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A Survey of Series for $1/\pi$ Conjectured by Me

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Abstract

Since 2010, I have posed hundreds of conjectural series for $1/\pi$. I'll talk about how they were found as well as progress on these series.

Part I. Ramanujan Series for $\frac{1}{\pi}$ and my Philosophy on Series for $\frac{1}{\pi}$

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England *Modular equations and approximations to* π, Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

S. Ramanujan attributed his mathematical discoveries to inspirations from the God. He once said: **"An equation for me has no meaning, unless it represents a thought of God."**

General forms of Ramanujan-type series

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 rational Ramanujan-type series for $1/\pi$.

Two Examples recorded by Ramanujan:

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi}$$

(proved by S. Chowla in 1928),

$$\sum_{k=0}^{\infty}\frac{26390k+1103}{396^{4k}}\binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}$$

(proved by J. Borwein and P. Borwein in 1987).

Connections to modular forms

Many known series for $1/\pi$ have the form

$$\sum_{k=0}^{\infty} \frac{bk+c}{m^k} \binom{2k}{k} u_k = \frac{C}{\pi},$$

where $u_{-1} = 0$, $u_0 = 1$ and

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B) u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots),$$

and there are modular functions (i.e., meromorphic modular forms of weight 0) $x(\tau)$ and $\tilde{x}(\tau)$ such that

$$F(\tau) = \sum_{k=0}^{\infty} u_k (x(\tau))^k \quad \text{and} \quad \tilde{F}(\tau) = \sum_{k=0}^{\infty} \binom{2k}{k} u_k (\tilde{x}(\tau))^k$$

are modular forms of weights 1 and 2 respectively.

van Hamme's conjectures

For the two Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi} \text{ and } \sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

in 1997 van Hamme conjectured their following p-adic analogues:

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right) \pmod{p^4},$$

where p is an odd prime, and $(\frac{\cdot}{p})$ is the Legendre symbol.

All the p-adic analogue conjectures of van Hamme were proved before 2017. Following van Hamme's idea, Zudilin [JNT, 2009] proposed more p-adic analogues for Ramanujan-type series.

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases} (\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod{4}$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy\pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4}.$$

Another Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

Two conjectural series for $1/\pi^2$

Motivated by corresponding congruences, I made the following conjecture in 2010.

Conjecture (Z.-W. Sun, Sci. China Math. 54 (2011)] We have

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2},$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

Remark. In 2004 H.H. Chan, S.H. Chan and Z. Liu [Adv. Math.] proved that

$$\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{8}{\sqrt{3}\pi}.$$

Part II. Series for $\frac{1}{\pi}$ involving $T_n(b, c)$

Central trinomial coefficients

For each $n = 0, 1, 2, \ldots$, the central trinomial coefficient T_n is defined as the coefficient of x^n in the expansion of $(x^2 + x + 1)^n$. It is easy to see that $T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}$. For any prime $p > 3$ and $k \in \{1, \ldots, p-1\}$ we have $T_{p-k} \equiv (\frac{p}{3}) T_{k-1}/(-3)^{k-1} \pmod p$.

Conjecture (Sun, Jan. 22, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2 T_k$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p}) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2 T_k \equiv p\left(20 + 24\left(\frac{p}{3}\right)(2 - 3^{p-1})\right) \pmod{p^3}.$$

Two new series for π involving central trinomial coefficients

Conjecture (Sun, 2019) For any prime $p > 3$, we have

$$p^2\sum_{k=1}^{p-1}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} \equiv 11\left(\frac{p}{3}\right)+\frac{p}{2}\left(13-35\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$p^2\sum_{k=1}^{p-1}\frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2(k-1)3^{k-1}} \equiv -\frac{1}{2}\left(\frac{p}{3}\right)-\frac{p}{8}\left(7+\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Two new series for π involving central trinomial coefficients

Conjecture (Sun, 2019) For any prime $p > 3$, we have

$$p^2\sum_{k=1}^{p-1}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} \equiv 11\left(\frac{p}{3}\right)+\frac{p}{2}\left(13-35\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$p^2\sum_{k=1}^{p-1}\frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2(k-1)3^{k-1}} \equiv -\frac{1}{2}\left(\frac{p}{3}\right)-\frac{p}{8}\left(7+\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Conjecture (Sun, Dec. 7, 2019). We have

$$\sum_{k=1}^{\infty}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} = \frac{5\pi}{\sqrt{3}}+6\log 3,$$

$$\sum_{k=2}^{\infty}\frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2(k-1)3^{k-1}} = \frac{21-2\sqrt{3}\,\pi-9\log 3}{12}.$$

Remark. As the two series converge very fast, it is easy to check the two identities numerically.

