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# Some of my number-theoretic conjectures and related progress

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# Abstract and References

In recent years the author raised hundreds of number-theoretic conjectures some of which involve representations of integers, binary quadratic forms, congruences for sums of binomial coefficients, and series for powers of $\pi$. In this talk we will introduce my main conjectures and related progress, and announce prizes for proofs of some conjectures.

References

1. Z. W. Sun, *Open Conjectures on Congruences*, `arXiv:0911.5665`
2. Z. W. Sun, *List of conjectural series for powers of $\pi$ and other constants*, `arXiv:1102.5649`.
3. Z. W. Sun, *Conjectures and results on $x^2$ mod $p^2$ with $4p = x^2 + dy^2$*, `arXiv:1103.4325`.

# Part A. On Representations of Integers

# Mixed Sums of Triangular Numbers and Other Terms

Triangular numbers: $T_x = x(x+1)/2 \ (x \in \mathbb{Z})$.

**Conjecture** (Sun [J. Combin. Number Theory 1(2009)]). Any positive integer $n \neq 216$ can be written in the form $p + T_x$ with $p$ zero or a prime.

*Remark*. This has been verified for $n \leqslant 10^{12}$. Prize \$1000 for the first rigorous proof.

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

**Theorem** (B. K. Oh and Sun [JNT 2009]). Any positive integer $n$ can be written as the sum of a square, an *odd* square and a triangular number.

# Mixed Sums of Squares and Triangular Numbers

**Ken Ono and K. Soundararajan** [Invent. Math. 130(1997)]: Under GRH (the generalized Riemann hypothesis), we have Ramanujan's assertion that any positive odd integer greater than 2719 can be represented by the form $x^2 + y^2 + 10z^2$.

$$2n + 1 = (2x)^2 + (2y + 1)^2 + 10z^2 \iff n = 2x^2 + 4T_y + 5z^2.$$

**Another Version of the Ono-Soundararajan Theorem**. Under GRH, $2x^2 + 5y^2 + 4T_z$ represents all integers greater than 1359.

**B. Kane and Z. W. Sun** [Trans. AMS 362(2010), 6425–6455] determined completely when the general form $ax^2 + by^2 + cT_z$ $(a, b, c \in \mathbb{Z}^+)$ represents sufficiently large integers and established similar results for the forms $ax^2 + bT_y + cT_z$ and $aT_x + bT_y + cT_z$. In particular, *the form $ax^2 + y^2 + T_z$ represents sufficiently large integers if and only if each odd prime divisor of $a$ is congruent to 1 or 3 modulo 8.*

**Tools**: The theory of ternary quadratic forms and modular forms.

# Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons.

For $m = 3, 4, 5, \ldots$, the $m$-gonal numbers are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Note that $p_3(n) = T_n$ and $p_4(n) = n^2$.

Those $p_5(n) = n(3n-1)/2$ $(n \in \mathbb{N})$ are called *pentagonal numbers*. Those $p_6(n) = n(2n-1) = T_{2n-1}$ $(n \in \mathbb{N})$ are called *hexagonal numbers*.

**Fermat's Assertion** (proved by Cauchy in 1813). Any natural number $n$ can be written as the sum of $m$ $m$-gonal numbers.

# Diagonal Representations by Polygonal Numbers

$$n = \underline{p_3(x_1) + p_3(x_2) + p_3(x_3)}$$
$$n = \underline{p_4(x_1)} + p_4(x_2) + p_4(x_3) + p_4(x_4)$$
$$n = p_5(x_1) + \underline{p_5(x_2)} + p_5(x_3) + p_5(x_4) + p_5(x_5)$$
$$n = p_6(x_1) + p_6(x_2) + \underline{p_6(x_3)} + p_6(x_4) + p_6(x_5) + p_6(x_6)$$

**Diagonal Representation**:

$$n = \underline{p_4(x_1)} + \underline{p_5(x_2)} + \underline{p_6(x_3)}$$

**Conjecture** (Z. W. Sun, 2009). Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written in the form

$$p_{m+1}(x_1)+p_{m+2}(x_2)+p_{m+3}(x_3)+r \ \ (x_1, x_2, x_3 \in \mathbb{N},\ r \in \{0, \ldots, m-3\}).$$

