohn, *Advanced Number Theory*, Dover Publ., New York, 1962.

[DH] W. Duke and K. Hopkins, *Quadratic reciprocity in a finite group*, Amer. Math. Monthly **112** (2005), 251–256.

[G04] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd Edition, Springer, 2004.

[K99] C. Krattenthaler, *Advanced determinant calculus*, Séminaire Lotharinggien Combin. **42** (1999), Article B42q, 67pp.

[K05] C. Krattenthaler, *Advanced determinant calculus: a complement*, Linear Algebra Appl. **411** (2005), 68–166.

[M61] L. J. Mordell, *The congruence* $((p-1)/2)!\equiv\pm1 \pmod p$, Amer. Math. Monthly **68** (1961), 145–146.

[P06] H. Pan, *A remark on Zoloterav's theorem*, preprint, `arXiv:0601026`, 2006.

[Sl] N. J. A. Sloane, Sequences A002895, A006077, A053175, A086618 in OEIS, `http://oeis.org`.

[St90] J. R. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. in Math. **83** (1990), 96–131.

[Su13] Z.-W. Sun, Sequences A225611, A225776, A226163, A227609, A227968, A228005, A228095, A228143, A228289, A228304, A228456 in OEIS, `http://oeis.org`.

[V12] M. Vsemirnov, *On the evaluation of R. Chapman's "evil determinant"*, Linear Algebra Appl. **436** (2012), 4101–4106.

[V13] M. Vsemirnov, *On R. Chapman's "evil determinant": case* $p\equiv1 \pmod 4$, Acta Arith. **159** (2013), 331–344.

[Z] G. Zolotarev, *Nouvelle démonstration de la loi de réciprocité de Legendre*, Nouvelles Ann. Math. 11(1872), 354–362.