Generalized central trinomial coefficients

Recall that

$$T_n := [x^n](x^2+x+1)^n \text{ (the coefficient of } x^n \text{ in } (x^2+x+1)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Note that central binomial coefficients are those

$$\binom{2n}{n} = [x^n](x+1)^{2n} = [x^n](x^2+2x+1)^n \quad (n \in \mathbb{N}).$$

For real numbers b and c, we define the generalized central trinomial coefficient

$$T_n(b,c) := [x^n](x^2+bx+c)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k} c^k.$$

Series for $1/\pi$ involving $T_k(b,c)$ (2011)

I view $T_n(b,c)$ as natural extensions of the central binomial coefficients.

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

`Type I.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2T_k(b,c)/m^k$.

`Type II.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k}T_k(b,c)/m^k$.

`Type III.` $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k}T_k(b,c)/m^k$.

`Type IV.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2T_{2k}(b,c)/m^k$.

`Type V.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)/m^k$.

In October 2011, I found 10 conjectural series for $1/\pi$ of two new types:

`Type VI.` $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

`Type VII.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, W. Zudilin.

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

My unsolved conjectural series of type VII

Conjecture (Sun, 2011). (i) For any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n\binom{2n-1}{n-1}}\sum_{k=0}^{n-1}(2800512k+435257)434^{2(n-1-k)}\binom{2k}{k}T_k(73,576)^2$$

is an odd integer, and

$$n\binom{2n-1}{n-1} \Bigm| \sum_{k=0}^{n-1}(24k+5)28^{2(n-1-k)}\binom{2k}{k}T_k(4,9)^2.$$

(ii) We have

$$\sum_{k=0}^{p-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2=\frac{10406669}{2\sqrt{6}\,\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}).$$

Conjecture (Sun). (i) If $p > 3$ is a prime with $p \neq 7, 11, 17, 31$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(73,576)^2}{434^{2k}}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{17}) = 1,\ p = x^2 + 102y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{17}) = 1, (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 2x^2 + 51y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1, (\frac{2}{p}) = (\frac{p}{17}) = -1,\ p = 3x^2 + 34y^2, \\ 24x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{3}) = (\frac{p}{17}) = -1,\ p = 6x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-102}{p}) = -1, \end{cases}$$

where x and y are integers.

(ii) For any odd prime $p \neq 7, 31$, we have

$$\sum_{k=0}^{p-1} \frac{2800512k + 435257}{434^{2k}} \binom{2k}{k} T_k(73,576)^2$$

$$\equiv p \left(466752 \left(\frac{-6}{p} \right) - 31495 \right) \pmod{p^2}.$$

Conjecture (Sun). (i) For any prime $p > 7$, we have

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{T_k(4,9)^2}{28^{2k}}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{5}) = 1,\ p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1, (\frac{2}{p}) = (\frac{p}{5}) = -1,\ p = 3x^2 + 10y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{3}) = (\frac{p}{5}) = -1,\ p = 2x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{5}) = 1, (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 5x^2 + 6y^2, \\ p\delta_{p,7} \pmod{p^2} & \text{if } (\frac{-30}{p}) = -1. \end{cases}$$

where x and y are integers.

(ii) For any odd prime $p \neq 7$, we have

$$\sum_{k=0}^{p-1} \frac{24k+5}{28^{2k}} \binom{2k}{k} T_k(4,9)^2 \equiv p \left(\frac{-6}{p}\right) \left(4 + \left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

Remark. All my conjectural series came from **combinations of philosophy, intuition, inspiration, experience and computation!**

My 2019 conjectural series of type VIII

In November 2019, I introduced a new type series for $1/\pi$.

`Type VIII.` $\sum_{k=0}^{\infty}(a+dk)T_k(b_1,c_1)T_k(b_2,c_2)^2/m^k = C/\pi$.