Moreover, $\{r(n): \ n = 1, 2, 3, \ldots\} = \{1, 2, 3, \ldots\}$, where

$$r(n) = |\{(x, y, z) \in \mathbb{N}^3 : \ n = p_4(x) + p_5(y) + p_6(z)\}|.$$

**Remark**. \$500 for the first rigorous proof.

# Part B. On Combinatorial Congruences

# On Fibonacci quotient mod $p$

**Z. W. Sun** [Sci. China Math. 53(2010)]) determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod $p^2$, where $p$ is an odd prime and $m\not\equiv 0 \pmod p$. In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}\equiv\left(\frac{p}{5}\right)\left(1-2F_{p-(\frac{p}{5})}\right) \pmod{p^2}\ \ (p\neq 5),$$

where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence with $F_0=0,\ F_1=1$ and $F_{n+1}=F_n+F_{n-1}$ $(n=1,2,3,\ldots)$.

**Theorem** (Sun and Tauraso [Adv. in Appl. Math. 2010]). For any odd prime $p\neq 5$ we have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{2k}{k}\equiv -5\frac{F_{p-(\frac{p}{5})}}{p} \pmod p.$$

# Determination of $F_{p-(\frac{p}{5})}$ mod $p^3$

**Conjecture** (Sun and Tauraso [Adv. in Appl. Math. 2010]). Let $p \neq 2, 5$ be a prime and let $a \in \mathbb{Z}^+$. Then

$$\sum_{k=0}^{p^a-1}(-1)^k\binom{2k}{k} \equiv \left(\frac{p^a}{5}\right)\left(1-2F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3}.$$

Recently Hao Pan and Z. W. Sun confirmed the conjecture. The proof is very sophisticated and we use an important technique of Andrew Granville [2004]. In the same paper Pan and Sun also proved the following conjecture of Roberto Tauraso:

$$\sum_{k=1}^{p-1}\frac{L_k}{k^2} \equiv 0 \pmod{p} \quad \text{for any prime } p > 3,$$

where $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ $(n = 1, 2, 3, \ldots)$.

# Another congruence for $F_{p-(\frac{p}{5})}$ mod $p^3$

**Theorem** (Z. W. Sun, 2010) Let $p \neq 2, 5$ be a prime. Then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p}{5}\right)\left(1 + \frac{F_{p-(\frac{p}{5})}}{2}\right) \pmod{p^3}.$$

The proof is also very sophisticated and quite difficult. One of the lemmas is as follows.

**Lemma** (Conjectured by Sun and proved by Tauraso) Let $p > 5$ be a prime and let $H_k^{(2)} = \sum_{0<j\leqslant k} 1/j^2$ for $k = 0, 1, 2, \ldots$. Then

$$\sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} H_k^{(2)} \equiv \left(\frac{p}{5}\right)\frac{5}{2}\left(\frac{F_{p-(\frac{p}{5})}}{p}\right)^2 \pmod{p},$$

$$\sum_{k=0}^{p-1} (-2)^k \binom{2k}{k} H_k^{(2)} \equiv \frac{2}{3} q_p(2)^2 \pmod{p}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

# A further conjecture

**Conjecture** (Sun, 2010) Let $p$ be an odd prime and let $a \in \mathbb{Z}^+$.
(i) If $p^a \equiv 1, 2 \pmod 5$, or $a > 1$ and $p \not\equiv 3 \pmod 5$,

$$\sum_{k=0}^{\lfloor 4p^a/5 \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod 5$, or $a > 1$ and $p \not\equiv 2 \pmod 5$, then

$$\sum_{k=0}^{\lfloor 3p^a/5 \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

(ii) If $p \equiv 1, 7 \pmod{10}$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor 7p^a/10 \rfloor} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p \equiv 1, 3 \pmod{10}$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor 9p^a/10 \rfloor} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

# A conjecture on 5-adic valuations

For a prime $p$ the $p$-adic valuation $\nu_p(m)$ of an integer $m$ refers to $\sup\{a \in \mathbb{N} : \ p^a \mid m\}$.

**Conjecture** (Z. W. Sun). For each $n = 1, 2, 3, \ldots$, we have

$$\nu_5\left(\sum_{k=0}^{n-1} F_{2k+1}\binom{2k}{k}\right) \geqslant \nu_5\left((2n+1)n^2\binom{2n}{n}\right).$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geqslant b$ then the sum

$$\frac{1}{5^{2a}}\sum_{k=0}^{5^a-1} F_{2k+1}\binom{2k}{k}$$

modulo $5^b$ only depends on $b$.

# Central binomial coefficients and 3-adic valuations

**Theorem** (Strauss, Shallit and Zagier, 1992)

$$\nu_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right) = 2\nu_3(n) + \nu_3\left(\binom{2n}{n}\right) \text{ for } n = 1, 2, 3, \ldots.$$

**Theorem** (Sun and Tauraso [Int. J. Number Theory 7(2011)]) Let $p > 3$ be a prime and let $a$ be a positive integer. Then

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d} \equiv \left(\frac{p^a-d}{3}\right) + 2p^a \sum_{0<k<d} \frac{(-1)^{k-1}}{k}\left(\frac{d-k}{3}\right) \pmod{p^2}.$$

*Remark*. Note that $\binom{2k}{k} = \binom{2k}{k,k}$, where multi-nomial coefficients are given by

$$\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} = \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}.$$

# On $p$-adic valuations

**Conjecture** (Z. W. Sun [Acta Arith. 148(2011)]). For any odd prime $p$ and positive integer $n$ we have

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\bigg) \geqslant \nu_p\left(n\binom{2n}{n}\right).$$

*Remark*. I'd like to offer \$200 for the first rigorous proof.

**Theorem** (Z. W. Sun [Acta Arith. 148(2011)]). For any prime $p$ we have

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k} \equiv pB_{p-1} + (-1)^{p-1} - 2p \pmod{p^2},$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. Also, an integer $n > 1$ is a prime if and only if

$$\sum_{k=0}^{n-1}\binom{(n-1)k}{k,\ldots,k} \equiv 0 \pmod{n}.$$

## A conjecture on $\sum_{k=1}^{p-1}\binom{2k}{k}/k^3$ mod $p^4$

**Conjecture** (Z. W. Sun, Sci. China Math., in press). For any prime $p>7$ we have

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^3}\equiv-\frac{2}{p^2}H_{p-1}-\frac{13}{27}\sum_{k=1}^{p-1}\frac{1}{k^3}\pmod{p^4},$$

where $H_n$ denotes the harmonic number $\sum_{k=1}^{n}1/k$.

*Remark*. It is known that $H_{p-1}/p^2\equiv-B_{p-3}/3\pmod p$ for any prime $p>3$ and $\sum_{k=1}^{p-1}1/k^3\equiv-\frac{6}{5}p^2B_{p-5}\pmod{p^3}$ for each prime $p>5$.

**Motivation**.

$$\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}=\frac{17}{36}\zeta(4).$$

**Progress**. (K. Hessami Pilehrood and T. Hessami Pilehrood, 2011) The conjectural congruence holds mod $p$.

## On $\sum_{k=1}^{(p-1)/2}(-1)^k/(k^3\binom{2k}{k}) \bmod p$

**Conjecture** (Z. W. Sun, Sci. China Math., in press). For any prime $p > 5$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3\binom{2k}{k}} \equiv -2B_{p-3} \pmod{p}.$$

**Motivation**.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3\binom{2k}{k}} = -\frac{2}{5}\zeta(3).$$

**Related result** (Sun [Sci. China Math. 53(2010)]): For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2}\frac{4}{3}E_{p-3} \pmod{p},$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers. (Compare this with the identity $\sum_{k=1}^{\infty} 1/(k^2\binom{2k}{k}) = \pi^2/18$.

# On divisibility of binomial coefficients

Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \ (n = 0, 1, 2, \ldots).$$

**An Observation of Sun**. Let $k$ and $\ell$ be positive integers. If all prime factors of $k$ divide $\ell$, then $\ell n + 1 \mid \binom{kn+\ell n}{\ell n}$ for every $n = 0, 1, 2, \ldots$.