Conjecture (Sun, Nov. 2019). We have

$$\sum_{k=0}^{\infty}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2=\frac{55\sqrt{15}}{9\pi}, \qquad \text{(VIII1)}$$

$$\sum_{k=0}^{\infty}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2=\frac{1452\sqrt{5}}{\pi}, \qquad \text{(VIII2)}$$

$$\sum_{k=0}^{\infty}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2=\frac{189\sqrt{15}}{2\pi}, \qquad \text{(VIII3)}$$

$$\sum_{k=0}^{\infty}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2=\frac{6795\sqrt{5}}{\pi}. \qquad \text{(VIII4)}$$

My conjectural series of type IX

In August 2020, I wanted to find a new type series for $1/\pi$.

`Type IX.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k(b_1,c_1)T_k(b_2,c_2)/m^k = C/\pi$.

For $|m| \leqslant 650$ I could not find such a series. To make the search more efficient, later I required that c_1, c_2 are squares if they are positive.

Conjecture (Sun, August 7, 2020). We have

$$\sum_{k=0}^{\infty}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)=\frac{5390}{\pi} \qquad \text{(IX1)}$$

and

$$\sum_{k=0}^{\infty}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)=\frac{98}{3\pi}(10+7\sqrt{5}). \quad \text{(IX2)}$$

Congruences related to (IX1)

Conjecture (Sun, August 2020). (i) For any integer $n>1$,

$$n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(4290k+367)3136^{n-1-k}\binom{2k}{k}T_k(14,1)T_k(17,16).$$

(ii) Let p be an odd prime with $p\neq 7$. Then

$$\sum_{k=0}^{p-1}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)$$
$$\equiv\frac{p}{2}\left(1430\left(\frac{-1}{p}\right)+39\left(\frac{3}{p}\right)-735\right)\pmod{p^2}.$$

(iii) For any prime $p>7$, we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(14,1)T_k(17,16)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2\,(x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2\,(x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases}$$

Congruences related to (IX2)

Conjecture (Sun, August 7, 2020). (i) For any prime $p > 7$,

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(2,81)T_k(14,81)$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) Let p be an odd prime with $p \neq 7$. Then

$$\sum_{k=0}^{p-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$

$$\equiv\frac{p}{3}\left(270\left(\frac{-1}{p}\right)-104\left(\frac{-2}{p}\right)+245\left(\frac{-5}{p}\right)\right) \pmod{p^2}.$$

Series involving Domb numbers

The Domb numbers are defined by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \quad (n = 0, 1, 2, \ldots).$$

In 2004, H. H. Chan, S. H. Chan and Z. Liu [Adv. Math. 186 (2004)] proved that

$$\sum_{k=0}^{\infty} \frac{5k+1}{64^k} D_k = \frac{8}{\sqrt{3}\,\pi}.$$

M. D. Rogers [Ramanujan J. 18 (2009)] found a similar identity:

$$\sum_{k=0}^{\infty} \frac{5k+1}{(-32)^k} D_k = \frac{2}{\pi}.$$

In 2019, Z.-W. Sun introduced the numbers

$$S_n(b,c) = \sum_{k=0}^{n} \binom{n}{k}^2 T_k(b,c) T_{n-k}(b,c) \quad (n = 0, 1, 2, \ldots)$$

which can be viewed as an extension of Domb numbers.

Series involving $S_n(b, c)$

Theorem (Z.-W. Sun [Electron. Res. Arch. 28 (2020)]) We have

$$\sum_{k=0}^{\infty}\frac{7k+3}{24^k}S_k(1,-6)=\frac{15}{\sqrt{2}\,\pi},\ \sum_{k=0}^{\infty}\frac{12k+5}{(-28)^k}S_k(1,7)=\frac{6\sqrt{7}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{84k+29}{80^k}S_k(1,-20)=\frac{24\sqrt{15}}{\pi},\ \sum_{k=0}^{\infty}\frac{3k+1}{(-100)^k}S_k(1,25)=\frac{25}{8\pi},$$

$$\sum_{k=0}^{\infty}\frac{228k+67}{224^k}S_k(1,-56)=\frac{80\sqrt{7}}{\pi},\ \sum_{k=0}^{\infty}\frac{399k+101}{(-676)^k}S_k(1,169)=\frac{2535}{8\pi},$$

$$\sum_{k=0}^{\infty}\frac{2604k+563}{2600^k}S_k(1,-650)=\frac{850\sqrt{39}}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{39468k+7817}{(-6076)^k}S_k(1,1519)=\frac{4410\sqrt{31}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{41667k+7879}{9800^k}S_k(1,-2450)=\frac{40425\sqrt{6}}{4\pi},$$

$$\sum_{k=0}^{\infty}\frac{74613k+10711}{(-530^2)^k}S_k(1,265^2)=\frac{1615175}{48\pi}.$$