**Conjecture** (Sun, 2010). Let $k$ and $\ell$ be positive integers. If

$$\ell n + 1 \,\Big|\, \binom{kn + \ell n}{\ell n}$$

for all sufficiently large positive integers $n$, then each prime factor of $k$ divides $\ell$.

# Part C. Series for powers of $\pi$ and $L(2, (\frac{\cdot}{3}))$

# Fast convergent series for $L(2,(\frac{\cdot}{3}))$

With a Dirichlet character $\chi$, the series

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\quad (\mathrm{Re}(s)>1).$$

converges very slow.

**Conjecture** (Z. W. Sun, 2010). Set

$$K=L\left(2,\left(\frac{\cdot}{3}\right)\right)=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}=0.781302412896486296867187429624\ldots.$$

Then

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K, \tag{1}$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^{k}}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K. \tag{2}$$

**Progress**. (1) was recently confirmed by K. Hessami Pilehrood and T. Hessami Pilehrood (arXiv:1104.3659) via the Hurwitz $\zeta(s,x)$.

# Fast covergent series for $\pi^2$

**Conjecture** (Z. W. Sun, 2010). We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} =\frac{\pi^2}{2}, \tag{1}$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} =8\pi^2, \tag{2}$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} =12\pi^2. \tag{3}$$

**Progress**. (2) has been confirmed by J. Guillera via applying a Barnes-integrals strategy of the WZ-method.

**Theorem** (Sun, 2010). We have

$$\sum_{n=1}^{\infty}\frac{2^n H_{n-1}^{(2)}}{n\binom{2n}{n}} = \frac{\pi^3}{48}, \quad \text{where } H_{n-1}^{(2)} = \sum_{k=1}^{n-1}\frac{1}{k^2}.$$

# Conjectural series for $\zeta(3)$ and $1/\pi^2$

**Conjecture** (Z. W. Sun). We have

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3), \qquad (*)$$

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2},$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

**Related $p$-adic congruences of $(*)$:** For any odd prime $p$ we have

$$\sum_{k=0}^{p-1}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2-\frac{7}{2}p^5B_{p-3} \pmod{p^6},$$

$$\sum_{k=0}^{(p-1)/2}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2+6\left(\frac{-1}{p}\right)p^4E_{p-3} \pmod{p^5}.$$

# Conjectural series for $1/\pi$

I have made over 100 conjectural series for $1/\pi$ of the form $\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi}$, where $a_k,b,c,m\in\mathbb{Z}$, $bm$ is nonzero, and $C^2$ is rational. Such series are closely related to the theory of modular forms, and the full list of my conjectural series is available from `arXiv:1102.5649`.

**Examples of my conjectural series**:

$$\sum_{k=0}^{\infty}\frac{280k-139}{912^{2k}}\binom{4k}{2k}\binom{2k}{k}a_k=\frac{95\sqrt{399}}{\pi},$$

where $a_k$ denotes the coefficient of $x^k$ in $(x^2+12098x+1)^k$;

$$\sum_{k=0}^{\infty}\frac{1054k+233}{480^k}\binom{2k}{k}P_k(8)=\frac{520}{\pi},\qquad(*)$$

$$\sum_{k=0}^{\infty}\frac{224434k+32849}{5760^k}\binom{2k}{k}P_k(18)=\frac{93600}{\pi},$$

where $P_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k x^{2k-n}$. \$520 for a proof of $(*)$.

# Part D. On $x^2$ mod $p^2$ with $p = x^2 + dy^2$

# On $p = x^2 + 7y^2$ and $4p = x^2 + 11y^2$

**Conjecture** (Z. W. Sun [JNT 131(2011)]). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod{7}.\end{cases}$$

*Remark*. Recently Z. H. Sun confirmed the congruence in the case $(\frac{p}{7})=-1$ via Legendre polynomials.

**Conjecture** (Z. W. Sun [JNT 131(2011)]). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$

$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1 \ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1,\ \text{i.e., } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

# Apéry numbers and polynomials

Define Apéry polynomials by

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n = 0, 1, 2, \ldots).$$

Those Apéry numbers $A_n = A_n(1)$ play important roles in Apéry's proof of the irrationality of $\zeta(3)$.

**Theorem** (Z. W. Sun, 2011) Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(-2) \equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture** (Z. W. Sun, 2010). For any odd prime $p$, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

# On congruences involving $D_k(x)^3$

$$D_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}.$$

In combinatorics, $D_n$ is the number of lattice paths from $(0,0)$ to $(n,n)$ with steps $(1,0),(0,1)$ and $(1,1)$. We define

$$D_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k.$$

$P_n(x) = D_n((x-1)/2)$ is the Legendre polynomial of degree $n$.

**Conjecture** (Sun [JNT, in press]). Let $p$ be an odd prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1} D_k(-3)^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p = x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3; \end{cases}$$

$$\sum_{k=0}^{p-1} D_k(3)^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p = x^2+15y^2, \\ 12x^2-2p \pmod{p^2} & \text{if } p = 3x^2+5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

## On $S_n(x)=\sum_{k=0}^n\binom nk x^k$

Define polynomials

$$S_n(x) := \sum_{k=0}^{n}\binom{n}{k}^4 x^k \quad (n=0,1,2,\ldots).$$

**Conjecture** (Sun). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(12)$$

$$\equiv \begin{cases} (-1)^{[3|x]}(4x^2-2p) \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x,\ 3\mid xy),\\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x,\ 3\nmid xy),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

$$\sum_{k=0}^{p-1}(4k+3)S_k(12) \equiv p\left(1+2\left(\frac{3}{p}\right)\right) \pmod{p^2}.$$

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)S_k(12)\in\mathbb{Z} \qquad \text{for all } n=1,2,3,\ldots.$$

## On $p = x^2 + 5y^2$ ($\mathbb{Q}(\sqrt{-5})$ has class number 2)

**Conjecture** (Sun). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(-4) \equiv \sum_{k=0}^{p-1} S_k(-64)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \&\ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-5}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (8k+7) S_k(-64) \equiv p \left(\frac{p}{15}\right) \left(3 + 4\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k+7) S_k(-64) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

# On $p = x^2 + 30y^2$ ($\mathbb{Q}(\sqrt{-30})$ has class number 4)

**Conjecture** (Sun). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(36)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{5}) = 1,\ p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1,\ (\frac{2}{p}) = (\frac{p}{5}) = -1,\ p = 3x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1,\ (\frac{p}{3}) = (\frac{p}{5}) = -1,\ p = 2x^2 + 15y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1,\ (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 5x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1} (8k+7) S_k(36) \equiv p\left(\frac{p}{15}\right)\left(3 + 4\left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

$$\frac{1}{n}\sum_{k=0}^{n-1} (8k+7) S_k(36) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

*Remark*. I would like to offer \$300 for the first correct proof.

## On $p = x^2 + 190y^2$ ($\mathbb{Q}(\sqrt{-190})$ has class number 4)

**Conjecture** (Sun) Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(5776)$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if}, (\frac{2}{p}) = (\frac{p}{5}) = (\frac{p}{19}) = 1, p = x^2+190y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{5}) = (\frac{p}{19}) = -1, p = 2x^2+95y^2, \\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{p}{19}) = 1, (\frac{2}{p}) = (\frac{p}{5}) = -1, p = 5x^2+38y^2, \\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1, (\frac{2}{p}) = (\frac{p}{19}) = -1, p = 10x^2+19y^2 \\ 0 \pmod{p^2} & \text{if } (\frac{-190}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1}(816k+769)S_k(5776) \equiv p\left(\frac{p}{95}\right)\left(361+408\left(\frac{p}{19}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(816k+769)S_k(5776) \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

## On $p = x^2 + 105y^2$

**Conjecture** (Sun, 2011).

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

*Remark*. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number eight.
I'd like to offer \$1050 for the first correct proof of the conjecture.

# Related series for $1/\pi$

**Conjecture** (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

For my general philosophy about connections between series for $1/\pi$ and $x^2$ mod $p^2$ with $p=x^2+dy^2$, see the survey

Z. W. Sun, *Conjectures and results on $x^2$ mod $p^2$ with $4p=x^2+dy^2$*, `arXiv:1103.4325`.

Thank you!

You are welcome to solve my conjectures!