Part III. Series for $1/\pi$ involving $F_n(x)$, $p_n(x)$ and $W_n(x)$

Series for $1/\pi$ involving $F_n(x)$

An identity of MacMahon implies that the polynomial

$$F_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}x^{n-k}$$

at $x=-4$ coincides with the Franel number $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$.

Sun [J. Nanjing Univ. Math. Biquarterly 31 (2014)] listed ten conjectural series for $1/\pi$ involving $F_n(x)$, eight of them were confirmed by S. Cooper, J. G. Wan and W. Zudilin (2017), but the following two remain open:

$$\sum_{k=0}^{\infty}\frac{357k+103}{2160^k}\binom{2k}{k}F_k(-324)=\frac{90}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{k}{3645^k}\binom{2k}{k}F_k(486)=\frac{10}{3\pi}.$$

More series for $1/\pi$ involving $F_n(x)$

Below are more such series discovered in 2020.

Conjecture (Sun, Colloq. Math. 173 (2023)). We have the following identities:

$$\sum_{k=0}^{\infty}\frac{6k+1}{(-1728)^k}\binom{2k}{k}F_k(-324)=\frac{24}{25\pi}\sqrt{375+120\sqrt{10}},$$

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-160)^k}\binom{2k}{k}F_k(-20)=\frac{\sqrt{30}}{5\pi}\cdot\frac{5+\sqrt[3]{145+30\sqrt{6}}}{\sqrt[6]{145+30\sqrt{6}}},$$

$$\sum_{k=0}^{\infty}\frac{1290k+289}{27648^k}\binom{2k}{k}F_k(-2160)=\frac{96\sqrt{15}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{804k+49}{276480^k}\binom{2k}{k}F_k(12096)=\frac{120\sqrt{15}}{\pi},$$

$$\sum_{k=0}^{\infty}(24k+5)\left(\frac{2}{135}\right)^k F_k\left(-\frac{27}{8}\right)=\frac{3}{2\pi}(5\sqrt{6}+4\sqrt{15}).$$

Series for $1/\pi$ involving $p_n(x)$

In March 2011, I introduced the polynomials

$$p_n(x)=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^{n-k}\quad (n=0,1,2,\ldots)$$

and proved that $\sum_{k=0}^{\infty}k\binom{2k}{k}p_k(4)/128^k=\sqrt{2}/\pi$,

$$\sum_{k=0}^{\infty}\frac{8k+1}{576^k}\binom{2k}{k}p_k(4)=\frac{9}{2\pi},\ \sum_{k=0}^{\infty}\frac{8k+1}{(-4032)^k}\binom{2k}{k}p_k(4)=\frac{9\sqrt{7}}{8\pi}.$$

via Ramanujan-type series for $1/\pi$. I noted that

$$\binom{2n}{n}p_n(4)=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{k}{n-k}(-64)^{n-k}.$$

Series for $1/\pi$ involving $p_n(x)$

Conjecture (Sun, 2011) We have

$$\sum_{k=0}^{\infty}\frac{17k-224}{(-225)^k}\binom{2k}{k}p_k(-14)=\frac{1800}{\pi},\ \sum_{k=0}^{\infty}\frac{15k-256}{17^{2k}}\binom{2k}{k}p_k(18)=\frac{2312}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k-11}{(-576)^k}\binom{2k}{k}p_k(-32)=\frac{90}{\pi},\ \sum_{k=0}^{\infty}\frac{3k-2}{640^k}\binom{2k}{k}p_k(36)=\frac{5\sqrt{10}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k-67}{(-3136)^k}\binom{2k}{k}p_k(-192)=\frac{490}{\pi},\ \sum_{k=0}^{\infty}\frac{7k-24}{3200^k}\binom{2k}{k}p_k(196)=\frac{125\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{5k-32}{(-6336)^k}\binom{2k}{k}p_k(-392)=\frac{495}{2\pi},\ \sum_{k=0}^{\infty}\frac{66k-427}{6400^k}\binom{2k}{k}p_k(396)=\frac{1000\sqrt{11}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{34k-7}{(-18432)^k}\binom{2k}{k}p_k(-896)=\frac{54\sqrt{2}}{\pi},\ \sum_{k=0}^{\infty}\frac{24k-5}{136^{2k}}\binom{2k}{k}p_k(900)=\frac{867}{16\pi}.$$

These identities remain open!

Series for $1/\pi$ involving $p_n(x)$

Conjecture (Sun, 2019). For any prime $p \neq 2, 5$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{100^k} p_k\left(\frac{9}{4}\right)$$
$$\equiv \begin{cases} (\frac{-1}{p})(4x^2-2p) \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e.,}\ p\equiv 3,5,6 \pmod 7, \end{cases}$$

and

$$\sum_{k=0}^{pn-1} \frac{12k+1}{100^k}\binom{2k}{k} p_k\left(\frac{9}{4}\right) - \left(\frac{-1}{p}\right) p \sum_{r=0}^{n-1} \frac{12r+1}{100^r}\binom{2r}{r} p_r\left(\frac{9}{4}\right)$$

divided by $(pn)^2\binom{2n}{n}$ is p-adic integral for all $n \in \mathbb{Z}^+$. Moreover,

$$\sum_{k=0}^{\infty} \frac{12k+1}{100^k}\binom{2k}{k} p_k\left(\frac{9}{4}\right) = \frac{75}{4\pi}.$$

Franel numbers of order four

Franel numbers of order 4: $f_n^{(4)} = \sum_{k=0}^{n} \binom{n}{k}^4$ $(n \in \mathbb{N})$.

By Zeilberger's algorithm, the sequence $(f_n^{(4)})_{n\geqslant 0}$ satisfies the following recurrence first claimed by Franel:

$$(n+2)^3 f_{n+2}^{(4)} = 4(1+n)(3+4n)(5+4n) f_n^{(4)} + 2(3+2n)(7+9n+3n^2) f_{n+1}^{(4)}.$$

In 2005 Y. Yang used mofular forms of level 10 to discover the following curious identity relating Franel numbers of order four to Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty} \frac{4k+1}{36^k} f_k^{(4)} = \frac{18}{\sqrt{15}\,\pi}.$$

More this kind of identities were deduced by S. Cooper in 2012 via modular forms.

$W_n(x)$

For $n \in \mathbb{N}$ the polynomial

$$W_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} x^k$$
$$= \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} x^k$$

at $x = -1$ coincides with $(-1)^n f_n^{(4)}$.

In 2011 the author proposed ten identities of the form

$$\sum_{k=0}^{\infty} \frac{ak+b}{m^k} W_k\left(\frac{1}{m}\right) = \frac{C}{\pi},$$

where a, b, m are integers with $am \neq 0$, and C^2 is rational. They were later confirmed by Cooper et al. in 2017.

Six series found in August 2020 have been proved

Theorem (Z.-W. Sun, Colloq. Math. 173 (2023)).

$$\sum_{k=0}^{\infty}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)=\frac{215\sqrt{15}}{12\pi},$$

$$\sum_{k=0}^{\infty}\frac{1360k+389}{(-60)^k}W_k\left(\frac{16}{15}\right)=\frac{205\sqrt{15}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)=\frac{10125\sqrt{7}}{56\pi},$$

$$\sum_{k=0}^{\infty}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)=\frac{260480\sqrt{5}}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)=\frac{1314625\sqrt{2}}{12\pi},$$

$$\sum_{k=0}^{\infty}\frac{41673840k+4777111}{5780^k}W_k\left(\frac{1444}{1445}\right)=\frac{147758475}{\sqrt{95}\,\pi}.$$

An auxiliary theorem

Auxiliary Theorem (Sun, Colloq. Math. 173 (2023)). Let a, b and x be complex numbers with $|x-1| \geqslant 7.5$. Then

$$\frac{10}{x}(x-1)^2(x-2)\sum_{n=0}^{\infty}\frac{an+b}{(4x)^n}W_n\left(1-\frac{1}{x}\right)$$
$$=\sum_{k=0}^{\infty}(2ax(5x-7)k+a(10x-13)+10b(x-1)(x-2))\frac{f_k^{(4)}}{(4x-4)^k}.$$

Combining this with known identities of the form

$$\sum_{k=0}^{\infty}\frac{ak+b}{m^k}f_k^{(4)}=\frac{c}{\pi}$$

we obtain the six identities in our theorem.

Open series involving $W_n(x)$

Conjecture (Sun, August 2020). We have

$$\sum_{k=0}^{\infty}\frac{4k+1}{6^k}W_k\left(-\frac{1}{8}\right)=\frac{\sqrt{72+42\sqrt{3}}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)=\frac{387\sqrt{3}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)=\frac{1652\sqrt{3}}{9\pi},$$

$$\sum_{k=0}^{\infty}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)=\frac{1184\sqrt{35}}{5\pi},$$

$$\sum_{k=0}^{\infty}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)=\frac{1134\sqrt{35}}{125\pi},$$

$$\sum_{k=0}^{\infty}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)=\frac{49356\sqrt{39}}{5\pi},$$

More conjectural series involving $W_n(x)$

and

$$\sum_{k=0}^{\infty} \frac{9360k-1343}{1300^k} W_k\left(\frac{900}{13}\right) = \frac{21515\sqrt{39}}{3\pi},$$

$$\sum_{k=0}^{\infty} \frac{56355k+2443}{(-5776)^k} W_k\left(-\frac{83521}{361}\right) = \frac{4669535\sqrt{2}}{68\pi},$$

$$\sum_{k=0}^{\infty} \frac{5928k+253}{5780^k} W_k\left(\frac{1156}{5}\right) = \frac{28951\sqrt{2}}{4\pi}.$$

Series involving $\sum_{k=0}^{n} 5^k \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 / \binom{n}{k}$

Conjecture (Z.-W. Sun, Jan. 2012) (i) For any prime $p > 3$ we have

$$\sum_{n=0}^{p-1} \frac{28n+5}{576^n} \binom{2n}{n} \sum_{k=0}^{n} 5^k \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} \equiv p \left(\frac{-1}{p}\right) \left(3 + 2\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

(ii) We have the identity

$$\sum_{n=0}^{\infty} \frac{28n+5}{576^n} \binom{2n}{n} \sum_{k=0}^{n} 5^k \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = \frac{9}{\pi}(2+\sqrt{2}).$$

Series involving $\sum_{k=0}^{n}\left(-\frac{25}{16}\right)^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2/\binom{n}{k}$

Conjecture (Sun). For any prime $p>5$, we have

$$\left(\frac{-1}{p}\right)\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{576^n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1,\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1,\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1,\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

Conjecture (Sun, August 2020).

$$\sum_{n=0}^{\infty}\frac{182n+31}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}\left(-\frac{25}{16}\right)^k=\frac{189}{2\pi}.$$

Part IV. Series for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$ not of the Ramanujan type

Recent conjectural hypergeometric series for $1/\pi$ and $1/\pi^2$

Conjecture (Z.-W. Sun, Oct.-Nov. 2023). We have

$$\sum_{k=0}^{\infty}\frac{(66k^2+37k+4)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}{(2k+1)729^k}=\frac{27\sqrt{3}}{2\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{(92k^3+54k^2+12k+1)\binom{2k}{k}^7}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}=\frac{12}{\pi^2}.$$

Remark. Note that the two series are **not of the Ramanujan type**. As they converge fast, it is easy to check the two identities numerically.

Main Papers containing my Conjectural Series:

1. Z.-W. Sun, *List of conjectural series for powers of* π *and other constants*, in: Ramanujan's Identities, Press of Harbin Institute of Tech., 2020, Chapter 5, 205–261. [This is arXiv:1102.5649v47.]

2. Z.-W. Sun, *New series for powers of* π *and related congruences*, Electron. Res. Arch. **28** (2020), 1273–1342.

3. Z.-W. Sun, New Conjectures in Number Theory and Combinatorics, Harbin Institute of Technology Press, 2021 (in Chinese).

4. Z.-W. Sun, *Some new series for* $1/\pi$ *motivated by congruences*, Colloq. Math. **173** (2023), 89–109.

5. Z.-W. Sun, *New series involving binomial coefficients*, arXiv:2307.03086, 2023.

Thank you!